1 Fourier Series

Periodic Function A function is called "periodic" function of period p if $f(x+np) = f(x)$, $\forall x$, $\exists n > 0$, n: integer. If both f and g are functions of period p, then $af + bg$ is also a function of period p.

Fourier Series Periodic functions of period 2π can be represented in terms of trigonometric series as

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)} \qquad (1)$$

If the series converges, it is called "Fourier series" of $f(x)$, where a_0, a_n, b_n are called "Fourier coefficients". a_0, a_n, b_n are given by the "Euler formulas":

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx; \qquad \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\begin{Bmatrix} \cos nx \\ \sin nx \end{Bmatrix}dx, n = 1,2,\ldots} \qquad (2)$$

Derivation of the Euler Formulas

Preliminary Orthogonality of trigonometric system

(1) $\int_{-\pi}^{\pi}\cos nx \cos mx dx = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases} = \pi\delta_{nm}$

"Kronecker Delta" : $\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

Proof $\int_{-\pi}^{\pi}\cos nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(n+m)x + \cos(n-m)x]dx = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases}$

(2) $\int_{-\pi}^{\pi}\sin nx \sin mx dx = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases} = \pi\delta_{nm}$

Proof $\int_{-\pi}^{\pi}\sin nx \sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi}[-\cos(n+m)x + \cos(n-m)x]dx = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases}$

(3) $\int_{-\pi}^{\pi}\sin nx \cos mx dx = 0$

Proof $\int_{-\pi}^{\pi}\sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(n+m)x + \sin(n-m)x]dx = 0$

Now, integrating both sides of (1) from $-\pi$ to π yields

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx = 2\pi a_0 \to a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

Likewise, multiplying both sides of (*) by cos nx and integrating both sides from $-\pi$ to π yields

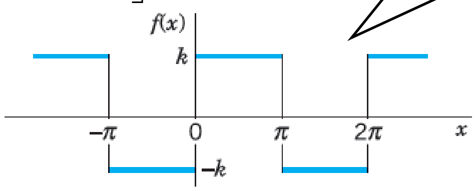
$$\int_{-\pi}^{\pi} f(x)\cos nx dx = \int_{-\pi}^{\pi}\left[a_0 \cos nx + \sum_{m=1}^{\infty}\cos nx(a_m \cos mx + b_m \sin mx)\right]dx = \pi a_n$$

Likewise, $\int_{-\pi}^{\pi} f(x)\sin nx dx = \int_{-\pi}^{\pi}\left[a_0 \sin nx + \sum_{m=1}^{\infty}\sin nx(a_m \cos mx + b_m \sin mx)\right]dx = \pi b_n$.

EX s1

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Odd function



Clearly, $a_0 = 0$ and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\, dx + \int_{0}^{\pi} k \cos nx\, dx\right] = \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\, dx + \int_{0}^{\pi} k \sin nx\, dx\right] = \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\frac{\cos nx}{n}\Big|_{0}^{\pi}\right].$$

$$b_n = \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi). \quad = \begin{cases} 4k/n\pi & n: odd \\ 0 & n: even \end{cases}$$

$$\therefore f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

Odd functions include only sine terms

NOTE The integral range does not need to be from $-\pi$ to π, it can be "arbitrary" as long as it covers an entire period, e.g., 0 to 2π and vice versa.

2 Functions of any period $p = 2L$

Assume $g(v)$ is a function of period 2π, then $g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$.

Now, let $x = Lv/\pi$ (or $v = \pi x/L$) (called "change of scale"), then let

$$g(v) = g(v + 2n\pi) = g\left(\frac{\pi}{L}x + 2n\pi\right) = g\left(\frac{\pi}{L}[x + 2nL]\right)$$

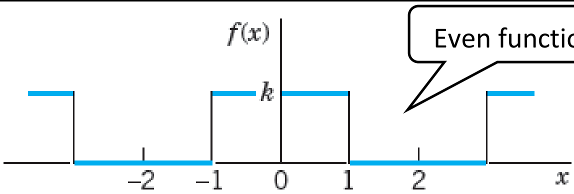
Period $2L$ with respect to x

Let $g(v) = f(x)$, then $f(x+2nL) = f(x)$, i.e., periodic function of $2L$.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \text{ with } a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx; \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{L}\int_{-L}^{L} f(x)\begin{Bmatrix} \cos\frac{n\pi x}{L} \\ \sin\frac{n\pi x}{L} \end{Bmatrix}dx, n = 1,2,\ldots \quad (3)$$

EX s2

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$



Even function

$a_0 = 2k/4 = k/2$, $b_n = 0$

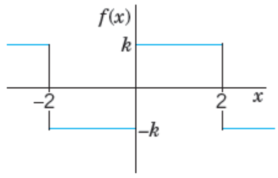
$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}. \quad = \begin{cases} (-1)^{\ell}\, 2k/n\pi & n: odd, n = 2\ell + 1 \\ 0 & n: even \end{cases}$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right)$$

Even functions include only cosine terms

EX s3 EX s1 with period changed to 4

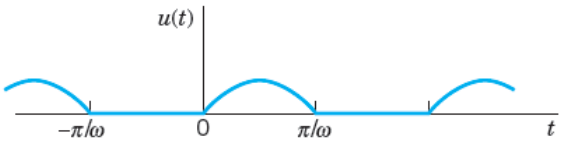
$$f(x) = \begin{cases} -k & \text{if} \quad -2 < x < 0 \\ k & \text{if} \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2$$



$$g(v) = \frac{4k}{\pi}\left(\sin v + \frac{1}{3}\sin 3v + \frac{1}{5}\sin 5v + \cdots\right) \xrightarrow{v = \pi x/2} f(x) = \frac{4k}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right)$$

EX s4 half-wave rectifier

$$u(t) = \begin{cases} 0 & \text{if} \quad -L < t < 0, \\ E\sin\omega t & \text{if} \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \qquad L = \frac{\pi}{\omega}$$



$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\sin\omega t\, dt = \frac{E}{\pi}$$

Clearly, $n=1 \rightarrow a_1=0$

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t\cos n\omega t\, dt = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\sin(1+n)\omega t + \sin(1-n)\omega t]\, dt$$

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} = \frac{E}{2\pi}\left(\frac{-\cos(1+n)\pi + 1}{1+n} + \frac{-\cos(1-n)\pi + 1}{1-n}\right)$$

$n: odd, a_n = 0; n: even, a_n = \frac{E}{2\pi}\left(\frac{2}{1+n}+\frac{2}{1-n}\right) = \frac{E}{\pi}\frac{2}{1-n^2}$; $b_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t \sin n\omega t dt = \begin{cases} E/2 & n=1 \\ 0 & else \end{cases}$

3 Even, Odd Functions & Cosine, Sine Series

EVEN: $f_e(-x)=f_e(x)$, $b_n=0$, $f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ with $a_0 = \frac{1}{L}\int_0^L f_e(x)dx; a_n = \frac{2}{L}\int_0^L f_e(x)\cos\frac{n\pi x}{L}dx, n = 1,2,\ldots$

ODD: $f_o(-x)=-f_o(x)$, $a_n=0$, $f_o(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ with $b_n = \frac{2}{L}\int_0^L f_o(x)\sin\frac{n\pi x}{L}dx, n = 1,2,\ldots$.

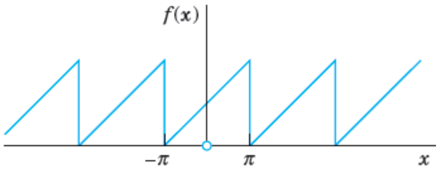
{Even, Odd} functions become Fourier {Cosine, Sine} series, respectively. Thus, for any function f,

$$f(x) = f_e(x) + f_o(x) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}_{even} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}_{odd}$$

EX s4b Consider $w(t) = u(t)-(E/2)\sin \omega t$, one can see that $w(t) = (E/2)\sin \omega t$, $p = \pi/\omega$ becomes an even function.

It follows that $a_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega}\frac{E}{2}\sin\omega t dt; a_n = \frac{2\omega}{\pi}\int_0^{\pi/\omega}\frac{E}{2}\sin\omega t\cos 2n\omega t dt, n = 1,2,\ldots$

EX s5 Sawtooth wave



$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

Consider $f(x) = f_1(x) + f_2(x)$ where $f_1(x) = x$, $p=2\pi$ (f_1 is an odd function), $f_2(x) = \pi$, then

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx = \frac{2}{\pi}\left[\frac{-x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\, dx\right] = -\frac{2}{n}\cos n\pi.$$

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right)$$

Condition for Existence and Convergence

Existence $\int_{-L}^{L}|f(x)|dx < \infty$ (Absolutely integrable)

Convergence

DIRICHLET CONDITIONS

Theorem 2-1: Suppose that

(i) $f(x)$ is defined and single-valued except possibly at a finite number of points in $(-L, L)$

(ii) $f(x)$ is periodic with period $2L$

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$

Then the series (1) with coefficients (2) or (3) converges to

(a) $f(x)$ if x is a point of continuity

(b) $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

UNIFORM CONVERGENCE

Suppose that we have an infinite series $\sum_{n=1}^{\infty} u_n(x)$. We define the Rth *partial sum* of the series to be the sum of the first R terms of the series, i.e.

$$S_R(x) = \sum_{n=1}^{R} u_n(x) \tag{7}$$

Now by definition the infinite series is said to *converge* to $f(x)$ in some interval if given any positive number ϵ, there exists for each x in the interval a positive number N such that

$$|S_R(x) - f(x)| < \epsilon \quad \text{whenever } R > N \tag{8}$$

The number N depends in general not only on ϵ but also on x. We call $f(x)$ the *sum* of the series.

An important case occurs when N depends on ϵ but *not* on the value of x in the interval. In such case we say that the series converges *uniformly* or is *uniformly convergent* to $f(x)$.

Theorem 2-2: If each term of an infinite series is continuous in an interval (a, b) and the series is uniformly convergent to the sum $f(x)$ in this interval, then

1. $f(x)$ is also continuous in the interval
2. the series can be integrated term by term, i.e.

$$\int_a^b \left\{\sum_{n=1}^{\infty} u_n(x)\right\} dx \quad = \quad \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx \qquad (9)$$

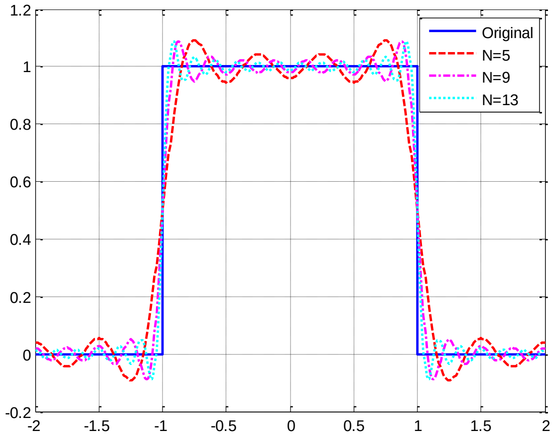
Theorem 2-3: If each term of an infinite series has a derivative and the series of derivatives is uniformly convergent, then the series can be differentiated term by term, i.e.

$$\frac{d}{dx}\sum_{n=1}^{\infty} u_n(x) \quad = \quad \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x) \qquad (10)$$

Theorem 2-4 **(*Weierstrass M test*):** If there exists a set of constants M_n, $n = 1, 2, \ldots$, such that for all x in an interval $|u_n(x)| \leq M_n$, and if furthermore $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly in the interval. Incidently, the series is also *absolutely convergent*, i.e. $\sum_{n=1}^{\infty} |u_n(x)|$ converges, under these conditions.

PARSEVAL's IDENTITY states that $\boxed{\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}$ if a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions. For the proof, consider $\int_{-L}^{L}\{f(x)\}^2 dx = \int_{-L}^{L}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)\right] f(x)dx$, then use Euler's formula.

Gibbs Phenomenon



The sinusoidal components of the signal that occur at multiples of the fundamental frequency are called harmonics. In general, for well-behaved (continuous) periodic signals, a sufficiently large number of harmonics can be used to approximate the signal reasonably well. For periodic signals with discontinuities, however, such as a periodic square wave, even a large number of harmonics will not be sufficient to reproduce the square wave exactly, resulting in the appearance of so-called "overshoot". This effect is known as Gibbs phenomenon and it manifests itself in the form of ripples of increasing frequency and closer to the transitions of the square signal. An illustration of Gibbs phenomenon is shown in the figure on the right. The figure shows the result of adding 5, 9 and 13 harmonics. The overshoots tend to shift toward the discontinuities, but their magnitude do not change much.

EX s4c (Half-wave rectifier) $u(t) = \begin{cases} E\sin\omega t & 0 < t < L \\ 0 & -L < t < 0 \end{cases}, p = 2L, L = \frac{\pi}{\omega}$

$$u(t) = \frac{E}{2}\sin\omega t + \frac{E}{\pi}\left[1 - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t - \cdots\right] \ldots(*)$$

$$u'(t) = \begin{cases} E\omega\cos\omega t & 0 < t < L \\ 0 & -L < t < 0 \end{cases}, p = 2L, L = \frac{\pi}{\omega}$$

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\omega\cos\omega t\,dt = 0; a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\omega\cos\omega t\cos n\omega t\,dt = \begin{cases} E\omega/2 & n=1 \\ 0 & else \end{cases}$$

$$b_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\omega\cos\omega t\sin n\omega t\,dt = \frac{\omega^2 E}{2\pi}\int_0^{\pi/\omega}\left[\sin(n+1)\omega t + \sin(n-1)\omega t\right]dt$$

$$= \frac{\omega^2 E}{\pi}\left[-\left.\frac{\cos(n+1)\omega t}{(n+1)\omega}\right|_0^{\pi/\omega} - \left.\frac{\cos(n-1)\omega t}{(n-1)\omega}\right|_0^{\pi/\omega}\right]$$

$$n: odd, b_n = 0; n: even, b_n = \frac{E\omega}{2\pi}\left(\frac{2}{n+1} + \frac{2}{n-1}\right) = \frac{E\omega}{\pi}\frac{2n}{n^2-1} \rightarrow$$

$$u'(t) = \frac{E\omega}{2}\cos\omega t + \frac{2E\omega}{\pi}\left[\frac{2}{3}\sin 2\omega t + \frac{4}{15}\sin 4\omega t + \frac{6}{35}\sin 6\omega t - \cdots\right]$$

<u>EX s6</u> Consider $f(x) = x$ on the interval $[-\pi,\pi]$ → Fourier series: $f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx \ldots(*)$

From Parseval's formula: $\sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n}\right]^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx \rightarrow 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi}\left(\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right) = \frac{2\pi^2}{3} \rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

<u>EX s7</u> Consider $f(x)=x$, $g(x) = \int_{-\pi}^{x} f(u)du = \int_{-\pi}^{x} u\,du = \frac{x^2}{2} - \frac{\pi^2}{2}$

→ Fourier series: $g(x) = \frac{\pi^2}{6} - \frac{\pi^2}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx$

Integrating term by term of (*) in the previous example yields

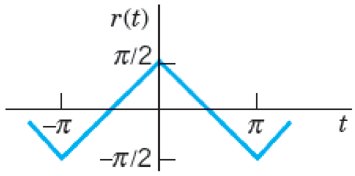
$$\int_{-\pi}^{x} f(u)du = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\int_{-\pi}^{x}\sin nu\,du = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left.\frac{-\cos nu}{n}\right|_{-\pi}^{x} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{2}{n^2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - \frac{\pi^2}{3}$$

4 Application of Fourier Series to ODE for steady-state solution

Consider ODE when the input (external force) becomes **<u>periodic</u>**, e.g., $my''+cy'+ky = r(t)$, where $r(t)$ is periodic. The idea is to represent $r(t)$ in terms of Fourier series, and then solve the ODE for each term. For example,

<u>Example</u>

$$y'' + 0.05y' + 25y = r(t) \qquad r(t) = \begin{cases} t + \frac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \frac{\pi}{2} & \text{if } 0 < t < \pi, \end{cases} \qquad r(t+2\pi) = r(t)$$



First Fourier series

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

Next solve

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

n^{th} term becomes

$y_n = A_n\cos nt + B_n\sin nt.$ with $A_n = \frac{4(25-n^2)}{n^2\pi D_n}$, $B_n = \frac{0.2}{n\pi D_n}$, where $D_n = (25-n^2)^2 + (0.05n)^2$

<u>Alternative Approach</u> Using the transfer function $Q(s) = (s^2+0.05s+25)^{-1}$. For sinusoidal steady state response,

$$Q(jn) = \left(-n^2 + j0.05n + 25\right)^{-1} = \frac{25-n^2-j0.05n}{\left(25-n^2\right)^2 + (0.05n)^2}; y_n = \frac{4}{n^2\pi}\mathrm{Re}\,Q(jn)e^{jnt} = \frac{4}{n^2\pi}\left[\frac{25-n^2}{D_n}\cos nt + \frac{0.05n}{D_n}\sin nt\right]$$

The total solution is given by $y = y_1 + y_3 + y_5 + \ldots$

5 Application to PDE

(a) Heat Equation

Consider the heat conduction along a heat-conducting homogeneous rod with temperature distribution $u(x, t)$.

Assumptions for one dimensional (1-D) case:

1. homogeneous rod with uniform cross section A and constant density ρ [kg/m^3], specific heat σ [J/kg·K], thermal conductivity κ [W/m·K]
2. insulated laterally so heat flows only in x-direction only
3. temperature is constant at all points of a cross section

The heat equation in this case is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, (t \geq 0, 0 \leq x \leq L) \ldots (1); c^2 = \frac{\kappa}{\sigma\rho} \text{ [m}^2\text{/s] (for three dimensional (3-D) case: } \frac{\partial u}{\partial t} = c^2 \nabla^2 u, (t \geq 0) \text{)}$$

with initial condition: $u(x, 0) = f(x)$ $(0 \leq x \leq L) \ldots (2)$,

and boundary conditions : $u(0, t) = 0$, $u(L, t) = 0$, $(t \geq 0) \ldots$ (3)

By method of separation of variables: $u(x, t) = F(x) \cdot G(t)$

A constant is required for the RHS in order to satisfy for all t and x.

$$(1) \Rightarrow F\dot{G} = c^2 F''G \Rightarrow \underbrace{\frac{\dot{G}}{c^2 G}}_{t\text{-dependence}} = \underbrace{\frac{F''}{F}}_{x\text{-dependence}} = \begin{cases} p^2 \Rightarrow \dot{G} = c^2 p^2 G \Rightarrow G(t) = Ce^{c^2 p^2 t} \times \\ 0 \Rightarrow \dot{G} = 0 \Rightarrow G(t) = C \times \\ -p^2 \Rightarrow \dot{G} = -c^2 p^2 G \Rightarrow G(t) = Ce^{-c^2 p^2 t} \end{cases}$$

$$F'' + p^2 F = 0 \ldots (4), \dot{G} + c^2 p^2 G = 0 \ldots (5)$$

$$(4) \Rightarrow F(x) = A\cos px + B\sin px \xrightarrow{(3)} F(0)G(t) = 0, F(L)G(t) = 0 \Rightarrow F(0) = F(L) = 0 \Rightarrow A = 0, \sin pL = 0$$

$$\Rightarrow pL = n\pi \Rightarrow p = \frac{n\pi}{L}, n = 1,2,\ldots \Rightarrow F_n(x) = \sin\frac{n\pi x}{L}, n = 1,2,\ldots$$

$$(5) \Rightarrow \dot{G} + \lambda_n^2 G = 0, \lambda_n = \frac{cn\pi}{L} \Rightarrow G_n(t) = B_n e^{-\lambda_n^2 t}, n = 1,2,\ldots$$

Eigenfunction $u_n(x,t) = F_n(x) \cdot G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}, n = 1,2,\ldots$ with corresponding eigenvalue λ_n

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t} \xrightarrow{(2)} u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L} dx.$$

Fourier sine series

(b) Wave Equation

Consider a 1-D wave equation, which represents an up-down movement of a "thin" string.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, (t \geq 0, 0 \leq x \leq L) \ldots (1); \text{(for three dimensional (3-D) case: } \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, (t \geq 0) \text{)}$$

with initial conditions (initial deflection, velocity): $u(x, 0) = f(x)$ …(2a), $\partial u(x,0)/\partial t = g(x)$…(2b), $(0 \leq x \leq L)$

and boundary conditions : $u(0, t) = 0$, $u(L, t) = 0$, $(t \geq 0) \ldots$ (3)

By method of separation of variables: $u(x, t) = F(x) \cdot G(t)$

$$(1) \Rightarrow F\ddot{G} = c^2 F''G \Rightarrow \underbrace{\frac{\ddot{G}}{c^2 G}}_{t\text{-dependence}} = \underbrace{\frac{F''}{F}}_{x\text{-dependence}} = \begin{cases} p^2 \Rightarrow F'' = p^2 F \Rightarrow F(x) = Ae^{px} + Be^{-px} \xrightarrow{(3)} A = B = 0 \times \\ 0 \Rightarrow F'' = 0 \Rightarrow F(x) = ax + b \xrightarrow{(3)} a = b = 0 \times \\ -p^2 \Rightarrow F'' = -p^2 F \Rightarrow F(x) = Ae^{jpx} + Be^{-jpx} \end{cases}$$

$$F'' + p^2 F = 0 \ldots (4), \ddot{G} + c^2 p^2 G = 0 \ldots (5)$$

$$(4) \Rightarrow F(x) = A\cos px + B\sin px \xrightarrow{(3)} F(0)G(t) = 0, F(L)G(t) = 0 \Rightarrow F(0) = F(L) = 0 \Rightarrow A = 0, \sin pL = 0$$

$$\Rightarrow pL = n\pi \Rightarrow p = \frac{n\pi}{L}, n = 1,2,\ldots \Rightarrow F_n(x) = \sin\frac{n\pi x}{L}, n = 1,2,\ldots$$

$$(5) \Rightarrow \ddot{G} + \lambda_n^2 G = 0, \lambda_n = \frac{cn\pi}{L} \Rightarrow G_n(t) = A_n \cos\lambda_n t + B_n \sin\lambda_n t, n = 1,2,\ldots$$

Eigenfunction $u_n(x,t)=\left(A_n\cos\lambda_n t+B_n\sin\lambda_n t\right)\sin\frac{n\pi x}{L}, n=1,2,\ldots$ with corresponding eigenvalue λ_n

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\lambda_n t+B_n\sin\lambda_n t\right)\sin\frac{n\pi x}{L}$$

$$\xrightarrow{(2a)} u(x,0)=f(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}\Rightarrow A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$\xrightarrow{(2b)} \left.\frac{\partial u}{\partial x}\right|_{t=0}=g(x)=\sum_{n=1}^{\infty}B_n\lambda_n\sin\frac{n\pi x}{L}\Rightarrow B_n\lambda_n=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

6 Complex Fourier Series

Using Euler's Formula, given by $\cos x=\frac{e^{jx}+e^{-jx}}{2};\sin x=\frac{e^{jx}-e^{-jx}}{j2}$, then

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right)=a_0+\sum_{n=1}^{\infty}\left(a_n\frac{e^{j\frac{n\pi}{L}x}+e^{-j\frac{n\pi}{L}x}}{2}+b_n\frac{e^{j\frac{n\pi}{L}x}-e^{-j\frac{n\pi}{L}x}}{j2}\right)$$

$$=a_0+\sum_{n=1}^{\infty}\left(\frac{a_n}{2}-\frac{jb_n}{2}\right)e^{j\frac{n\pi}{L}x}+\sum_{n=1}^{\infty}\left(\frac{a_n}{2}+\frac{jb_n}{2}\right)e^{-j\frac{n\pi}{L}x}=\sum_{n=-\infty}^{\infty}c_n e^{j\frac{n\pi}{L}x}$$

where $c_0=a_0$, $c_n=(a_n-jb_n)/2$, $c_{-n}=c_n^*$, $n=1,2,\ldots$

$$c_0=a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx; c_n=\frac{a_n-jb_n}{2}=\frac{1}{2L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx-\frac{j}{2L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx=\frac{1}{2L}\int_{-L}^{L}f(x)e^{-j\frac{n\pi}{L}x}dx$$

<u>EX s2b</u> EX2 revisited.

$$c_n=\frac{1}{4}\int_{-1}^{1}ke^{-j\frac{n\pi}{L}x}dx=\frac{k}{4}\left(-\frac{2}{jn\pi}\right)e^{-j\frac{n\pi}{2}x}\Bigg|_{-1}^{1}=\frac{k}{j2n\pi}\left(e^{j\frac{n\pi}{2}}-e^{-j\frac{n\pi}{2}}\right)=\frac{k}{n\pi}\sin\frac{n\pi}{2}=\begin{cases}k/2 & n=0\\(-1)^{\ell}k/n\pi & n:odd, n=2\ell+1\\0 & n:even\end{cases}$$

Thus, $a_0=k/2$; $a_n=(-1)^{\ell}2k/n\pi$, $n=2\ell+1$, $\ell=0,1,2,\ldots$

<u>EX s4c</u> Half-wave rectifier with E=1. Note that $L=\pi/\omega$.

$$c_n=\frac{\omega}{2\pi}\int_0^{\pi/\omega}\sin\omega t e^{-jn\omega t}dt=\frac{\omega}{2\pi}\int_0^{\pi/\omega}\left(\frac{e^{j\omega t}-e^{-j\omega t}}{j2}\right)e^{-jn\omega t}dt=\frac{\omega}{j4\pi}\int_0^{\pi/\omega}\left(e^{-j(n-1)\omega t}-e^{-j(n+1)\omega t}\right)dt$$

$$c_n=\frac{\omega}{j4\pi}\int_0^{\pi/\omega}\left(e^{-j(n-1)\omega t}-e^{-j(n+1)\omega t}\right)dt=\frac{\omega}{j4\pi}\left[\frac{e^{-j(n-1)\omega t}}{-j(n-1)\omega}\Bigg|_0^{\pi/\omega}-\frac{e^{-j(n+1)\omega t}}{-j(n+1)\omega}\Bigg|_0^{\pi/\omega}\right]$$ n: odd, $n\neq1$ $\rightarrow c_n=0$,

$n=1$; $c_1=\frac{\omega}{j4\pi}\int_0^{\pi/\omega}\left(1-e^{-j2\omega t}\right)dt=\frac{\omega}{j4\pi}\frac{\pi}{\omega}=-\frac{j}{4}$; $\rightarrow a_1=0$, $b_1=½$

n:even$\rightarrow$ $c_n=\frac{\omega}{j4\pi}\left[\frac{-2}{-j(n-1)\omega}-\frac{-2}{-j(n+1)\omega}\right]=-\frac{1}{2\pi}\left[\frac{1}{n-1}-\frac{1}{n+1}\right]=-\frac{1}{\pi}\frac{1}{n^2-1}; a_0=c_0=\frac{1}{\pi}; a_n=c_n+c_{-n}=-\frac{2}{\pi}\frac{1}{n^2-1}$

7 Fourier Transform

Consider a periodic function $f_L(x)$, $f_L(x)=\sum_{n=-\infty}^{\infty}c_n e^{j\frac{n\pi}{L}x}$, $p=2L$. Now, set $w_n=\frac{n\pi}{L}, \Delta w=w_{n+1}-w_n=\frac{\pi}{L}, \frac{1}{L}=\frac{\Delta w}{\pi}$.

Let $L\rightarrow\infty$, assuming $f(x)=\lim_{L\rightarrow\infty}f_L(x)$ is "absolutely integrable", i.e.,

$$\lim_{a\rightarrow-\infty}\int_a^0|f(x)|dx+\lim_{b\rightarrow\infty}\int_0^b|f(x)|dx\left(\text{P.V.}\int_{-\infty}^{\infty}|f(x)|dx\right)$$ exists.

P.V. stands for Cauchy's principal value

then $\Delta w\rightarrow0$, $w_n\rightarrow w$ (becomes "continuous"), the infinite sum of the Fourier series of $f_L(x)$ becomes infinite integral as $\Delta w\rightarrow dw$ in integrand:

$$f(x)=\lim_{L\to\infty} f_L(x)=\lim_{L\to\infty}\sum_{n=-\infty}^{\infty}\left(\frac{1}{2L}\int_{-L}^{L} f_L(v)e^{-j\frac{n\pi}{L}v}dv\right)e^{j\frac{n\pi}{L}x}=\lim_{L\to\infty}\sum_{n=-\infty}^{\infty}\left(\frac{\Delta w}{2\pi}\int_{-L}^{L} f_L(v)e^{-jw_n v}dv\right)e^{jw_n x}$$

$$f(x)=\int_{-\infty}^{\infty}\left(\frac{dw}{2\pi}\int_{-\infty}^{\infty} f(v)e^{-jwv}dv\right)e^{jwx}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\underbrace{\int_{-\infty}^{\infty} f(v)e^{-jwv}dv}_{\hat{f}(w)}\right)e^{jwx}dw$$

In engineering, Fourier Transform pair : $\boxed{\mathfrak{F}[f]=\hat{f}(w)=\int_{-\infty}^{\infty} f(x)e^{-jwx}dx \leftrightarrow \mathfrak{F}^{-1}[\hat{f}]=f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)e^{jwx}dw}$

In mathematics, generally define $\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-jwx}dx \leftrightarrow f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{jwx}dw$

Note also that conventionally variable pairs (x, w), (t (time), ω (frequency)) are used.

EX t1 unit gate function $f(x)=rect(x)=\begin{cases}1 & |x|<0.5\\ 0.5 & |x|=0.5\\ 0 & |x|>0.5\end{cases}$

$$\hat{f}(w)=\int_{-0.5}^{0.5}e^{-jwx}dx=\left.\frac{e^{-jwx}}{-jw}\right|_{-0.5}^{0.5}=\frac{e^{-jw/2}-e^{jw/2}}{-jw}=\frac{e^{jw/2}-e^{-jw/2}}{jw}=\frac{2\sin(w/2)}{w}=\frac{\sin(w/2)}{w/2}=\text{sinc}\,(w/2)\,.$$

EX t2 $f(x)=\begin{cases}e^{-ax} & x>0\\ 0 & else\end{cases}, a>0$ $=e^{-ax}u(x)$, $u(x)$: unit step function

$$\hat{f}(w)=\int_0^{\infty}e^{-ax}e^{-jwx}dx=\int_0^{\infty}e^{-(a+jw)x}dx=\left.\frac{e^{-(a+jw)x}}{-(a+jw)}\right|_0^{\infty}=\frac{1}{a+jw}\,.$$

Some Properties

- Linearity : $\mathfrak{F}[af+bg]=a\mathfrak{F}[f]+b\mathfrak{F}[g]$
- Symmetry or Duality : $f(x)\leftrightarrow\hat{f}(w)\to\hat{f}(x)\leftrightarrow 2\pi f(-w)$

Proof $f(-x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)e^{-jwx}dw\xrightarrow{x\to w,w\to x}f(-w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(x)e^{-jwx}dx\to 2\pi f(-w)=\int_{-\infty}^{\infty}\hat{f}(x)e^{-jwx}dx$.

- Scaling : $f(x)\leftrightarrow\hat{f}(w)\to f(ax)\leftrightarrow\frac{1}{|a|}\hat{f}\left(\frac{w}{a}\right)$

Proof $a>0$: $\mathfrak{F}\left[f(ax)\right]=\int_{-\infty}^{\infty} f(ax)e^{-jwx}dx\overset{v=ax}{=}\int_{-\infty}^{\infty} f(v)e^{-jwv/a}\frac{dv}{a}=\frac{1}{a}\int_{-\infty}^{\infty} f(v)e^{-jwv/a}dv=\frac{1}{|a|}\hat{f}\left(\frac{w}{a}\right)$

$a<0$, $a=-b$, $b>0$: $\mathfrak{F}\left[f(ax)\right]=\int_{-\infty}^{\infty} f(ax)e^{-jwx}dx\overset{v=-bx}{=}\int_{\infty}^{-\infty} f(v)e^{-jwv/(-b)}\frac{dv}{-b}=\frac{1}{b}\int_{-\infty}^{\infty} f(v)e^{-jwv/a}dv=\frac{1}{|a|}\hat{f}\left(\frac{w}{a}\right)$

EX t1b gate function $f(x)=rect\left(\frac{x}{a}\right)=\begin{cases}1 & |x|<a/2\\ 0.5 & |x|=a/2\\ 0 & |x|>a/2\end{cases}$

$$\hat{f}(w)=\int_{-a/2}^{a/2}e^{-jwx}dx=\left.\frac{e^{-jwx}}{-jw}\right|_{-a/2}^{a/2}=\frac{e^{-jwa/2}-e^{jwa/2}}{-jw}=\frac{e^{jwa/2}-e^{-jwa/2}}{jw}=\frac{2\sin(wa/2)}{w}=a\frac{\sin(wa/2)}{wa/2}=a\text{sinc}\,(wa/2)$$

- Time shift : $f(x)\leftrightarrow\hat{f}(w)\to f(x-x_0)\leftrightarrow\hat{f}(w)\underbrace{e^{-jwx_0}}_{\text{phaseshift}}$
- Frequency shift : $f(x)\leftrightarrow\hat{f}(w)\to\underbrace{f(x)e^{jw_0x}}_{\text{modulation}}\leftrightarrow\hat{f}(w-w_0)$

Special Cases

- Delta Function : $f(x)=\delta(x); \hat{f}(w)=\int_{-\infty}^{\infty}\delta(x)e^{-jwx}dx=1\rightarrow \boxed{\delta(x)\leftrightarrow 1}$

Furthermore, $\delta(x)=\mathfrak{F}^{-1}[1]=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{jwx}dw;$

Likewise, from duality $\boxed{\delta(x)\leftrightarrow 1\rightarrow 1\leftrightarrow 2\pi\delta(-w)=2\pi\delta(w)}\therefore \delta(w)=\dfrac{1}{2\pi}\mathfrak{F}[1]=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{-jwx}dx$

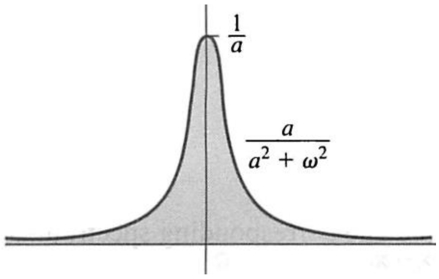
- Harmonics : $f(x)=e^{jw_0x}\rightarrow \hat{f}(w)=\int_{-\infty}^{\infty}e^{jw_0x}e^{-jwx}dx=\int_{-\infty}^{\infty}e^{-j(w-w_0)x}dx=2\pi\delta(w-w_0)$
- Unit step function : $f(x)=u(x)$ [= 1 ($x\geq 0$) and = 0 ($x<0$)]

Let $u(x)=\lim_{a\to 0}e^{-ax}u(x)$, $\hat{u}(w)=\mathfrak{F}[\lim_{a\to 0}e^{-ax}u(x)]=\lim_{a\to 0}\int_{0}^{\infty}e^{-ax}e^{jwx}dw=\lim_{a\to 0}\dfrac{1}{a+jw}=\lim_{a\to 0}\left[\dfrac{a-jw}{a^2+w^2}\right]$

Imaginary part : $\lim_{a\to 0}\left[\dfrac{-jw}{a^2+w^2}\right]=\dfrac{1}{jw}$

Real part : $\int_{-\infty}^{\infty}\dfrac{a}{a^2+w^2}dw=\tan^{-1}\dfrac{w}{a}\Big|_{-\infty}^{\infty}=\pi;\lim_{a\to 0}\dfrac{a}{a^2+w^2}=\pi\delta(w)$

$\hat{u}(w)=\pi\delta(w)+1/jw$



<u>Parseval's Theorem</u> : $\int_{-\infty}^{\infty}|f(x)|^2dx=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(w)\right|^2dw$

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}f(x)f^*(x)dx=\int_{-\infty}^{\infty}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)e^{jwx}dw}_{f(x)}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}^*(v)e^{-jvx}dvdx}_{f^*(x)}$$

$$=\int_{-\infty}^{\infty}\hat{f}(w)\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}^*(v)\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-j(v-w)x}dxdvdw=\int_{-\infty}^{\infty}\hat{f}(w)\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}^*(v)\delta(v-w)dvdw=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)\hat{f}^*(w)dw$$

<u>Significance</u> Total energy in x (or time) domain = total energy in transform (or frequency) domain

<u>Fourier transform of derivative</u> : $f(x)\leftrightarrow\hat{f}(w)\rightarrow f'(x)\leftrightarrow jw\hat{f}(w)$

<u>Proof</u> $\mathfrak{F}[f'(x)]=\int_{-\infty}^{\infty}f'(x)e^{-jwx}dx=\underbrace{f(x)e^{-jwx}\Big|_{-\infty}^{\infty}}_{|x|\to\infty\Rightarrow f(x)\to 0}+jw\int_{-\infty}^{\infty}f(x)e^{-jwx}dx=jw\hat{f}(w)$

It follows that $f(x)\leftrightarrow\hat{f}(w)\rightarrow f''(x)\leftrightarrow jw\mathfrak{F}[f'(x)]=jw\left[jw\hat{f}(w)\right]=-w^2\hat{f}(w)$

<u>EX t3</u> Consider $f(x)=xe^{-x^2}$; $xe^{-x^2}=-\dfrac{1}{2}(e^{-x^2})'\therefore\hat{f}(w)=-\dfrac{1}{2}\mathfrak{F}[(e^{-x^2})']=-\dfrac{jw}{2}\mathfrak{F}[e^{-x^2}]$

$$\mathfrak{F}[e^{-x^2}]=\int_{-\infty}^{\infty}e^{-x^2}e^{-jwx}dx=\int_{-\infty}^{\infty}e^{-(x+\frac{jw}{2})^2}e^{-\frac{w^2}{4}}dx=e^{-\frac{w^2}{4}}\int_{-\infty}^{\infty}e^{-(x+\frac{jw}{2})^2}dx=e^{-\frac{w^2}{4}}\int_{-\infty}^{\infty}e^{-z^2}dz=\sqrt{\pi}e^{-\frac{w^2}{4}}$$

Thus, $\hat{f}(w)=-\dfrac{jw}{2}\mathfrak{F}[e^{-x^2}]=-\dfrac{jw}{2}\sqrt{\pi}e^{-\frac{w^2}{4}}$

<u>Convolution</u> $\boxed{(f*g)(x)=\int_{-\infty}^{\infty}f(p)g(x-p)dp}\rightarrow f(x)\leftrightarrow\hat{f}(w),g(x)\leftrightarrow\hat{g}(w)\rightarrow f*g\leftrightarrow\hat{f}\hat{g}$

$$\mathfrak{F}[f*g]=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(p)g(x-p)dp\right)e^{-jwx}dx=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}g(x-p)e^{-jwx}dx\right)f(p)dp$$

$$\overset{x-p=q}{=}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}g(q)e^{-jw(q+p)}dq\right)f(p)dp=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}g(q)e^{-jwq}dq\right)e^{-jwp}f(p)dp=\int_{-\infty}^{\infty}\hat{g}(w)e^{-jwp}f(p)dp=\hat{g}(w)\hat{f}(w)$$

- Duality of convolution : $f(x)\leftrightarrow\hat{f}(w),g(x)\leftrightarrow\hat{g}(w)\rightarrow fg\leftrightarrow\dfrac{1}{2\pi}\hat{f}*\hat{g}$

<u>Exercise</u> Prove this property.

EX t4 Let $f(x) = g(x) = \text{rect}(x)$, then for $|x| > 1$, $f*g=0$, $-1< x<0$, $f * g(x) = \int_{-0.5}^{x+0.5} dp = x+1$, $0<x<1$,

$$f * g(x) = \int_{x-0.5}^{0.5} dp = 1-x \therefore \mathfrak{F}[f*g] = \int_{-1}^{0}(x+1)e^{-jwx}dx + \int_{0}^{1}(1-x)e^{-jwx}dx = I_1 + I_2 - I_3$$

$$I_1 = \int_{-1}^{1} e^{-jwx}dx = \left.\frac{e^{-jwx}}{-jw}\right|_{-1}^{1} = \frac{e^{-jw}-e^{jw}}{-jw} = \frac{\sin w}{w};\ I_2 = \int_{-1}^{0} xe^{-jwx}dx = \left.\frac{xe^{-jwx}}{-jw}\right|_{-1}^{0} + \frac{1}{jw}\int_{-1}^{0} e^{-jwx}dx = \frac{-e^{jw}}{jw} + \frac{1-e^{jw}}{w^2}$$

$$I_3 = \int_{0}^{1} xe^{-jwx}dx = \left.\frac{xe^{-jwx}}{-jw}\right|_{0}^{1} + \frac{1}{jw}\int_{0}^{1} e^{-jwx}dx = \frac{e^{-jw}}{-jw} + \frac{e^{-jw}-1}{w^2}$$

$$\mathfrak{F}[f*g] = \frac{e^{-jw}-e^{jw}}{-jw} + \frac{-e^{jw}}{jw} + \frac{1-e^{jw}}{w^2} - \frac{e^{-jw}}{-jw} - \frac{e^{-jw}-1}{w^2} = \frac{1}{w^2}\left(2-e^{jw}-e^{-jw}\right) = -\frac{1}{w^2}\left(e^{jw/2}-e^{-jw/2}\right)^2$$

$$= -\frac{1}{w^2}\left(j2\sin\frac{w}{2}\right)^2 = \frac{4}{w^2}\left(\sin\frac{w}{2}\right)^2 = \frac{1}{(w/2)^2}\left(\sin\frac{w}{2}\right)^2 = \text{sinc}^2\frac{w}{2}$$

Using the "even" function property yields

$$\mathfrak{F}[f*g] = 2\int_{0}^{1}(1-x)\cos wx dx = \left.\frac{2\sin wx}{w}\right|_{0}^{1} - \left.\frac{2x\sin wx}{w}\right|_{0}^{1} + \frac{2}{w}\int_{0}^{1}\sin wx dx = -\left.\frac{2\cos wx}{w^2}\right|_{0}^{1} = \frac{2-2\cos w}{w^2} = \frac{4\sin^2(w/2)}{w^2}$$

Alternatively, since $\mathfrak{F}[rect] = \text{sinc}(w/2), \mathfrak{F}[rect * rect] = \text{sinc}^2(w/2)$.

8 Application of Fourier Transform to PDE

(a) Heat Equation Consider

$\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}, (t \ge 0, -\infty < x < \infty)\ldots(1)$ with initial condition: $u(x, 0) = f(x)$ …(2),

Recall $\hat{u}(w) = \mathfrak{F}[u(x)] = \int_{-\infty}^{\infty} u(x)e^{-jwx}dx, \mathfrak{F}[u'(x)] = jw\mathfrak{F}[u], \mathfrak{F}[u''(x)] = -w^2\mathfrak{F}[u]$

Take Fourier transform of (1) yields

$$\mathfrak{F}\left[\frac{\partial u}{\partial t}\right] = c^2\mathfrak{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -c^2w^2\mathfrak{F}[u];\ \mathfrak{F}\left[\frac{\partial u}{\partial t}\right] = \int_{-\infty}^{\infty}\frac{\partial u}{\partial t}e^{-jwx}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} ue^{-jwx}dx = \frac{\partial}{\partial t}\mathfrak{F}[u]$$

$$\underbrace{\mathfrak{F}\left[\frac{\partial u}{\partial t}\right] = -c^2w^2\mathfrak{F}[u]}_{PDE} \rightarrow \underbrace{\frac{\partial \hat{u}(w,t)}{\partial t} = -c^2w^2\hat{u}(w,t)}_{ODE}\ldots(3)$$

Take Fourier transform of (2) yields $\hat{u}(w,0) = \hat{f}(w)\ldots(4)$

From (3) $\hat{u}(w,t) = Ce^{-c^2w^2t}$; from (4) $\hat{u}(w,0) = C = \hat{f}(w) \rightarrow \hat{u}(w,t) = \hat{f}(w)e^{-c^2w^2t}$

Take inverse Fourier transform yields $u(x,t) = \mathfrak{F}^{-1}[\hat{u}(w,t)] = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(w,t)e^{jwx}dw = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{jwx}dw$

(b) Wave Equation Consider

$\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}, (t \ge 0, -\infty < x < \infty)\ldots(1)$; subject to

with initial conditions : $u(x, 0) = f(x)$ …(2a), $\partial u(x,0)/\partial t = 0$…(2b),

and boundary conditions : $u \to 0$, $u_x \to 0$, as $|x| \to \infty$ ($t \ge 0$)… (3)

where $f(x)$ assumed to have Fourier transform

$$\mathfrak{F}[u_{tt}] = \hat{u}_{tt} = c^2\mathfrak{F}[u_{xx}] = -c^2w^2\hat{u} \rightarrow \hat{u}_{tt} = -c^2w^2\hat{u} \rightarrow \hat{u}(w,t) = A(w)\cos cwt + B(w)\sin cwt$$

For t=0, $\hat{u}(w,0) = A(w) = \hat{f}(w), \hat{u}_t(w,0) = cwB(w) = 0$, Thus $\hat{u}(w,t) = \hat{f}(w)\cos cwt = \hat{f}(w)\dfrac{e^{jcwt}+e^{-jcwt}}{2}$

$u(x,t) = \mathfrak{F}^{-1}[\hat{u}(w,t)] \overset{\text{frequency shift}}{=} \dfrac{1}{2}\left[f(x-ct) + f(x+ct)\right]$ "d'Alembert's solution"

9 Poisson's Sum Formula

Consider a "periodic" impulse train with period p=2L, $s(x)=\sum_{k=-\infty}^{\infty}\delta(x-2kL)$ (Called comb, Ш (Shah) function).

Fourier series of $s(x)$ is given by

$$c_n=\frac{1}{2L}\int_{-L}^{L}s(x)e^{-j\frac{n\pi}{L}x}dx=\frac{1}{2L}\int_{-L}^{L}\sum_{k=-\infty}^{\infty}\delta(x-2kL)e^{-j\frac{n\pi}{L}x}dx=\frac{1}{2L}\text{ and }s(x)=\frac{1}{2L}\sum_{n=-\infty}^{\infty}e^{j\frac{n\pi}{L}x}=\sum_{k=-\infty}^{\infty}\delta(x-2kL)$$

Now, consider Fourier transform of $s(x)$

$$\hat{s}(w)=\int_{-\infty}^{\infty}\frac{1}{2L}\sum_{n=-\infty}^{\infty}e^{j\frac{n\pi}{L}x}e^{-jwx}dx=\frac{1}{2L}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j(w-\frac{n\pi}{L})x}dx=\frac{2\pi}{2L}\sum_{n=-\infty}^{\infty}\delta(w-\frac{n\pi}{L})=\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\delta(w-\frac{n\pi}{L})$$

Let $h(x)=g(x)*s(x)=g(x)*\sum_{k=-\infty}^{\infty}\delta(x-2kL)=\sum_{k=-\infty}^{\infty}g(x-2kL)\leftarrow$"periodic version" of $g(x)$

FT of $h(x)$: $\hat{h}(w)=\hat{g}(w)\hat{s}(w)=\hat{g}(w)\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\delta(w-\frac{n\pi}{L})=\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\hat{g}(w)\delta(w-\frac{n\pi}{L})=\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\hat{g}(\frac{n\pi}{L})\delta(w-\frac{n\pi}{L})$

$\hat{g}(w)$ sampled at $n\pi/L$. Take inverse FT of $\hat{h}(w)$ yields

$$\sum_{k=-\infty}^{\infty}g(x-2kL)=\frac{1}{2L}\sum_{n=-\infty}^{\infty}\hat{g}(\frac{n\pi}{L})e^{j\frac{n\pi}{L}x}$$ $\leftarrow$Poisson's Sum Formula

<u>Fourier series vs Fourier transform</u>

Since $h(x)=\sum_{k=-\infty}^{\infty}g(x-2kL)=\sum_{n=-\infty}^{\infty}c_n e^{j\frac{n\pi}{L}x}=\frac{1}{2L}\sum_{n=-\infty}^{\infty}\hat{g}(\frac{n\pi}{L})e^{j\frac{n\pi}{L}x}$, $c_n=\frac{1}{2L}\hat{g}(\frac{n\pi}{L})$.

<u>EX t5</u> Consider Fourier series in EX s2. First, find the Fourier transform of $g(x)=k\,\mathrm{rect}(x/2)$. From the scaling property (EX t1b with a=2), $\hat{g}(w)=2k\,\mathrm{sinc}(w)$. Now, consider the periodic version of $g(x)$ with period p = 2L, L=2.

$$c_n=\frac{1}{4}\hat{g}(\frac{n\pi}{2})=\frac{k}{2}\mathrm{sinc}(\frac{n\pi}{2})=\frac{k}{n\pi}\sin\frac{n\pi}{2}=\begin{cases}k/2 & n=0\\(-1)^{\ell}k/n\pi & n:odd, n=2\ell+1\\0 & n:even\end{cases}$$

<u>EX t6</u> Consider a half-wave rectifier. First, let $g(t)=\sin\omega t[u(t)-u(t-\pi/\omega)]$,

$$\hat{g}(w)=\int_0^{\pi/\omega}\sin\omega t e^{-jwt}dt=\int_0^{\pi/\omega}\left(\frac{e^{j\omega t}-e^{-j\omega t}}{j2}\right)e^{-jwt}dt=\frac{1}{j2}\int_0^{\pi/\omega}\left(e^{-j(w-\omega)t}-e^{-j(w+\omega)t}\right)dt$$

$$=\frac{1}{j2}\left[\frac{e^{-j(w-\omega)t}}{-j(w-\omega)}\Bigg|_0^{\pi/\omega}-\frac{e^{-j(w+\omega)t}}{-j(w+\omega)}\Bigg|_0^{\pi/\omega}\right]=\frac{1}{j2}\left[\frac{e^{-j(w-\omega)\pi/\omega}-1}{-j(w-\omega)}-\frac{e^{-j(w+\omega)\pi/\omega}-1}{-j(w+\omega)}\right]$$

$$=\frac{1}{j2}\left[e^{-j(w-\omega)\pi/2\omega}\frac{e^{-j(w-\omega)\pi/2\omega}-e^{j(w-\omega)\pi/2\omega}}{-j(w-\omega)}-e^{-j(w+\omega)\pi/2\omega}\frac{e^{-j(w+\omega)\pi/2\omega}-e^{j(w+\omega)\pi/2\omega}}{-j(w+\omega)}\right]$$

$$=\frac{1}{j}\left[e^{-j(w-\omega)\pi/2\omega}\frac{\sin(w-\omega)\pi/2\omega}{(w-\omega)}-e^{-j(w+\omega)\pi/2\omega}\frac{\sin(w+\omega)\pi/2\omega}{(w+\omega)}\right]$$

$$=\frac{1}{j}\frac{\pi}{2\omega}\left[e^{-j(w-\omega)\pi/2\omega}\frac{\sin(w-\omega)\pi/2\omega}{(w-\omega)\pi/2\omega}-e^{-j(w+\omega)\pi/2\omega}\frac{\sin(w+\omega)\pi/2\omega}{(w+\omega)\pi/2\omega}\right]$$

$$=\frac{1}{j}\frac{\pi}{2\omega}\left[e^{-j(w-\omega)\pi/2\omega}\mathrm{sinc}\frac{(w-\omega)\pi}{2\omega}-e^{-j(w+\omega)\pi/2\omega}\mathrm{sinc}\frac{(w+\omega)\pi}{2\omega}\right]$$

Alternatively, let $g(t)=\sin\omega t\cdot rect\left(\frac{\omega(t-\pi/2\omega)}{\pi}\right)=\left(\frac{e^{j\omega t}-e^{-j\omega t}}{j2}\right)\cdot rect\left(\frac{\omega(t-\pi/2\omega)}{\pi}\right)$

$$\mathscr{F}\left(\frac{e^{j\omega t}-e^{-j\omega t}}{j2}\right)=\frac{2\pi}{j2}[\delta(w-\omega)-\delta(w+\omega)];\mathscr{F}\left[rect\left(\frac{\omega(t-\pi/2\omega)}{\pi}\right)\right]=e^{-jw\pi/2\omega}\frac{\pi}{\omega}\operatorname{sinc}\frac{\pi w}{2\omega}$$

Using $fg\leftrightarrow\frac{1}{2\pi}\hat{f}*\hat{g}$ yields $\hat{g}(w)=\frac{1}{j2}\frac{\pi}{\omega}\left[e^{-j(w-\omega)\pi/2\omega}\operatorname{sinc}\frac{\pi(w-\omega)}{2\omega}-e^{-j(w+\omega)\pi/2\omega}\operatorname{sinc}\frac{\pi(w+\omega)}{2\omega}\right]$

Now, let $h(t)=g(t)*s(t)=g(t)*\sum_{k=-\infty}^{\infty}\delta(t-2kL)=\sum_{k=-\infty}^{\infty}g(t-2kL);L=\frac{\pi}{\omega}$

$$c_n=\frac{1}{2L}\hat{g}(\frac{n\pi}{L})=\frac{\omega}{2\pi}\hat{g}(n\omega)=\frac{\omega}{2\pi}\frac{1}{j2}\frac{\pi}{\omega}\left[e^{-j(n\omega-\omega)\pi/2\omega}\operatorname{sinc}\frac{\pi(n\omega-\omega)}{2\omega}-e^{-j(n\omega+\omega)\pi/2\omega}\operatorname{sinc}\frac{\pi(n\omega+\omega)}{2\omega}\right]$$

$$=\frac{1}{j4}\left[e^{-j\frac{(n-1)\pi}{2}}\operatorname{sinc}\frac{(n-1)\pi}{2}-e^{-j\frac{(n+1)\pi}{2}}\operatorname{sinc}\frac{(n+1)\pi}{2}\right]$$

n=0, $c_0=\frac{1}{j4}\left[e^{j\frac{\pi}{2}}\operatorname{sinc}\frac{-\pi}{2}-e^{-j\frac{\pi}{2}}\operatorname{sinc}\frac{\pi}{2}\right]=\frac{1}{j4}\left[\frac{j}{\pi/2}+\frac{j}{\pi/2}\right]=\frac{1}{\pi}$

$c_1=\frac{1}{j4}\left[\operatorname{sinc}0-e^{-j\frac{2\pi}{2}}\operatorname{sinc}\frac{2\pi}{2}\right]=\frac{1}{j4}$, n: odd, $n\neq 1 \rightarrow c_n=0$,

n:even: $c_n=\frac{1}{j4}\left[e^{-j\frac{(n-1)\pi}{2}}\operatorname{sinc}\frac{(n-1)\pi}{2}-e^{-j\frac{(n+1)\pi}{2}}\operatorname{sinc}\frac{(n+1)\pi}{2}\right]=\frac{1}{j4}\left[\frac{-j2}{(n-1)\pi}-\frac{-j2}{(n+1)\pi}\right]=-\frac{1}{\pi}\frac{1}{n^2-1}$

<u>Discrete (Sampled) Signal</u>

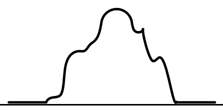
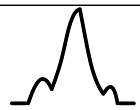
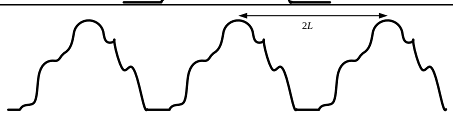
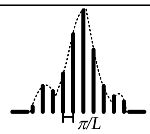
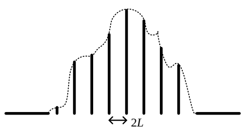
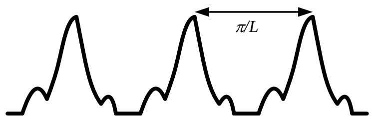
Let $f_C(x)$ be a "continuous" signal, and $f_S(x)$ be a "sampled" signal with sampling period p=2L.

$$f_S(x)=f_C(x)s(x)=f_C(x)\sum_{k=-\infty}^{\infty}\delta(x-2kL)=\sum_{k=-\infty}^{\infty}f_C(x)\delta(x-2kL)$$

Using $fg\leftrightarrow\frac{1}{2\pi}\hat{f}*\hat{g}$, $\hat{f}_S(w)=\frac{1}{2\pi}\hat{f}_C(w)*\hat{s}(w)=\frac{1}{2\pi}\hat{f}_C(w)*\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\delta(w-\frac{n\pi}{L})=\frac{1}{2L}\hat{f}_C(w)*\sum_{n=-\infty}^{\infty}\delta(w-\frac{n\pi}{L})$

which means $\hat{f}_C(w)$ is repeated every π/L, in other words, $\hat{f}_S(w)$ is a "periodic" version of $\hat{f}_C(w)$. This is called Discrete Time Fourier Transform (DTFT).

<u>Summary</u>

x (or time) domain	Example Waveform	Transform (frequency) domain	Example Spectrum
Continuous, aperiodic		Continuous, aperiodic	
Continuous, periodic		Discrete (Fourier series)	
Discrete (sampled signal)		Continuous, periodic (Periodic version of Fourier transform)	

10 Laplace Transform and Fourier Transform

<u>Laplace to Fourier</u>: Recall that Laplace transforms exist only for functions that satisfy the "growth restriction" condition, which requires region of convergence (ROC) consideration (i.e., region of s where the transform exists). Now, the conventional Laplace transform is defined for $f(t)$, $t\geq 0$, as

$F(s)=\int_0^\infty f(t)e^{-st}dt$,

which is sometimes called "unilateral" Laplace transform. Since $f(t)$ is assumed to be 0 for $t<0$,

$F(s)=\int_{-\infty}^\infty f(t)e^{-st}dt$.(←called "bilateral" Laplace transform)

Comparing to the Fourier transform for (t,ω) pair given by

$\hat{f}(\omega)=\int_{-\infty}^\infty f(t)e^{-j\omega t}dt$,

it follows that by substituting $s=j\omega$ in F(s), provided that $s=j\omega$ (imaginary axis) is in the ROC, one can obtain the Fourier transform, i.e., $\hat{f}(\omega)=F(j\omega)$.

EX1 Consider $f(t)=e^{-at}$, $a>0$, then $F(s)=(s+a)^{-1}$. $\hat{f}(\omega)=F(j\omega)=\left(j\omega+a\right)^{-1}$.

EX2 Consider $f(t) = \sin\omega t[u(t)-u(t-\pi/\omega)]$. Note that w is used as the variable for frequency domain here.

$$\hat{f}(w)=\frac{1}{j}\frac{\pi}{2\omega}\left[e^{-j(w-\omega)\pi/2\omega}\text{sinc}\frac{(w-\omega)\pi}{2\omega}-e^{-j(w+\omega)\pi/2\omega}\text{sinc}\frac{(w+\omega)\pi}{2\omega}\right]$$

$$=\frac{1}{j}\frac{\pi}{2\omega}\left[je^{-jw\pi/2\omega}\frac{-\cos(w\pi/2\omega)}{(w-\omega)\pi/2\omega}+je^{-jw\pi/2\omega}\frac{\cos(w\pi/2\omega)}{(w+\omega)\pi/2\omega}\right]$$

$$=e^{-jw\pi/2\omega}\frac{e^{jw\pi/2\omega}+e^{-jw\pi/2\omega}}{2}\left[-\frac{1}{w-\omega}+\frac{1}{w+\omega}\right]=\frac{1+e^{-jw\pi/\omega}}{2}\frac{2\omega}{\omega^2-w^2}=\frac{\omega}{\omega^2-w^2}\left(1+e^{-jw\pi/\omega}\right)$$

$$F(s)=\mathcal{L}\left\{\sin\omega t\left[u(t)-u(t-\frac{\pi}{\omega})\right]\right\}=\mathcal{L}\{\sin\omega t u(t)\}-\mathcal{L}\left\{\sin\omega t u(t-\frac{\pi}{\omega})\right\}=\frac{\omega}{s^2+\omega^2}+\mathcal{L}\left\{\sin\omega(t-\frac{\pi}{\omega})u(t-\frac{\pi}{\omega})\right\}$$

$$=\frac{\omega}{s^2+\omega^2}+\frac{\omega}{s^2+\omega^2}e^{-\frac{\pi s}{\omega}}$$

$$\hat{f}(w)=F(jw)=\frac{\omega}{-w^2+\omega^2}+\frac{\omega}{-w^2+\omega^2}e^{-\frac{j\pi w}{\omega}}=\frac{\omega}{\omega^2-w^2}\left(1+e^{-\frac{j\pi w}{\omega}}\right)$$

NOTE $u(t)\leftrightarrow 1/s$, but $\hat{u}(\omega)\neq 1/(j\omega)$! This is because $s>0$ (Re $s>0$) for $u(t)$.

Fourier to Laplace: Fourier transforms exist for functions that are "absolutely integrable". Now, consider a the Fourier transform of $f(t)$ which satisfies $f(t)=0$ for $t<0$. It follows that

$\hat{f}(\omega)=\int_0^\infty f(t)e^{-j\omega t}dt$.

This integral does not exist unless $f(t)$ is absolutely integrable. Thus, an additional term is introduced to guarantee the existence of integral, namely

$\hat{f}(\omega)=\int_0^\infty f(t)e^{-\sigma t}e^{-j\omega t}dt=\int_0^\infty f(t)e^{-st}dt=F(s)$.

Therefore, $s=\sigma+j\omega$ becomes complex and the ROC for s needs to be considered.

Inverse Laplace Transform

Recall the growth restriction dictates that Laplace transform F(s) exists if $|f(t)|<Me^{kt}$, for $t>0$, $\exists k,M$. Can choose $\gamma>k$ such that $Me^{-(\gamma-k)t}\xrightarrow{t\to\infty}0$. Therefore, for $f(t)=0$, $t<0$,

$$e^{-\gamma t}f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{j\omega t}\underbrace{\int_0^\infty e^{-\gamma\tau}f(\tau)e^{-j\omega\tau}d\tau}_{\mathfrak{F}[e^{-\gamma t}f(t)]}d\omega.$$

Moving the $e^{-\gamma t}$ term to the right hand side yields

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{\gamma t}e^{j\omega t}\int_0^\infty f(\tau)e^{-\gamma\tau}e^{-j\omega\tau}d\tau d\omega\overset{s=\gamma+j\omega}{=}\frac{1}{j2\pi}\int_{\gamma-j\infty}^{\gamma+j\infty}e^{st}\underbrace{\int_0^\infty f(\tau)e^{-s\tau}d\tau}_{F(s)}ds.$$

Thus, $\boxed{f(t)=\frac{1}{j2\pi}\int_{\gamma-j\infty}^{\gamma+j\infty}F(s)e^{st}ds}$ ←"Bromwich integral"

where $\gamma > \max_k \operatorname{Re} s_k$, s_k denotes the k^th pole of F(s), i.e., γ> Maximum of Real part of poles.

<u>Brief Introduction of Poles</u> Let F(s)=A(s)/B(s), and s_k denote a pole of F(s), then B(s_k)=0.
(Recall $e^{at}\leftrightarrow(s-a)^{-1}$, and a becomes a pole)

<u>EX</u> $F(s)=\frac{1}{s^3+2s^2-s-2}=\frac{1/6}{s-1}-\frac{1/2}{s+1}+\frac{1/3}{s+2}\rightarrow f(t)=\frac{e^t}{6}-\frac{e^{-t}}{2}+\frac{e^{-2t}}{3}$

Thus, s>1 is the ROC, and it follows that γ> 1. Note that each pole corresponds to an exponential term.

11 Fourier Cosine and Sine Transform

Recall that

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{jwx}\int_{-\infty}^{\infty}f(v)e^{-jwv}dvdw=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(v)e^{jw(x-v)}dvdw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(v)\left[\cos w(x-v)+j\sin w(x-v)\right]dvdw$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(v)\cos w(x-v)dvdw\left(\because\int_{-\infty}^{\infty}\sin w(x-v)dw=0,\int_{-\infty}^{\infty}\cos w(x-v)dw=2\int_{0}^{\infty}\cos w(x-v)dw\right)\text{where}$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(v)\left[\cos wx\cos wv+\sin wx\sin wv\right]dvdw$$

$$=\int_{0}^{\infty}\left[A(w)\cos wx+B(w)\sin wx\right]dw$$ ← "Fourier Integral"

$$A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\cos wv dv;\ B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\sin wv dv$$

If $f(x)$ is an "even" function, $B(w)$=0.

$$A(w)=\frac{2}{\pi}\int_{0}^{\infty}f(v)\cos wv dv;\ f(x)=\int_{0}^{\infty}A(w)\cos wx dw$$ ← "Fourier Cosine Integral"

If $f(x)$ is an "odd" function, $A(w)$=0.

$$B(w)=\frac{2}{\pi}\int_{0}^{\infty}f(v)\sin wv dv;\ f(x)=\int_{0}^{\infty}B(w)\sin wx dw$$ ← "Fourier Sine Integral"

Let $A(w)=\frac{2}{\pi}\hat{f}_c(w)$, then

$$\hat{f}_c(w)=\int_{0}^{\infty}f(x)\cos wx dx\xleftrightarrow{\mathfrak{F}_c}f(x)=\frac{2}{\pi}\int_{0}^{\infty}\hat{f}_c(w)\cos wx dw$$ ← "Fourier Cosine Transform (FCT)"

Let $B(w)=\frac{2}{\pi}\hat{f}_s(w)$, then

$$\hat{f}_s(w)=\int_{0}^{\infty}f(x)\sin wx dx\xleftrightarrow{\mathfrak{F}_s}f(x)=\frac{2}{\pi}\int_{0}^{\infty}\hat{f}_s(w)\sin wx dw$$ ← "Fourier Sine Transform (FST)"

Note that in mathamatics:

$$\hat{f}_c(w)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\cos wx dx\xleftrightarrow{\mathfrak{F}_c}f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}_c(w)\cos wx dw,$$

$$\hat{f}_s(w)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin wx dx\xleftrightarrow{\mathfrak{F}_s}f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}_s(w)\sin wx dw.$$

<u>Some Examples</u>

<u>EX1</u> $f(x)=\begin{cases}k & 0<x<a\\ 0 & x>a\end{cases}$

$$\hat{f}_c(w) = \int_0^a k\cos wx dx = \left.\frac{k\sin wx}{w}\right|_0^a = \frac{k\sin wa}{w} = ka\text{sinc}\,(wa).$$

$$\hat{f}_s(w) = \int_0^a k\sin wx dx = -\left.\frac{k\cos wx}{w}\right|_0^a = \frac{k}{w}[1-\cos wa].$$

<u>EX2</u> $f(x) = e^{-kx}$, $x > 0$, $k > 0$

$$\hat{f}_c(w) = \int_0^\infty e^{-kx}\cos wx dx = \left.\frac{e^{-kx}\sin wx}{w}\right|_0^\infty + \int_0^\infty \frac{k}{w}e^{-kx}\sin wx dx$$

$$= -\left.\frac{ke^{-kx}\cos wx}{w^2}\right|_0^\infty - \int_0^\infty \frac{k^2}{w^2}e^{-kx}\cos wx dx$$

$$\therefore \left(1+\frac{k^2}{w^2}\right)\int_0^\infty e^{-kx}\cos wx dx = \frac{k}{w^2} \Rightarrow \hat{f}_c(w) = \int_0^\infty e^{-kx}\cos wx dx = \frac{k/w^2}{1+k^2/w^2} = \frac{k}{k^2+w^2}.$$

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \hat{f}_c(w)\cos wx dw = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}dw, x>0, k>0.$$

It follows that $\int_0^\infty \frac{\cos wx}{k^2+w^2}dw = \frac{\pi}{2k}e^{-kx}, x>0, k>0$ … (*)

Likewise,

$$\hat{f}_s(w) = \int_0^\infty e^{-kx}\sin wx dx = -\left.\frac{e^{-kx}\cos wx}{w}\right|_0^\infty - \int_0^\infty \frac{k}{w}e^{-kx}\cos wx dx$$

$$= \frac{1}{w} - \left.\frac{ke^{-kx}\sin wx}{w^2}\right|_0^\infty - \int_0^\infty \frac{k^2}{w^2}e^{-kx}\sin wx dx$$

$$\therefore \left(1+\frac{k^2}{w^2}\right)\int_0^\infty e^{-kx}\sin wx dx = \frac{1}{w} \Rightarrow \hat{f}_s(w) = \int_0^\infty e^{-kx}\sin wx dx = \frac{1/w}{1+k^2/w^2} = \frac{w}{k^2+w^2}$$

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \hat{f}_s(w)\sin wx dw = \frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{k^2+w^2}dw, x>0, k>0$$

It follows that $\int_0^\infty \frac{w\sin wx}{k^2+w^2}dw = \frac{\pi}{2}e^{-kx}, x>0, k>0$ … (**)

Equations (*), (**) are called "Laplace Integrals"

<u>Cosine transform of derivative</u> : $f(x) \leftrightarrow \hat{f}_c(w) \to f'(x) \leftrightarrow w\hat{f}_s(w) - f(0)$

<u>Proof</u> $\mathfrak{F}_c[f'(x)] = \int_0^\infty f'(x)\cos wx dx = \underbrace{f(x)\cos wx|_0^\infty}_{x\to\infty \Rightarrow f(x)\to 0} + w\int_{-\infty}^\infty f(x)\sin wx dx = w\hat{f}_s(w) - f(0)$

<u>Sine transform of derivative</u> : $f(x) \leftrightarrow \hat{f}_s(w) \to f'(x) \leftrightarrow -w\hat{f}_c(w)$

<u>Proof</u> $\mathfrak{F}_s[f'(x)] = \int_0^\infty f'(x)\sin wx dx = \underbrace{f(x)\sin wx|_0^\infty}_{x\to\infty \Rightarrow f(x)\to 0} - w\int_{-\infty}^\infty f(x)\cos wx dx = -w\hat{f}_c(w)$

It follows that

$$\mathfrak{F}_c[f''(x)] = w\mathfrak{F}_s[f'(x)] - f'(0) = -w^2\mathfrak{F}_c[f(x)] - f'(0)$$

$$\mathfrak{F}_s[f''(x)] = -w\mathfrak{F}_c[f'(x)] = -w^2\mathfrak{F}_s[f(x)] + wf(0).$$

Application to PDE

Consider the heat equation given by:

$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, (t \ge 0, 0 \le x < \infty)$…(1) with conditions $u(x, 0) = f(x)$ …(2), $u(0,t) = 0$…(3)

Recall $\hat{u}(w) = \mathfrak{F}[u(x)] = \int_{-\infty}^{\infty} u(x)e^{-jwx}dx, \mathfrak{F}[u'(x)] = jw\mathfrak{F}[u], \mathfrak{F}[u''(x)] = -w^2\mathfrak{F}[u]$

Take Fourier sine transform of (1) yields

$$\mathfrak{F}_s\left[\frac{\partial u}{\partial t}\right] = \frac{\partial \hat{u}_s}{\partial t} = c^2\mathfrak{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right] = -c^2w^2\mathfrak{F}_s[u] + wu(0,t) = -c^2w^2\hat{u}_s \rightarrow \frac{\partial \hat{u}_s}{\partial t} = -c^2w^2\hat{u}_s \ldots(4)$$

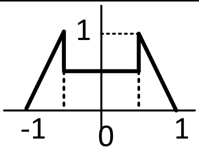
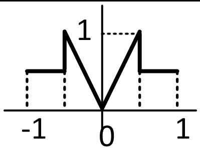
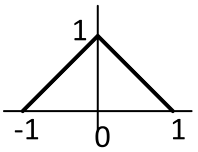
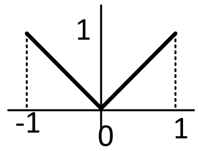
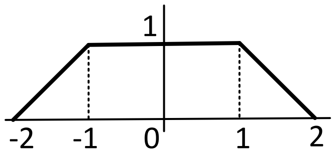
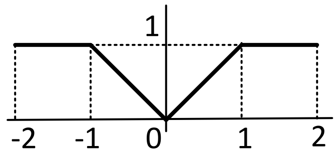
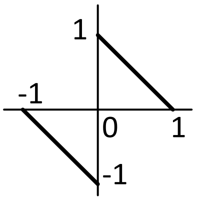
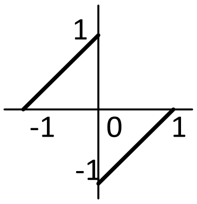
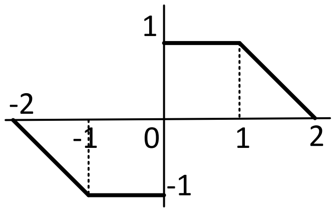
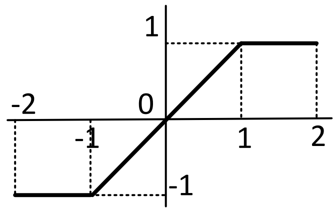
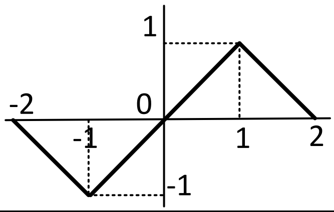
Take Fourier sine transform of (2) yields $\hat{u}_s(w,0) = \hat{f}_s(w) \ldots(5)$

From (4) $\hat{u}_s(w,t) = Ce^{-c^2w^2t}$; from (5) $\hat{u}_s(w,0) = C = \hat{f}_s(w) \rightarrow \hat{u}_s(w,t) = \hat{f}_s(w)e^{-c^2w^2t}$

Take inverse Fourier sine transform yields

$$u(x,t) = \mathfrak{F}_s^{-1}[\hat{u}(w,t)] = \frac{2}{\pi}\int_0^{\infty}\hat{u}(w,t)\sin wx\,dw = \frac{2}{\pi}\int_0^{\infty}\hat{f}_s(w)e^{-c^2w^2t}\sin wx\,dw.$$

12 Practice Problems

$f(x)$ or $f(t)$ [For periodic functions, only one period is shown.]		ODE
1 	2 	a. $y'+y = f(t)$ b. $y'+2y = f(t)$ c. $y'+3y = f(t)$ d. $y'+4y = f(t)$ e. $y''+3y'+2y = f(t)$ f. $y''+6y'+5y = f(t)$ g. $y''+5y'+4y = f(t)$ h. $y''+5y'+6y = f(t)$ i. $y''+4y'+4y = f(t)$
3 	4 	
5 	6 	
7 	8 	
9 	10 	
11 	12 $f(x) = x^2$ $(-1<x<1)$, $p = 2$ 13 $f(x) = x\lvert x\rvert$ $(-1<x<1)$, $p = 2$ 14 $f(x) = e^{-\lvert x\rvert}$ $(-1<x<1)$, $p = 2$	

<u>Question</u>

1. Find the Fourier series of $f(x)$ (or $f(t)$), then find the "steady-state" solution for the ODE.
2. Find the Fourier transform of $f(x)$ assuming that $p\rightarrow\infty$.
3. Compare the Fourier transform found in prob. 2 with the Fourier series found in prob. 1.
4. Find the Fourier transform of the functions given above when they consist of only "positive" half, then compare it with the corresponding Laplace transform.
5. Prove $f(x) \leftrightarrow \hat{f}(w), g(x) \leftrightarrow \hat{g}(w) \rightarrow fg \leftrightarrow \frac{1}{2\pi}\hat{f} * \hat{g}$.

6. Find the Fourier cosine transforms of:

7. Find the Fourier sine transforms of:

(a) $-x + a,\ 0 < x < a$ (b) e^{-ax^2} $(a > 0)$ (a) $-x + a,\ 0 < x < a$ (b) $x e^{-ax^2}$ $(a > 0)$

8. Show that

a. $$\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dx = \begin{cases} 0 & \text{if} \quad x < 0 \\ \pi/2 & \text{if} \quad x = 0 \\ \pi e^{-x} & \text{if} \quad x > 0 \end{cases}$$

b. $$\int_0^\infty \frac{\sin \pi w \sin xw}{1 - w^2}\, dw = \begin{cases} \frac{\pi}{2} \sin x & \text{if} \quad 0 \leqq x \leqq \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$$

c. $$\int_0^\infty \frac{w^3 \sin xw}{w^4 + 4}\, dw = \tfrac{1}{2}\pi e^{-x} \cos x \quad \text{if} \quad x > 0$$

d. $$\int_0^\infty \frac{\cos \frac{1}{2}\pi w}{1 - w^2} \cos xw\, dw = \begin{cases} \frac{1}{2}\pi \cos x & \text{if} \quad 0 < |x| < \frac{1}{2}\pi \\ 0 & \text{if} \quad |x| \geqq \frac{1}{2}\pi \end{cases}$$

9. Solve the following integral equations; (i.e., find $f(x)$)

(a) $\int_0^\infty f(x) \cos wx dx = e^{-w^2/4a}$ (b) $\int_0^\infty f(x) \sin wx dx = we^{-w^2/2}$

(c) $\int_{-\infty}^\infty f(x) e^{-jwx} dx = e^{-w^2/4a}$ (d) $\int_{-\infty}^\infty f(x) e^{jwx} dx = \text{sinc } wa$