1 Laplace Transform : Definition

If $f(t)$ is a function defined for all $t \geqq 0$, its **Laplace transform**[1] is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$; thus

$$\boxed{F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.}$$

Here, assume that the integral exists. The region where the integral exists is called "Region of Convergence" (ROC). Note that s is generally assumed to be complex, i.e., $s = \sigma + j\omega$.

Not only is the result $F(s)$ called the Laplace transform, but the operation just described, which yields $F(s)$ from a given $f(t)$, is also called the **Laplace transform**. It is an "**integral transform**"

$$F(s) = \int_0^\infty k(s, t) f(t)\, dt$$

with "**kernel**" $k(s, t) = e^{-st}$.

Furthermore, the given function $f(t)$ in (1) is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$; that is, we shall write

$$\boxed{f(t) = \mathcal{L}^{-1}(F).}$$

Ex 1

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

Solution. From (1) we obtain by integration

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \qquad (s > 0).$$

Such an integral is called an **improper integral** and, by definition, is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt.$$

Hence our convenient notation means

Region of Convergence

$$\int_0^\infty e^{-st}\, dt = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^T = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^0 \right] = \frac{1}{s} \qquad (s > 0).$$

Ex 2

Laplace Transform $\mathcal{L}(e^{at})$ of the Exponential Function e^{at}

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $\mathcal{L}(f)$.

Solution. Again by (1),

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{a - s} e^{-(s-a)t} \Big|_0^\infty ;$$

hence, when $s - a > 0$,

$$\mathcal{L}(e^{at}) = \frac{1}{s - a}.$$

THEOREM 1

Linearity of the Laplace Transform

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

Proof It follows the linearity of integral operation, i.e., integration is a "linear" operation.

Ex 3

Application of Theorem 1: Hyperbolic Functions

Find the transforms of cosh *at* and sinh *at*.

Solution. Since $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$ and $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$, we obtain from Example 2 and Theorem 1

$$\mathcal{L}(\cosh at) = \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}.$$

Ex 4

Cosine and Sine

Can use linearity by cos ωt = ($e^{j\omega t}$+ $e^{-j\omega t}$)/2; sin ωt = ($e^{j\omega t}$- $e^{-j\omega t}$)/j2. Can also use result from Ex 3 by substituting *a* = *jω*.

Derive the formulas

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

Solution. We write $L_c = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$. Integrating by parts and noting that the integral-free parts give no contribution from the upper limit ∞, we obtain

$$L_c = \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{e^{-st}}{-s} \cos \omega t \Big|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st} \sin \omega t \, dt = \frac{1}{s} - \frac{\omega}{s} L_s,$$

$$L_s = \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{e^{-st}}{-s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st} \cos \omega t \, dt = \frac{\omega}{s} L_c.$$

By substituting L_s into the formula for L_c on the right and then by substituting L_c into the formula for L_s on the right, we obtain

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right), \qquad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s}, \qquad L_c = \frac{s}{s^2 + \omega^2},$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right), \qquad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}, \qquad L_s = \frac{\omega}{s^2 + \omega^2}.$$

THEOREM 2

First Shifting Theorem, *s*-Shifting

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

or, if we take the inverse on both sides,

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s - a)\}.$$

PROOF We obtain $F(s - a)$ by replacing s with $s - a$ in the integral in (1), so that

$$F(s - a) = \int_0^\infty e^{-(s-a)t} f(t) \, dt = \int_0^\infty e^{-st}[e^{at}f(t)] \, dt = \mathcal{L}\{e^{at}f(t)\}.$$

If $F(s)$ exists (i.e., is finite) for s greater than some k, then our first integral exists for $s - a > k$. Now take the inverse on both sides of this formula to obtain the second formula in the theorem. (**CAUTION!** $-a$ in $F(s - a)$ but $+a$ in $e^{at}f(t)$.)

Ex 5

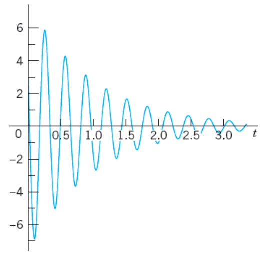
s-Shifting: Damped Vibrations. Completing the Square

From Example 4 and the first shifting theorem we immediately obtain formulas 11 and 12 in Table 6.1,

$$\mathscr{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}, \qquad \mathscr{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}.$$

For instance, use these formulas to find the inverse of the transform

$$\mathscr{L}(f) = \frac{3s-137}{s^2+2s+401}.$$



Solution. Applying the inverse transform, using its linearity (Prob. 24), and completing the square, we obtain

$$f = \mathscr{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathscr{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}.$$

We now see that the inverse of the right side is the damped vibration (Fig. 114)

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t).$$

Existence and Uniqueness of Laplace Transforms

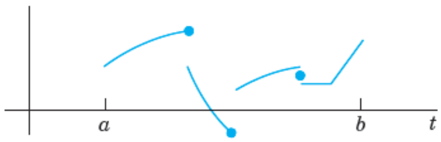
This is not a big *practical* problem because in most cases we can check the solution of an ODE without too much trouble. Nevertheless we should be aware of some basic facts.

A function $f(t)$ has a Laplace transform if it does not grow too fast, say, if for all $t \geqq 0$ and some constants M and k it satisfies the "**growth restriction**"

$$|f(t)| \leqq Me^{kt}. \tag{2}$$

(The growth restriction (2) is sometimes called "growth of exponential order," which may be misleading since it hides that the exponent must be kt, not kt^2 or similar.)

$f(t)$ need not be continuous, but it should not be too bad. The technical term (generally used in mathematics) is *piecewise continuity*. $f(t)$ is **piecewise continuous** on a finite interval $a \leqq t \leqq b$ where f is defined, if this interval can be divided into *finitely many* subintervals in each of which f is continuous and has finite limits as t approaches either endpoint of such a subinterval from the interior. This then gives **finite jumps** as in Fig. 115 as the only possible discontinuities, but this suffices in most applications, and so does the following theorem.



THEOREM 3

Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies (2) for all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathscr{L}(f)$ exists for all $s > k$.

PROOF Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis. From (2), assuming that $s > k$ (to be needed for the existence of the last of the following integrals), we obtain the proof of the existence of $\mathscr{L}(f)$ from

$$|\mathscr{L}(f)| = \left|\int_0^\infty e^{-st}f(t)\,dt\right| \leqq \int_0^\infty |f(t)|e^{-st}\,dt \leqq \int_0^\infty Me^{kt}e^{-st}\,dt = \frac{M}{s-k}.$$

Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points (see Ref. [A14] in App. 1). Hence we may say that the inverse of a given transform is essentially unique. In particular, if two *continuous* functions have the same transform, they are completely identical.

Example (Region of Convergence)

(i) $f(t) = 1$, $|f(t)| \le 1 \to k = 0 \to s > 0$ (ii) $f(t) = e^{at}$, $|f(t)| \le Me^{at} \to k = a \to s > a$

(iii) $f(t) = t^n$, $|f(t)| \le Me^{t} \to k = 1 \to s > 1$ (iv) $f(t) = \cos t$ (or $\sin t$), $|f(t)| \le 1 \to k = 0 \to s > 0$

2 Transform of Derivatives & Integrals

Thm1

Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

(1) $$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

(2) $$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

Formula (1) holds if $f(t)$ is continuous for all $t \geqq 0$ and satisfies the growth restriction (2) in Sec. 6.1 and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Similarly, (2) holds if f and f' are continuous for all $t \geqq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$.

Proof

We prove (1) first under the *additional assumption* that f' is continuous. Then, by the definition and integration by parts,

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\, dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt.$$

Since f satisfies (2) in Sec. 6.1, the integrated part on the right is zero at the upper limit when $s > k$, and at the lower limit it contributes $-f(0)$. The last integral is $\mathcal{L}(f)$. It exists for $s > k$ because of Theorem 3 in Sec. 6.1. Hence $\mathcal{L}(f')$ exists when $s > k$ and (1) holds.

If f' is merely piecewise continuous, the proof is similar. In this case the interval of integration of f' must be broken up into parts such that f' is continuous in each such part.

The proof of (2) now follows by applying (1) to f'' and then substituting (1), that is

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s[s\mathcal{L}(f) - f(0)] = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

EX1

Transform of a Resonance Term (Sec. 2.8)

Let $f(t) = t \sin \omega t$. Then $f(0) = 0$, $f'(t) = \sin \omega t + \omega t \cos \omega t$, $f'(0) = 0$, $f'' = 2\omega \cos \omega t - \omega^2 t \sin \omega t$. Hence by (2),

$$\mathcal{L}(f'') = 2\omega \frac{s}{s^2 + \omega^2} - \omega^2 \mathcal{L}(f) = s^2 \mathcal{L}(f), \qquad \text{thus} \qquad \mathcal{L}(f) = \mathcal{L}(t \sin \omega t) = \frac{2\omega s}{(s^2 + \omega^2)^2}.$$

EX2 $f(t) = t$, $f'(t) = 1$, $f(0) = 0 \to \mathcal{L}(f') = 1/s = s\mathcal{L}(f) - f(0) = s\mathcal{L}(f)$, $\mathcal{L}(f) = 1/s^2$.

$f(t) = t^2$, $f'(t) = 2t$, $f(0) = 0 \to \mathcal{L}(f') = 2/s^2 = s\mathcal{L}(f) - f(0) = s\mathcal{L}(f)$, $\mathcal{L}(f) = 2/s^3$.

Theorem3

Laplace Transform of Integral

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geqq 0$ and satisfies a growth restriction (2), Sec. 6.1. Then, for $s > 0$, $s > k$, and $t > 0$,

$$\textbf{(4)} \qquad \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

Proof

Denote the integral in (4) by $g(t)$. Since $f(t)$ is piecewise continuous, $g(t)$ is continuous, and (2), Sec. 6.1, gives

$$|g(t)| = \left|\int_0^t f(\tau)\,d\tau\right| \leqq \int_0^t |f(\tau)|\,d\tau \leqq M\int_0^t e^{k\tau}\,d\tau = \frac{M}{k}(e^{kt} - 1) \leqq \frac{M}{k}e^{kt} \qquad (k > 0).$$

This shows that $g(t)$ also satisfies a growth restriction. Also, $g'(t) = f(t)$, except at points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise continuous on each finite interval and, by Theorem 1, since $g(0) = 0$ (the integral from 0 to 0 is zero)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}.$$

Division by s and interchange of the left and right sides gives the first formula in (4), from which the second follows by taking the inverse transform on both sides. ■

EX3

Using Theorem 3, find the inverse of $\dfrac{1}{s(s^2 + \omega^2)}$ and $\dfrac{1}{s^2(s^2 + \omega^2)}$.

Solution. From Table 6.1 in Sec. 6.1 and the integration in (4) (second formula with the sides interchanged) we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega^2}\right\} = \frac{\sin \omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + \omega^2)}\right\} = \int_0^t \frac{\sin \omega t}{\omega}\,d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

This is formula 19 in Sec. 6.9. Integrating this result again and using (4) as before, we obtain formula 20 in Sec. 6.9:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + \omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1 - \cos \omega\tau)\,d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin \omega\tau}{\omega^3}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin \omega\tau}{\omega^3}.$$

It is typical that results such as these can be found in several ways. In this example, try partial fraction reduction. ■

3 Differential Equation, Initial Value Problem

Consider initial value problem

$$\textbf{(5)} \qquad y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

where a and b are constant. Here $r(t)$ is the given **input** (*driving force*) applied to the mechanical or electrical system and $y(t)$ is the **output** (*response to the input*) to be obtained.
In Laplace's method we do three steps:

Step 1. Setting up the subsidiary equation. This is an algebraic equation for the transform $Y = \mathcal{L}(y)$ obtained by transforming (5) by means of (1) and (2), namely,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2. Solution of the subsidiary equation by algebra. We divide by $s^2 + as + b$ and use the so-called **transfer function**

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}. \tag{6}$$

(Q is often denoted by H, but we need H much more frequently for other purposes.) This gives the solution

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s). \tag{7}$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; hence

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

and this explains the name of Q. Note that ***Q depends neither on $r(t)$ nor on the initial conditions*** (but only on a and b).

Step 3. Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$. We reduce (7) (usually by ***partial fractions*** as in calculus) to a sum of terms whose inverses can be found from the tables (e.g., in Sec. 6.1 or Sec. 6.9) or by a CAS, so that we obtain the solution $y(t) = \mathcal{L}^{-1}(Y)$ *of* (5).

EX4

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution. *Step 1.* From (2) and Table 6.1 we get the subsidiary equation [with $Y = \mathcal{L}(y)$]

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2, \qquad \text{thus} \qquad (s^2 - 1)Y = s + 1 + 1/s^2.$$

Step 2. The transfer function is $Q = 1/(s^2 - 1)$, and (7) becomes

$$Y = (s + 1)Q + \frac{1}{s^2}Q = \frac{s + 1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)}.$$

Simplification of the first fraction and an expansion of the last fraction gives

$$Y = \frac{1}{s - 1} + \left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

EX6
Shifted
Data

This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$. For such a problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \qquad \tilde{y}(0) = \tfrac{1}{2}\pi, \qquad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$. Using (2) and Table 6.1 and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the "shifted" initial value problem is

$$s^2\tilde{Y} - s \cdot \tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \qquad \text{thus} \qquad (s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

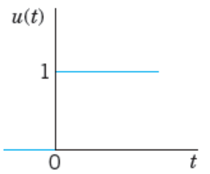
The inverse of the first two terms can be seen from Example 3 (with $\omega = 1$), and the last two terms give cos and sin,

$$\begin{aligned}\tilde{y} = \mathcal{L}^{-1}(\tilde{Y}) &= 2(\tilde{t} - \sin \tilde{t}) + \tfrac{1}{2}\pi(1 - \cos \tilde{t}) + \tfrac{1}{2}\pi \cos \tilde{t} + (2 - \sqrt{2}) \sin \tilde{t} \\ &= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2} \sin \tilde{t}.\end{aligned}$$

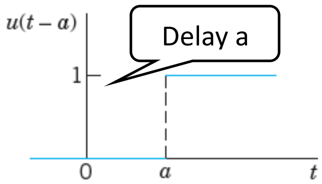
Now $\tilde{t} = t - \frac{1}{4}\pi$, $\sin \tilde{t} = \dfrac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

$$y = 2t - \sin t + \cos t.$$

Unit Step Function (Heaviside Function) $u(t - a)$



The **unit step function** or **Heaviside function** $u(t - a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a formula:



(1) $$u(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geqq 0).$$

$$\mathcal{L}\{u(t - a)\} = \int_0^\infty e^{-st}u(t - a)\,dt = \int_a^\infty e^{-st} \cdot 1\,dt = -\frac{e^{-st}}{s}\bigg|_{t=a}^{\infty}; \qquad \boxed{\mathcal{L}\{u(t - a)\} = \frac{e^{-as}}{s}}$$

Theorem 1
Time-shifting

*If $f(t)$ has the transform $F(s)$, then the "**shifted function**"*

(3) $$\tilde{f}(t) = f(t - a)u(t - a) = \begin{cases} 0 & \text{if } t < a \\ f(t - a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}\{f(t)\} = F(s)$, then

(4) $$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s).$$

Or, if we take the inverse on both sides, we can write

(4*) $$f(t - a)u(t - a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.$$

Proof $$e^{-as}F(s) = e^{-as}\int_0^\infty f(\tau)e^{-s\tau}d\tau = \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau \overset{t=\tau+a}{=} \int_a^\infty f(t-a)e^{-st}dt = \int_0^\infty \underbrace{f(t-a)u(t-a)}_{\tilde{f}(t)}e^{-st}dt$$

EX 1
Time
Shift

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } \quad t > \frac{1}{2}\pi. \end{cases} \qquad \text{(Fig. 122)}$$

Solution. *Step 1.* In terms of unit step functions,

$$f(t) = 2(1 - u(t-1)) + \tfrac{1}{2}t^2(u(t-1) - u(t - \tfrac{1}{2}\pi)) + (\cos t)u(t - \tfrac{1}{2}\pi).$$

Indeed, $2(1 - u(t-1))$ gives $f(t)$ for $0 < t < 1$, and so on.

Step 2. To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t-a)u(t-a)$. Thus, $2(1 - u(t-1))$ remains as it is and gives the transform $2(1 - e^{-s})/s$. Then

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{L}\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\right\} = \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{\frac{1}{2}\left(t - \frac{1}{2}\pi\right)^2 + \frac{\pi}{2}\left(t - \frac{1}{2}\pi\right) + \frac{\pi^2}{8}\right)u\left(t - \frac{1}{2}\pi\right)\right\}$$

$$= \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{(\cos t)\,u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t - \frac{1}{2}\pi\right)\right)u\left(t - \frac{1}{2}\pi\right)\right\} = -\frac{1}{s^2+1}e^{-\pi s/2}.$$

Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2+1}e^{-\pi s/2}.$$

If the conversion of $f(t)$ to $f(t-a)$ is inconvenient, replace it by

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}. \tag{4**}$$

(4**) follows from (4) by writing $f(t-a) = g(t)$, hence $f(t) = g(t+a)$ and then again writing f for g. Thus,

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}t^2 + t + \frac{1}{2}\right\} = e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)$$

as before. Similarly for $\mathcal{L}\{\frac{1}{2}t^2u(t - \frac{1}{2}\pi)\}$. Finally, by (4**),

$$\mathcal{L}\left\{\cos t\, u\left(t - \frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t + \frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\{-\sin t\} = -e^{-\pi s/2}\frac{1}{s^2+1}.$$ ■

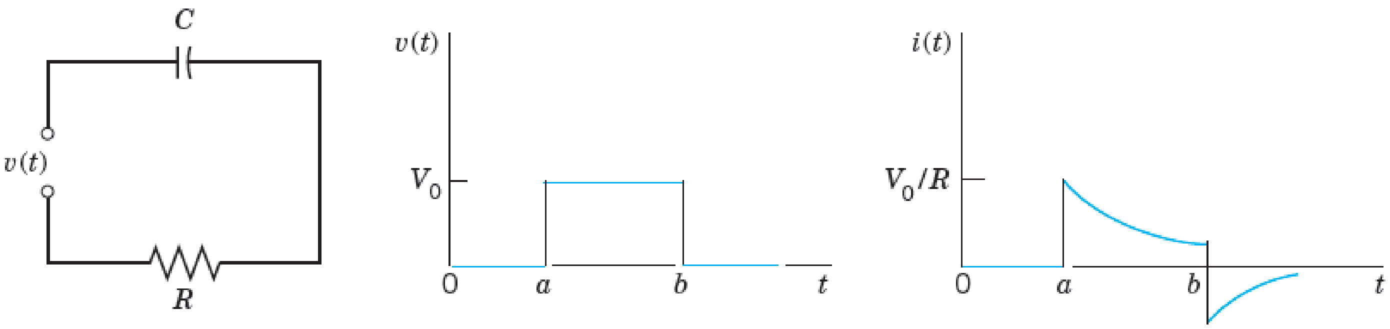
EX 3
Time
Shift

Response of an *RC*-Circuit to a Single Rectangular Wave

Find the current $i(t)$ in the RC-circuit in Fig. 124 if a single rectangular wave with voltage V_0 is applied. The circuit is assumed to be quiescent before the wave is applied.

Solution. The input is $V_0[u(t-a) - u(t-b)]$. Hence the circuit is modeled by the integro-differential equation (see Sec. 2.9 and Fig. 124)

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t) = V_0[u(t-a) - u(t-b)].$$



Using Theorem 3 in Sec. 6.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s}\left[e^{-as} - e^{-bs}\right].$$

Solving this equation algebraically for $I(s)$, we get

V_0/R

$$I(s) = F(s)(e^{-as} - e^{-bs}) \quad \text{where} \quad F(s) = \frac{V_0/R}{s + 1/(RC)} \quad \text{and} \quad \mathcal{L}^{-1}(F) = \frac{V_0}{R}e^{-t/(RC)},$$

the last expression being obtained from Table 6.1 in Sec. 6.1. Hence Theorem 1 yields the solution (Fig. 124)

$$i(t) = \mathcal{L}^{-1}(I) = \mathcal{L}^{-1}\{e^{-as}F(s)\} - \mathcal{L}^{-1}\{e^{-bs}F(s)\} = \frac{V_0}{R}\left[e^{-(t-a)/(RC)}u(t-a) - e^{-(t-b)/(RC)}u(t-b)\right];$$

that is, $i(t) = 0$ if $t < a$, and

$$i(t) = \begin{cases} K_1 e^{-t/(RC)} & \text{if } a < t < b \\ (K_1 - K_2)e^{-t/(RC)} & \text{if } a > b \end{cases}$$

where $K_1 = V_0 e^{a/(RC)}/R$ and $K_2 = V_0 e^{b/(RC)}/R$.

Alternative Approach Let v_C be the voltage across the capacitor and assume $v_C(0)=0$, then

$$v_C + v_R = v_C + Ri = v_C + R\frac{dq}{dt} = v_C + RC\frac{dv_C}{dt} = v_S$$

Taking the Laplace transform yields

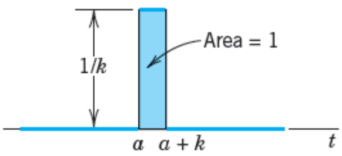
$$V_C + RCsV_C = V_S = V_0\left(e^{-as} - e^{-bs}\right)/s \rightarrow V_C = \frac{V_0\left(e^{-as} - e^{-bs}\right)/s}{1 + sRC} = \frac{V_0}{RC}\frac{\left(e^{-as} - e^{-bs}\right)}{s(s + 1/RC)} = V_0\left(e^{-as} - e^{-bs}\right)\left(\frac{1}{s} - \frac{1}{s + 1/RC}\right)$$

Thus, $v_C = V_0\left[\left(u(t-a) - u(t-b) - e^{-(t-a)/RC}u(t-a) + e^{-(t-b)/RC}u(t-b)\right)\right]$. And $i(t) = dq/dt = Cdv_C/dt$.

4 Dirac Delta Function, Unit Impulse Response

Short Impulse

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \leqq t \leqq a + k \\ 0 & \text{otherwise} \end{cases}$$



Dirac Delta Function

Unit Impulse Function

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a).$$

$$\delta(t-a) = \frac{du(t-a)}{dt}$$

Property

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_0^\infty \delta(t-a)\,dt = 1$$

Sifting Property

$$\int_0^\infty g(t)\delta(t-a)\,dt = g(a)$$

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]$$

$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\frac{1 - e^{-ks}}{ks}$$

Take limit as k-> 0

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}.$$

EX

Find the output voltage response in Fig. 135 if $R = 20\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, the input is $\delta(t)$ (a unit impulse at time $t = 0$), and current and charge are zero at time $t = 0$.

Solution. To understand what is going on, note that the network is an *RLC*-circuit to which two wires at *A* and *B* are attached for recording the voltage $v(t)$ on the capacitor. Recalling from Sec. 2.9 that current $i(t)$ and charge $q(t)$ are related by $i = q' = dq/dt$, we obtain the model

$$Li' + Ri + \frac{q}{C} = Lq'' + Rq' + \frac{q}{C} = q'' + 20q' + 10{,}000q = \delta(t).$$

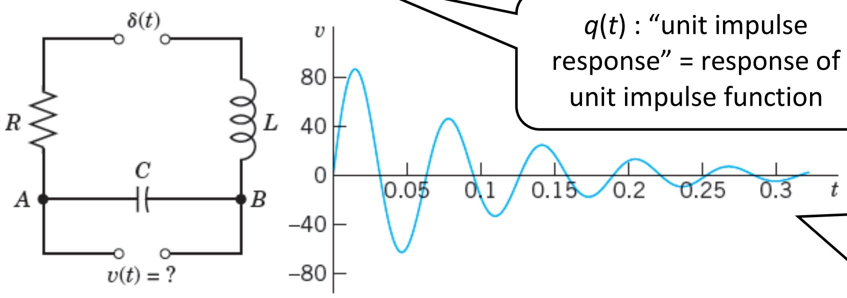
From (1) and (2) in Sec. 6.2 and (5) in this section we obtain the subsidiary equation for $Q(s) = \mathcal{L}(q)$

$$(s^2 + 20s + 10{,}000)Q = 1. \qquad \text{Solution} \qquad Q = \frac{1}{(s+10)^2 + 9900}.$$

By the first shifting theorem in Sec. 6.1 we obtain from Q damped oscillations for q and v; rounding $9900 \approx 99.50^2$, we get (Fig. 135)

$$q = \mathcal{L}^{-1}(Q) = \frac{1}{99.50} e^{-10t} \sin 99.50t \qquad \text{and} \qquad v = \frac{q}{C} = 100.5e^{-10t} \sin 99.50t.$$

$q(t)$: "unit impulse response" = response of unit impulse function



$\alpha = R/2L$ (damping attenuation)

$\omega_0 = 1/\sqrt{LC}$ (natural frequency)

$\zeta = \dfrac{\alpha}{\omega_0} = \dfrac{R}{2}\sqrt{\dfrac{C}{L}}$ (damping factor)

Here, ζ=0.1<1 -> Underdamped response

(ζ>1 -> Overdamped response;

ζ=1 -> Critically damped response

5 Convolution

$$\boxed{h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau}$$

Convolution Theorem

If two functions f and g satisfy the assumption in the existence theorem in Sec. 6.1, so that their transforms F and G exist, the product $H = FG$ is the transform of h given by (1). (Proof after Example 2.)

Use notation $h(t) = f(t)*g(t) = (f*g)(t)$.

Some Properties $f*g = g*f$ (Commutative), $f*(g_1+g_2)=f*g_1+f*g_2$ (distributive), $(f*g)*h = f*(g*h)$ (associative), $f*0 = 0*f = 0$

EX 1

Let $H(s) = 1/[(s-a)s]$. Find $h(t)$.

Solution. $1/(s-a)$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{a\tau}$ and $g(t-\tau) \equiv 1$ we thus obtain from (1) the answer

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1\, d\tau = \frac{1}{a}(e^{at} - 1).$$

$$H(s) = \mathcal{L}(h)(s) = \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{a} \cdot \frac{a}{s^2 - as} = \frac{1}{s-a} \cdot \frac{1}{s} = \mathcal{L}(e^{at})\mathcal{L}(1).$$

EX 2

Let $H(s) = 1/(s^2 + \omega^2)^2$. Find $h(t)$.

Solution. The inverse of $1/(s^2 + \omega^2)$ is $(\sin \omega t)/\omega$. Hence from (1) and the first formula in (11) in App. 3.1 we obtain

$$h(t) = \frac{\sin \omega t}{\omega} * \frac{\sin \omega t}{\omega} = \frac{1}{\omega^2}\int_0^t \sin \omega\tau \sin \omega(t-\tau)\, d\tau$$

$$= \frac{1}{2\omega^2}\int_0^t [-\cos \omega t + \cos(2\omega\tau - \omega t)]\, d\tau$$

$$= \frac{1}{2\omega^2}\left[-\tau \cos \omega t + \frac{\sin \omega\tau}{\omega}\right]_{\tau=0}^{t}$$

$$= \frac{1}{2\omega^2}\left[-t \cos \omega t + \frac{\sin \omega t}{\omega}\right]$$

Proof Let

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \qquad \text{and} \qquad G(s) = \int_0^\infty e^{-sp} g(p)\, dp.$$

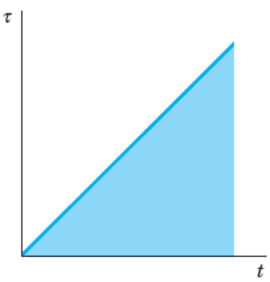
Let $p = t - \tau$

$$G(s) = \int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\, dt = e^{s\tau}\int_\tau^\infty e^{-st} g(t-\tau)\, dt.$$

τ in F and t in G vary independently. Hence we can insert the G-integral into the F-integral. Cancellation of $e^{-s\tau}$ and $e^{s\tau}$ then gives

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau)\, dt\, d\tau = \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau)\, dt\, d\tau.$$

Here we integrate for fixed τ over t from τ to ∞ and then over τ from 0 to ∞. This is the blue region in Fig. 141. Under the assumption on f and g the order of integration can be reversed (see Ref. [A5] for a proof using uniform convergence). We then integrate first over τ from 0 to t and then over t from 0 to ∞, that is,



$$F(s)G(s) = \int_0^\infty e^{-st} \int_0^t f(\tau) g(t-\tau)\, d\tau\, dt = \int_0^\infty e^{-st} h(t)\, dt = \mathcal{L}(h) = H(s).$$

Application to Nonhomogeneous Linear ODEs

$$y'' + ay' + by = r(t) \qquad (a, b \text{ constant}) \tag{2}$$

has the solution [(7) in Sec. 6.2]

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

with $R(s) = \mathcal{L}(r)$ and $Q(s) = 1/(s^2 + as + b)$ the transfer function. Inversion of the first term $[\cdots]$ provides no difficulty; depending on whether $\frac{1}{4}a^2 - b$ is positive, zero, or negative, its inverse will be a linear combination of two exponential functions, or of the form $(c_1 + c_2 t)e^{-at/2}$, or a damped oscillation, respectively. The interesting term is $R(s)Q(s)$ because $r(t)$ can have various forms of practical importance, as we shall see. If $y(0) = 0$ and $y'(0) = 0$, then $Y = RQ$, and the convolution theorem gives the solution

$$\boxed{y(t) = \int_0^t q(t-\tau) r(\tau)\, d\tau.}$$

$y(t) = q(t)*r(t)$

EX

Using convolution, determine the response of the damped mass–spring system modeled by

$$y'' + 3y' + 2y = r(t), \qquad r(t) = 1 \text{ if } 1 < t < 2 \text{ and } 0 \text{ otherwise}, \qquad y(0) = y'(0) = 0.$$

This system with an **input** (a driving force) ***that acts for some time only*** (Fig. 143) has been solved by partial fraction reduction in Sec. 6.4 (Example 1).

Solution by Convolution. The transfer function and its inverse are

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}, \qquad \text{hence} \qquad q(t) = e^{-t} - e^{-2t}.$$

Hence the convolution integral (3) is (except for the limits of integration)

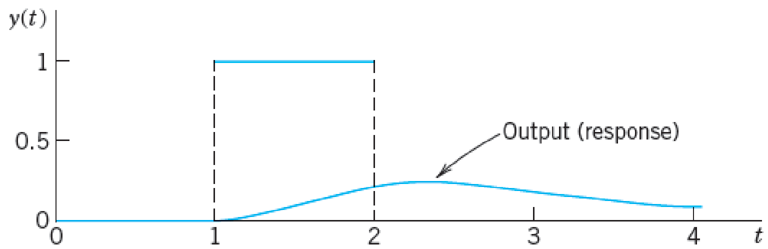
$$y(t) = \int q(t-\tau)\cdot 1\, d\tau = \int \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau = e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}.$$

Now comes an important point in handling convolution. $r(\tau) = 1$ if $1 < \tau < 2$ only. Hence if $t < 1$, the integral is zero. If $1 < t < 2$, we have to integrate from $\tau = 1$ (not 0) to t. This gives (with the first two terms from the upper limit)

$$y(t) = e^{-0} - \tfrac{1}{2}e^{-0} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}.$$

If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to t). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right).$$



Note: For $[(s+a)y(0)+y'(0)]Q(s)$ term, $\mathcal{L}^{-1}\{[ay(0)+y'(0)]Q(s)\}=[ay(0)+y'(0)]q(t)$, $\mathcal{L}^{-1}[sQ(s)]=q'(t)+q(0)\delta(t)$.

Sinusoidal Steady State Analysis

Consider a system with sinusoidal input (i.e., cos ωt) where the unit impulse response is given by $q(t)$ and all initial values are assumed to be zeroes. Then, the output is given in terms of convolution as

$$y(t) = q(t) * \cos\omega t = \int_0^t q(\tau)\cos\omega(t-\tau)d\tau = \underbrace{\int_0^\infty q(\tau)\cos\omega(t-\tau)d\tau}_{\text{sinusoidal steady state response}} - \underbrace{\int_t^\infty q(\tau)\cos\omega(t-\tau)d\tau}_{\text{transient response}}.$$

The transient response will go to 0 as t goes to ∞ <u>if the system is stable</u>. Let $y_{SSS}(t)$ denotes the sinusoidal steady state response,

$$y_{SSS}(t) = \int_0^\infty q(\tau)\cos\omega(t-\tau)d\tau = \frac{1}{2}\int_0^\infty q(\tau)\left[e^{j\omega(t-\tau)} + e^{-j\omega(t-\tau)}\right]d\tau = \frac{e^{j\omega t}}{2}\int_0^\infty q(\tau)e^{-j\omega\tau}d\tau + \frac{e^{-j\omega t}}{2}\int_0^\infty q(\tau)e^{j\omega\tau}d\tau$$

$$= \frac{e^{j\omega t}}{2}Q(j\omega) + \frac{e^{-j\omega t}}{2}Q(-j\omega) = \frac{e^{j\omega t}}{2}Q(j\omega) + \left[\frac{e^{j\omega t}}{2}Q(j\omega)\right]^* = \mathrm{Re}\left[Q(j\omega)e^{j\omega t}\right] = |Q(j\omega)|\cos(\omega t + \phi)$$

where $Q(j\omega)$ is the transfer function evaluated at $s = j\omega$, which is called system "frequency response".

<u>Ex1</u> Consider an RLC-series circuit, where the source is given by $v_S(t) = \cos\omega t$. The ODE is given by

$$Ri + L\frac{di}{dt} + v_C = v_S, i = C\frac{dv_C}{dt} \rightarrow LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = v_S$$

$$LCs^2V_C + RCsV_C + V_C = V_S \rightarrow Q(s) = \left(LCs^2 + RCs + 1\right)^{-1} \rightarrow Q(j\omega) = \left(-LC\omega^2 + jRC\omega + 1\right)^{-1}.$$

[From steady-state circuit analysis] $I = [R + j\omega L + (j\omega C)^{-1}]^{-1} = (j\omega C)^{-1}V_C \rightarrow V_C = [j\omega RC - \omega^2 LC + 1]^{-1}$.

<u>Ex2</u> Consider an RLC-parallel circuit, where the source is given by $i_S(t) = \cos\omega t$. The ODE is given by

$$\frac{v}{R} + C\frac{dv}{dt} + i_L = i_S, v = L\frac{di_L}{dt} \to LC\frac{d^2 i_L}{dt^2} + GL\frac{di_L}{dt} + i_L = i_S \ (G = 1/R)$$

$$LCs^2 I_L + GLsI_L + I_L = I_S \to Q(s) = \left(LCs^2 + GLs + 1\right)^{-1} \to Q(j\omega) = \left(-LC\omega^2 + jGL\omega + 1\right)^{-1}.$$

Note that this is the dual problem to Ex1, whose solution can be obtained from that of Ex1 by substituting $R \to 1/R = G$, $L \to C$, $C \to L$ with ω remains the same.

[From steady-state circuit analysis] $V = [G + j\omega C + (j\omega L)^{-1}]^{-1} = (j\omega L)^{-1} I_L \to I_L = [j\omega GL - \omega^2 LC + 1]^{-1}$.

6 System of ODEs

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t). \end{aligned} \qquad \begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s). \end{aligned} \qquad \begin{aligned} (a_{11} - s)Y_1 + \quad a_{12}Y_2 \quad &= -y_1(0) - G_1(s) \\ a_{21}Y_1 \quad + (a_{22} - s)Y_2 &= -y_2(0) - G_2(s). \end{aligned}$$

Can be written as $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \to \mathbf{AY} = -\mathbf{y}(0) - \mathbf{G}$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}; \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}; \mathbf{y}' = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}; \mathbf{y}(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}; \ \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}; \mathbf{G} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

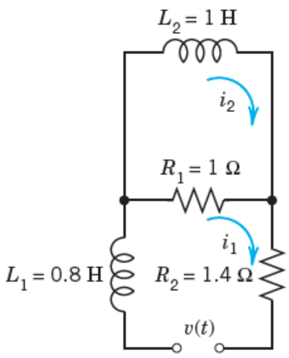
Find the currents $i_1(t)$ and $i_2(t)$ in the network in Fig. 145 with L and R measured in terms of the usual units (see Sec. 2.9), $v(t) = 100$ volts if $0 \leqq t \leqq 0.5$ sec and 0 thereafter, and $i(0) = 0$, $i'(0) = 0$.

Solution. The model of the network is obtained from Kirchhoff's Voltage Law as in Sec. 2.9. For the lower circuit we obtain

$$0.8i_1' + 1(i_1 - i_2) + 1.4i_1 = 100[1 - u(t - \tfrac{1}{2})]$$

and for the upper

$$1 \cdot i_2' + 1(i_2 - i_1) \qquad = 0.$$



Division by 0.8 and ordering gives for the lower circuit

$$i_1' + 3i_1 - 1.25i_2 = 125[1 - u(t - \tfrac{1}{2})]$$

and for the upper

$$i_2' - i_1 + \qquad i_2 = 0.$$

With $i_1(0) = 0$, $i_2(0) = 0$ we obtain from (1) in Sec. 6.2 and the second shifting theorem the subsidiary system

$$\begin{aligned} (s + 3)I_1 - 1.25I_2 &= 125\left(\frac{1}{s} - \frac{e^{-s/2}}{s}\right) \\ -I_1 + (s + 1)I_2 &= 0. \end{aligned}$$

Solving algebraically for I_1 and I_2 gives

$$I_1 = \frac{125(s + 1)}{s(s + \frac{1}{2})(s + \frac{7}{2})}(1 - e^{-s/2}),$$

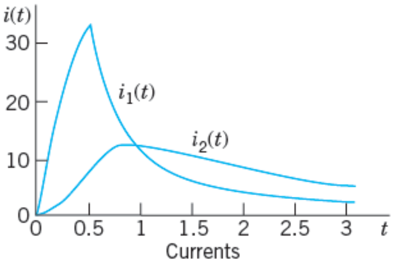
$$I_2 = \frac{125}{s(s + \frac{1}{2})(s + \frac{7}{2})}(1 - e^{-s/2}).$$

The right sides, without the factor $1 - e^{-s/2}$, have the partial fraction expansions

$$\frac{500}{7s} - \frac{125}{3(s + \frac{1}{2})} - \frac{625}{21(s + \frac{7}{2})}$$

and

$$\frac{500}{7s} - \frac{250}{3(s+\frac{1}{2})} + \frac{250}{21(s+\frac{7}{2})},$$



respectively. The inverse transform of this gives the solution for $0 \leqq t \leqq \frac{1}{2}$,

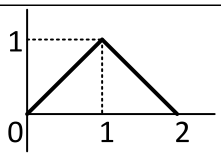
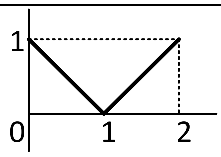
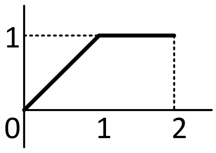
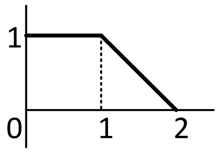
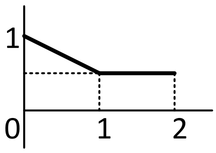
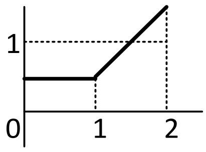
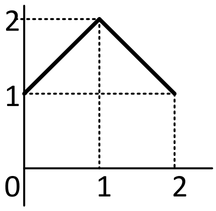
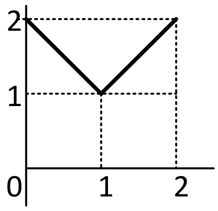
$$i_1(t) = -\tfrac{125}{3}e^{-t/2} - \tfrac{625}{21}e^{-7t/2} + \tfrac{500}{7}$$

$$i_2(t) = -\tfrac{250}{3}e^{-t/2} + \tfrac{250}{21}e^{-7t/2} + \tfrac{500}{7}.$$

According to the second shifting theorem the solution for $t > \frac{1}{2}$ is $i_1(t) - i_1(t-\frac{1}{2})$ and $i_2(t) - i_2(t-\frac{1}{2})$, that is,

$$\begin{aligned} i_1(t) &= -\tfrac{125}{3}(1-e^{1/4})e^{-t/2} - \tfrac{625}{21}(1-e^{7/4})e^{-7t/2} \\ i_2(t) &= -\tfrac{250}{3}(1-e^{1/4})e^{-t/2} + \tfrac{250}{21}(1-e^{7/4})e^{-7t/2} \end{aligned} \qquad (t > \tfrac{1}{2}).$$

7 Practice Problems

<table>
<tr><th colspan="2">$f(t)$</th><th>ODE</th></tr>
<tr><td>1 </td><td>2 </td><td rowspan="5">a. $y'+y = f(t)$, $y(0)=1$
b. $y'+2y = f(t)$, $y(0)=2$
c. $y'+3y = f(t)$, $y(0)=-1$
d. $y'+4y = f(t)$, $y(0)=-2$
e. $y''+5y'+6y = f(t)$, $y(0)=1$, $y'(0)=2$
f. $y''+6y'+8y = f(t)$, $y(0)=2$, $y'(0)=1$
g. $y''+5y'+4y = f(t)$, $y(0)=1$, $y'(0)=-1$
h. $y''+4y'+4y = f(t)$, $y(0)=-1$, $y'(0)=2$</td></tr>
<tr><td>3 </td><td>4 </td></tr>
<tr><td>5 </td><td>6 </td></tr>
<tr><td>7 </td><td>8 </td></tr>
<tr><td>9 $\sin(t)[u(t)-u(t-\pi)]$
11 $|\cos(t)|[u(t)-u(t-\pi)]$
13 $\sinh(t)[u(t)-u(t-\pi)]$
15 $\cosh(t)[u(t)-u(t-\pi)]$</td><td>10 $\sin(\pi t)[u(t)-u(t-1)]$
12 $|\cos(\pi t)|[u(t)-u(t-1)]$
14 $\sinh(\pi t)[u(t)-u(t-1)]$
16 $\cosh(\pi t)[u(t)-u(t-1)]$</td></tr>
</table>

Question

1. Find the Laplace transform of $f(t)$, then solve the ODE.
2. Find the unit impulse response of the system specified by the ODE, then use convolution to find the system output when the input is given by $f(t)$.