1. Orthogonal Curvilinear Coordinates

Let the rectangular coordinates (x, y, z) of any point be expressed as functions of (u_1, u_2, u_3) so that

$x = x(u_1, u_2, u_3)$, $y = y(u_1, u_2, u_3)$, $z = z(u_1, u_2, u_3)$ (1)

Suppose that Eq.(1) can be solved for u_1, u_2, u_3 in terms x, y, z, i.e.,

$u_1 = u_1(x, y, z)$, $u_2 = u_2(x, y, z)$, $u_3 = u_3(x, y, z)$ (2)

The surfaces $u_1=c_1$, $u_2=c_2$, $u_3=c_3$, where c_1, c_2, c_3 are constants, are called *coordinate surfaces* and each pair of these surfaces intersect in curves called *coordinate curves* or *lines* (Fig. 1). The coordinate surfaces as well as the base vectors for Cartesian coordinates, cylindrical coordinates, and spherical coordinates are shown in Fig. 2, Fig. 3, and Fig. 4, respectively.

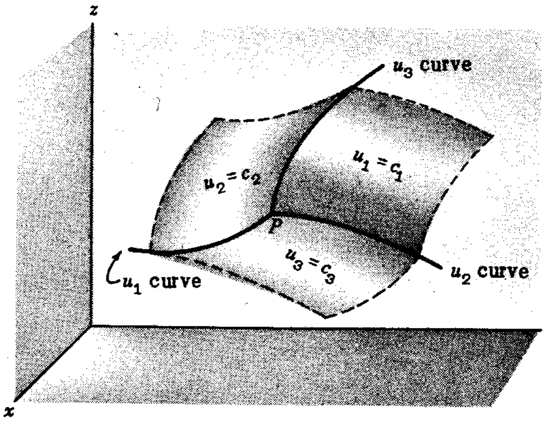

Fig. 1

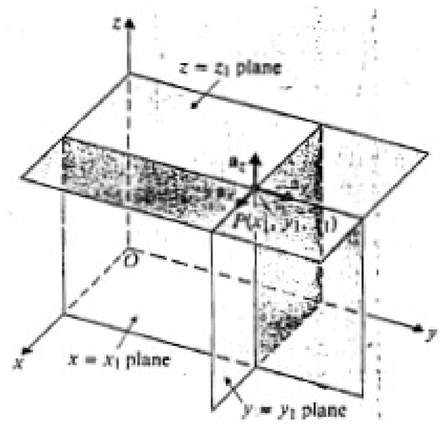

Fig. 2

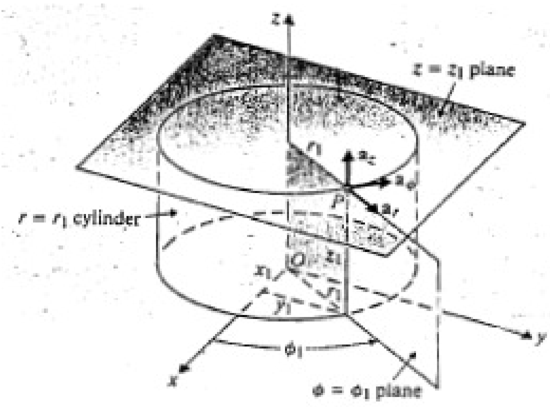

Fig. 3

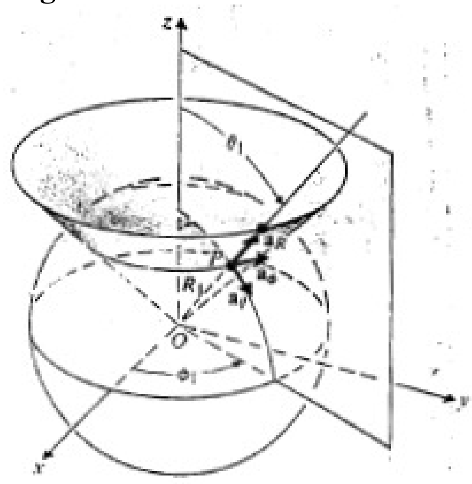

Fig. 4

Let $\mathbf{R} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ be the position vector of a point P. Then Eq.(1) can be written as $\boldsymbol{R} = \boldsymbol{R}(u_1, u_2, u_3)$. A **tangent vector** to the u_1 curve at P (for which u_2 and u_3 are constants) is $\frac{\partial \mathbf{R}}{\partial u_1}$. Then a **unit tangent vector** in this direction is given by

$$\mathbf{e}_1 = \frac{\dfrac{\partial \mathbf{R}}{\partial u_1}}{\left|\dfrac{\partial \mathbf{R}}{\partial u_1}\right|} = \frac{1}{h_1}\frac{\partial \mathbf{R}}{\partial u_1} \qquad (3)$$

so that $\frac{\partial \mathbf{R}}{\partial u_1} = h_1 \mathbf{e}_1$ where $h_1 = \left|\frac{\partial \mathbf{R}}{\partial u_1}\right|$.

Similarly, if $\mathbf{e}_2$ and $\mathbf{e}_3$ are unit tangent vectors along u_2 and u_3 curves at P respectively, then

$$\frac{\partial \mathbf{R}}{\partial u_2} = h_2 \mathbf{e}_2 \text{ and } \frac{\partial \mathbf{R}}{\partial u_3} = h_3 \mathbf{e}_3, \quad (4)$$

where $h_2 = \left|\frac{\partial \mathbf{R}}{\partial u_2}\right|$ and $h_3 = \left|\frac{\partial \mathbf{R}}{\partial u_3}\right|$. The quantities h_1, h_2, h_3 are called **scalar factors**. The unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are in the directions of increasing u_1, u_2, u_3 respectively (Fig. 5).

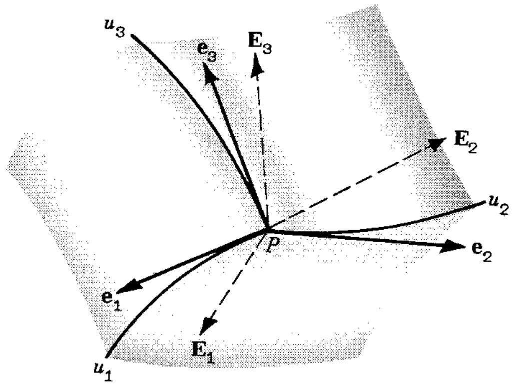

Fig. 5

<u>**Definition 1**</u>: The coordinates system (u_1, u_2, u_3) is said to be **orthogonal curvilinear coordinates** if and only if their unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are orthogonal.

2. Arc Length and Volume Elements

In an orthogonal curvilinear coordinates (u_1, u_2, u_3), from $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$, we have

$$d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u_1} du_1 + \frac{\partial \mathbf{R}}{\partial u_2} du_2 + \frac{\partial \mathbf{R}}{\partial u_3} du_3 = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3 \quad (5)$$

Then the differential of arc length ds is determined from

$$ds^2 = d\mathbf{R} \cdot d\mathbf{R} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2 \quad (6)$$

Referring to Fig. 6 the volume element for an orthogonal curvilinear coordinate system is given by

$$dV = \left|(h_1 du_1 \mathbf{e}_1) \cdot (h_2 du_2 \mathbf{e}_2) \times (h_3 du_3 \mathbf{e}_3)\right| = h_1 h_2 h_3 du_1 du_2 du_3 \quad (7)$$

since $\left|\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3\right| = 1$.

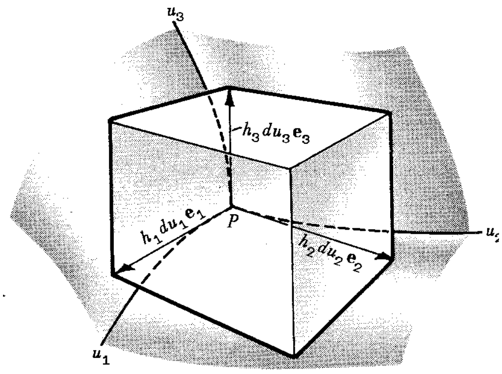

Fig. 6

3. ∇ Operator in Orthogonal Curvilinear Coordinates

(1) Gradient in Orthogonal Curvilinear Coordinates

For any scalar function Φ, we can express its gradient in orthogonal curvilinear coordinate system (u_1, u_2, u_3) as

$$\nabla\Phi = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3, \quad (8)$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors in the directions of increasing u_1, u_2, u_3 respectively. Since

$d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u_1} du_1 + \frac{\partial \mathbf{R}}{\partial u_2} du_2 + \frac{\partial \mathbf{R}}{\partial u_3} du_3 = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3$, we have

(1) $d\Phi = \nabla\Phi \cdot d\mathbf{R} = f_1 h_1 du_1 + f_2 h_2 du_2 + f_3 h_3 du_3$. But

(2) $d\Phi = \frac{\partial \Phi}{\partial u_1} du_1 + \frac{\partial \Phi}{\partial u_2} du_2 + \frac{\partial \Phi}{\partial u_3} du_3$,

equating (1) and (2) yields $f_1 = \frac{1}{h_1}\frac{\partial \Phi}{\partial u_1}; f_2 = \frac{1}{h_2}\frac{\partial \Phi}{\partial u_2}; f_3 = \frac{1}{h_3}\frac{\partial \Phi}{\partial u_3}$.

Then the gradient of Φ is given by

$$\nabla\Phi = \frac{\mathbf{e}_1}{h_1}\frac{\partial \Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial \Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial \Phi}{\partial u_3} \quad (9)$$

This indicates the operator equivalence

$$\nabla \equiv \frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3} \tag{10}$$

(2) Divergence in Orthogonal Curvilinear Coordinates

Consider the volume element ΔV (see Fig. 7) having edges $h_1\Delta u_1$, $h_2\Delta u_2$, $h_3\Delta u_3$. Let $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ and let $\mathbf{n}$ be the outward drawn unit normal to the surface ΔS of ΔV.

On the face $JKLP$, $\mathbf{n} = -\mathbf{e}_1$. Then we have approximately,

$$\iint_{JKLP} \mathbf{A}\cdot\mathbf{n}dS = (\mathbf{A}\cdot\mathbf{n}\,\text{at point P})(\text{Area of JKLP})$$

$$= [(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3)\cdot(-\mathbf{e}_1)](h_2h_3\Delta u_2\Delta u_3)$$

$$= -A_1h_2h_3\Delta u_2\Delta u_3$$

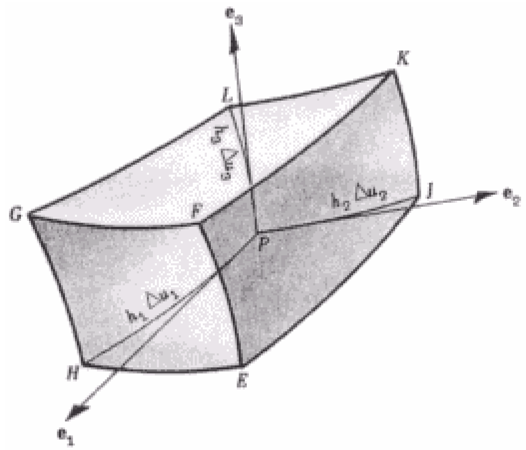

Fig. 7

On face $EFGH$, the surface integral is

$$A_1h_2h_3\Delta u_2\Delta u_3 + \frac{\partial}{\partial u_1}(A_1h_2h_3\Delta u_2\Delta u_3)\Delta u_1,$$

apart from infinitesimal of order higher than $\Delta u_1\Delta u_2\Delta u_3$. Then the net contribution to the surface integral from these two faces is

$$\iint_{JKLP} \mathbf{A}\cdot\mathbf{n}dS + \iint_{EFGH} \mathbf{A}\cdot\mathbf{n}dS = \frac{\partial}{\partial u_1}(A_1h_2h_3\Delta u_2\Delta u_3)\Delta u_1 = \frac{\partial}{\partial u_1}(A_1h_2h_3)\Delta u_1\Delta u_2\Delta u_3$$

The contribution from six faces of ΔV is

$$\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_1h_3) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]\Delta u_1\Delta u_2\Delta u_3$$

Dividing this by the volume $h_1h_2h_3\Delta u_1\Delta u_2\Delta u_3$ and taking the limit as Δu_1, Δu_2, Δu_3 approach zero, we find

$$\text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_1h_3) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right] \tag{11}$$

(3) Curl in Orthogonal Curvilinear Coordinates

Let us first calculate $(\nabla\times\mathbf{A})\cdot\mathbf{e}_1$. To do this, consider the surface S_1 normal to $\mathbf{e}_1$ at P, as shown in Fig. 8.

Denote the boundary of S_1 by C_1. Let $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$, we have

$$\oint_{C_1} \mathbf{A}\cdot d\mathbf{r} = \int_{PQ} \mathbf{A}\cdot d\mathbf{r} + \int_{QL} \mathbf{A}\cdot d\mathbf{r} + \int_{LM} \mathbf{A}\cdot d\mathbf{r} + \int_{MP} \mathbf{A}\cdot d\mathbf{r}$$

The following approximation holds

(1) $$\int_{PQ} \mathbf{A}\cdot d\mathbf{r} = (\mathbf{A}\,\text{at P})(h_2\Delta u_2\mathbf{e}_2) =$$

$$(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3)\cdot(h_2\Delta u_2\mathbf{e}_2) = A_2h_2\Delta u_2$$

Then

$$\int_{ML} \mathbf{A}\cdot d\mathbf{r} = A_2h_2\Delta u_2 + \frac{\partial}{\partial u_3}(A_2h_2\Delta u_2)\Delta u_3 \quad \text{or}$$

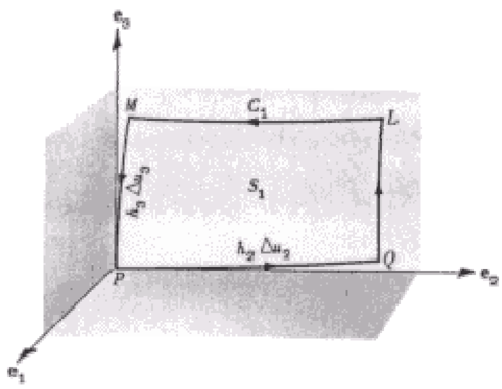

Fig. 8

(2) $$\int_{LM} \mathbf{A}\cdot d\mathbf{r} = -A_2h_2\Delta u_2 - \frac{\partial}{\partial u_3}(A_2h_2\Delta u_2)\Delta u_3$$

Similarly, $\int_{PM} \mathbf{A}\cdot d\mathbf{r} = (\mathbf{A}\text{ at P})(h_3\Delta u_3\mathbf{e}_3) = A_3h_3\Delta u_3$ or

(3) $\int_{MP} \mathbf{A}\cdot d\mathbf{r} = -A_3h_3\Delta u_3$ and

(4) $\int_{QL} \mathbf{A}\cdot d\mathbf{r} = A_3h_3\Delta u_3 + \frac{\partial}{\partial u_2}(A_3h_3\Delta u_3)\Delta u_2$

Adding (1), (2), (3) and (4) we have

$$\oint_{C_1} \mathbf{A}\cdot d\mathbf{r} = \frac{\partial}{\partial u_2}(A_3h_3\Delta u_3)\Delta u_2 - \frac{\partial}{\partial u_3}(A_2h_2\Delta u_2)\Delta u_3 = \left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right]\Delta u_2\Delta u_3$$

apart from infinitesimal of order higher than $\Delta u_2\Delta u_3$. Dividing by the area of S_1 equal to $h_2h_3\Delta u_2\Delta u_3$ and taking the limit as Δu_2 and Δu_3 approach zero,

$$(\nabla\times\mathbf{A})\cdot\mathbf{e}_1 = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right]$$

Similarly, by choosing area S_2 and S_3 perpendicular to $\mathbf{e}_2$ and $\mathbf{e}_3$ at P respectively, we find $(\nabla\times\mathbf{A})\cdot\mathbf{e}_2$ and $(\nabla\times\mathbf{A})\cdot\mathbf{e}_3$. This leads to the required result

$$\nabla\times\mathbf{A} = \frac{\mathbf{e}_1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right] + \frac{\mathbf{e}_2}{h_3h_1}\left[\frac{\partial}{\partial u_3}(A_1h_1) - \frac{\partial}{\partial u_1}(A_3h_3)\right] + \frac{\mathbf{e}_3}{h_1h_2}\left[\frac{\partial}{\partial u_1}(A_2h_2) - \frac{\partial}{\partial u_2}(A_1h_1)\right]$$

$$= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \qquad (12)$$

(4) Laplacian in Orthogonal Curvilinear Coordinates

To evaluate the Laplacian of a scalar function Φ in orthogonal curvilinear coordinates, we are making use the Eq.(11) by taking $\mathbf{A} = \nabla\Phi = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$

$$\nabla^2\Phi = \nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(f_1h_2h_3) + \frac{\partial}{\partial u_2}(f_2h_1h_3) + \frac{\partial}{\partial u_3}(f_3h_1h_2)\right]$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}h_2h_3\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}h_1h_3\right) + \frac{\partial}{\partial u_3}\left(\frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}h_1h_2\right)\right] \qquad (13)$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

4. Divergence, Curl and Laplacian in Curvilinear Coordinates

(1) Spherical coordinates

Fig. 9 shows the relationship between the spherical coordinates and the Cartesian coordinates. The spherical coordinates (r, θ, φ) of a point have the following relationships with rectangular coordinates:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad (14)$$

We have the following relations

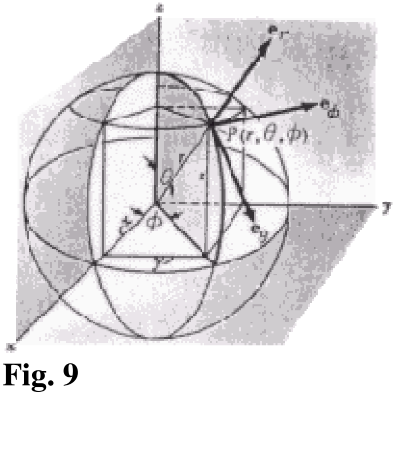

Fig. 9

$$\begin{cases} dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi = (\sin\theta\cos\phi)dr + (r\cos\theta\cos\phi)d\theta - (r\sin\theta\sin\phi)d\phi \\ dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi = (\sin\theta\sin\phi)dr + (r\cos\theta\sin\phi)d\theta + (r\sin\theta\cos\phi)d\phi \\ dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \phi}d\phi = (\cos\theta)dr - (r\sin\theta)d\theta \end{cases} \quad (15)$$

Hence, the unit vector along the r-direction is

$$h_r = \left|\frac{\partial \mathbf{R}}{\partial r}\right| = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2} = 1 \quad (16)$$

$$\begin{aligned} \hat{\mathbf{r}} &= \frac{\frac{\partial \mathbf{R}}{\partial r}}{\left|\frac{\partial \mathbf{R}}{\partial r}\right|} = \frac{\frac{\partial x}{\partial r}\hat{\mathbf{x}} + \frac{\partial y}{\partial r}\hat{\mathbf{y}} + \frac{\partial z}{\partial r}\hat{\mathbf{z}}}{\sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2}} \\ &= \frac{(\sin\theta\cos\phi)\hat{\mathbf{x}} + (\sin\theta\sin\phi)\hat{\mathbf{y}} + (\cos\theta)\hat{\mathbf{z}}}{\sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2}} \quad (17) \\ &= (\sin\theta\cos\phi)\hat{\mathbf{x}} + (\sin\theta\sin\phi)\hat{\mathbf{y}} + (\cos\theta)\hat{\mathbf{z}} \end{aligned}$$

Similarly the unit vector along the θ -direction is

$$h_\theta = \left|\frac{\partial \mathbf{R}}{\partial \theta}\right| = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = \sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (r\sin\theta)^2} = r \quad (18)$$

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= \frac{\frac{\partial \mathbf{R}}{\partial \theta}}{\left|\frac{\partial \mathbf{R}}{\partial \theta}\right|} = \frac{\frac{\partial x}{\partial \theta}\hat{\mathbf{x}} + \frac{\partial y}{\partial \theta}\hat{\mathbf{y}} + \frac{\partial z}{\partial \theta}\hat{\mathbf{z}}}{\sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2}} = \frac{(r\cos\theta\cos\phi)\hat{\mathbf{x}} + (r\cos\theta\sin\phi)\hat{\mathbf{y}} - (r\sin\theta)\hat{\mathbf{z}}}{\sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (r\sin\theta)^2}} \quad (19) \\ &= (\cos\theta\cos\phi)\hat{\mathbf{x}} + (\cos\theta\sin\phi)\hat{\mathbf{y}} - (\sin\theta)\hat{\mathbf{z}} \end{aligned}$$

and the unit vector along the ϕ -direction is

$$h_\phi = \left|\frac{\partial \mathbf{R}}{\partial \phi}\right| = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} = \sqrt{(r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2} = r\sin\theta \quad (20)$$

$$\begin{aligned} \hat{\boldsymbol{\phi}} &= \frac{\frac{\partial \mathbf{R}}{\partial \phi}}{\left|\frac{\partial \mathbf{R}}{\partial \phi}\right|} = \frac{\frac{\partial x}{\partial \phi}\hat{\mathbf{x}} + \frac{\partial y}{\partial \phi}\hat{\mathbf{y}} + \frac{\partial z}{\partial \phi}\hat{\mathbf{z}}}{\sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2}} = \frac{-(r\sin\theta\sin\phi)\hat{\mathbf{x}} + (r\sin\theta\cos\phi)\hat{\mathbf{y}}}{\sqrt{(r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2}} \quad (21) \\ &= -(\sin\phi)\hat{\mathbf{x}} + (\cos\phi)\hat{\mathbf{y}} \end{aligned}$$

Therefore,

Gradient:

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \quad (22)$$

Divergence:

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \quad (23)$$

Curl:

$$\nabla\times\mathbf{F}=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta F_\phi\right)-\frac{\partial F_\theta}{\partial\phi}\right]\hat{\mathbf{r}}+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi}-\frac{\partial}{\partial r}\left(rF_\phi\right)\right]\hat{\boldsymbol{\theta}}+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right)-\frac{\partial F_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{24}$$

Laplacian:

$$\nabla^2 T=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{25}$$

(2) Cylindrical coordinates

Fig. 10 shows the relationship between the cylindrical coordinates and the Cartesian coordinates. The cylindrical coordinates (ρ, ϕ, z) of a point have the following relationships with rectangular coordinates:

$$\begin{cases} x=\rho\cos\phi \\ y=\rho\sin\phi \\ z=z \end{cases} \tag{26}$$

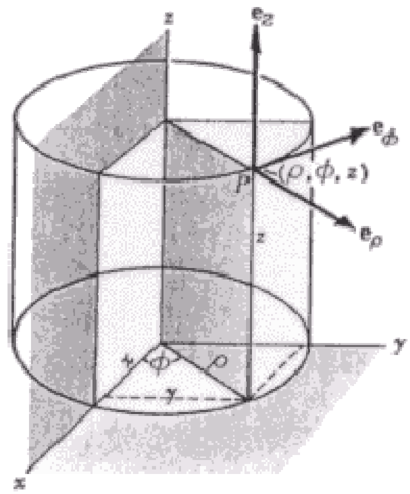

Fig. 10

We have the following relations

$$\begin{cases} dx=\dfrac{\partial x}{\partial\rho}d\rho+\dfrac{\partial x}{\partial\phi}d\phi+\dfrac{\partial x}{\partial z}dz=(\cos\phi)d\rho-(\rho\sin\phi)d\phi \\ dy=\dfrac{\partial y}{\partial\rho}d\rho+\dfrac{\partial y}{\partial\phi}d\phi+\dfrac{\partial y}{\partial z}dz=(\sin\phi)d\rho+(\rho\cos\phi)d\phi \\ dz=\dfrac{\partial z}{\partial\rho}d\rho+\dfrac{\partial z}{\partial\phi}d\phi+\dfrac{\partial z}{\partial z}dz=dz \end{cases} \tag{27}$$

Hence the unit vector along the ρ-direction is

$$h_\rho=\left|\frac{\partial\mathbf{R}}{\partial\rho}\right|=\sqrt{\left(\frac{\partial x}{\partial\rho}\right)^2+\left(\frac{\partial y}{\partial\rho}\right)^2+\left(\frac{\partial z}{\partial\rho}\right)^2}=\sqrt{\cos^2\phi+\sin^2\phi}=1 \tag{28}$$

$$\hat{\boldsymbol{\rho}}=\frac{\dfrac{\partial\mathbf{R}}{\partial\rho}}{\left|\dfrac{\partial\mathbf{R}}{\partial\rho}\right|}=\frac{\dfrac{\partial x}{\partial\rho}\hat{\mathbf{x}}+\dfrac{\partial y}{\partial\rho}\hat{\mathbf{y}}+\dfrac{\partial z}{\partial\rho}\hat{\mathbf{z}}}{\sqrt{\left(\dfrac{\partial x}{\partial\rho}\right)^2+\left(\dfrac{\partial y}{\partial\rho}\right)^2+\left(\dfrac{\partial z}{\partial\rho}\right)^2}}=\frac{(\cos\phi)\hat{\mathbf{x}}+(\sin\phi)\hat{\mathbf{y}}}{\sqrt{\cos^2\phi+\sin^2\phi}} \tag{29}$$

$$=(\cos\phi)\hat{\mathbf{x}}+(\sin\phi)\hat{\mathbf{y}}$$

Similarly the unit vector along the θ -direction is

$$h_\phi=\left|\frac{\partial\mathbf{R}}{\partial\phi}\right|=\sqrt{\left(\frac{\partial x}{\partial\phi}\right)^2+\left(\frac{\partial y}{\partial\phi}\right)^2+\left(\frac{\partial z}{\partial\phi}\right)^2}=\sqrt{(\rho\sin\phi)^2+(\rho\cos\phi)^2}=\rho \tag{30}$$

$$\hat{\boldsymbol{\phi}}=\frac{\dfrac{\partial\mathbf{R}}{\partial\phi}}{\left|\dfrac{\partial\mathbf{R}}{\partial\phi}\right|}=\frac{\dfrac{\partial x}{\partial\phi}\hat{\mathbf{x}}+\dfrac{\partial y}{\partial\phi}\hat{\mathbf{y}}+\dfrac{\partial z}{\partial\phi}\hat{\mathbf{z}}}{\sqrt{\left(\dfrac{\partial x}{\partial\phi}\right)^2+\left(\dfrac{\partial y}{\partial\phi}\right)^2+\left(\dfrac{\partial z}{\partial\phi}\right)^2}}=\frac{-(\rho\sin\phi)\hat{\mathbf{x}}+(\rho\cos\phi)\hat{\mathbf{y}}}{\sqrt{(\rho\sin\phi)^2+(\rho\cos\phi)^2}} \tag{31}$$

$$=-(\sin\phi)\hat{\mathbf{x}}+(\cos\phi)\hat{\mathbf{y}}$$

In the z-direction $h_z = 1$. Therefore

Gradient:

$$\nabla V=\hat{\boldsymbol{\rho}}\frac{\partial V}{\partial\rho}+\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial V}{\partial\phi}+\hat{\mathbf{z}}\frac{\partial V}{\partial z} \tag{32}$$

Divergence:

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \tag{33}$$

Curl:

$$\nabla \times \mathbf{F} = \frac{1}{\rho}\left[\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right]\hat{\boldsymbol{\rho}} + \left[\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right]\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho F_\phi\right) - \frac{\partial F_\rho}{\partial \phi}\right]\hat{\mathbf{z}} \tag{34}$$

Laplacian:

$$\nabla^2 T = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial T}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{35}$$

<u>Example 1</u> Verify the divergence theorem for a vector field $\mathbf{F} = \hat{\mathbf{r}}kr$ over the shell region enclosed by spherical surfaces at $r = R_1$ and $r = R_2$, where $R_1 < R_2$, centered at the origin.

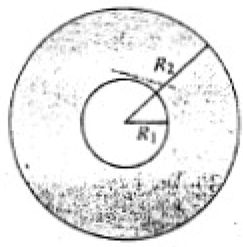

Fig. 11

<u>Example 2</u> Show that $\nabla \times \mathbf{A} = \mathbf{0}$ if

(a) $\mathbf{A} = \hat{\boldsymbol{\phi}}\frac{k}{\rho}$ in cylindrical coordinates, where k is a constant.

(b) $\mathbf{A} = \hat{\mathbf{r}}f(r)$ in spherical coordinates, where $f(r)$ is a function of only r.

<u>Example 3</u> Given a vector function $\mathbf{A} = \hat{\boldsymbol{\phi}}3\sin(\phi/2)$, verify the Stokes' theorem over the unit circle counterclockwise contour centered at (0,0) on the xy-plane.

<u>Example 4</u> Verify Stokes' theorem for a vector field $\mathbf{A} = \hat{\boldsymbol{\rho}}\rho\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi$ over the path shown in Fig. 12.

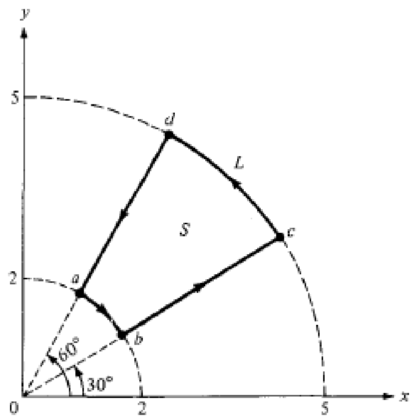

Fig. 12