Static Electric Fields

Electrostatics is the study of the effects of electric charges at rest, and the static electric fields, which are caused by stationary electric charges.

In the *deductive approach*, few fundamental relations for an idealized model are postulated as axioms, from which particular laws and theorems can be derived. Then the validity of the model and the axioms are verified by the experiments. The steps involved in building a theory based on an idealized model are as follows:

1. Define some basic quantities. ($\mathbf{E}$, q)
2. Specify the rules of operations. (Vector analysis)
3. Postulate some fundamental relations. (Divergence equation, Curl equation)

3-2 Electrostatics in Free Space

Here, electric field in free space is considered. Noted that the permittivity[1] of the free space, denoted by ε_0, is equal to $(1/36\pi)\times10^{-9} = 8.854\times10^{-12}$ (F/m).

First, **electric field intensity** is defined as the force per unit charge that a very small stationary test charge experiences when it is placed in a region where an electric field exists. That is,

$$\mathbf{E} = \lim_{q\to 0}\frac{\mathbf{F}}{q} \text{ (V/m)} \qquad (3\text{-}1)$$

Thus, $\mathbf{E}$ is proportional to and in the direction of the force $\mathbf{F}$. Notice that the unit Newton/Coulomb = V/m. An inverse relation of (3-1) gives

$$\mathbf{F} = q\mathbf{E} \text{ (N)} \qquad (3\text{-}2)$$

The two fundamental postulates of electrostatics in free space specify the divergence and the curl of $\mathbf{E}$. They are

$$\boxed{\nabla\cdot\mathbf{E} = \frac{\rho_v}{\varepsilon_0}} \qquad (3\text{-}3)$$

and

$$\boxed{\nabla\times\mathbf{E} = \mathbf{0}}, \qquad (3\text{-}4)$$

where ρ_v denotes the volume charge density with the unit (C/m^3). The definition of ρ_v is given by

$$\rho_v = \lim_{\Delta v\to 0}\frac{\Delta q}{\Delta v} \text{ (C/m}^3\text{)}$$

(3-4) asserts that *static electric fields are irrotational* whereas (3-3) implies that a static electric field is not solenoidal. These two equations are point relations or in differential forms. Taking the volume integral of both sides of (3-3) over a volume V yields

$$\int_V \nabla\cdot\mathbf{E}\,dv = \int_V \frac{\rho_v}{\varepsilon_0}dv = \frac{Q}{\varepsilon_0}$$

where Q is the total charge contained in V. Applying the divergence theorem, one obtains

$$\boxed{\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{Q}{\varepsilon_0}} \quad \text{(Gauss's law)} \qquad (3\text{-}6)$$

which is a form of **Gauss's law**. Likewise, taking the surface integral of both sides of (3-4) and applying Stokes' theorem yields

$$\boxed{\oint_C \mathbf{E}\cdot d\boldsymbol{\ell} = 0}, \qquad (3\text{-}7)$$

[1] **Permittivity** is a physical quantity that describes how an electric field affects, and is affected by, a dielectric medium, and is determined by the ability of a material to polarize in response to the field, and thereby reduce the total electric field inside the material. Thus, permittivity relates to a material's ability to transmit (or "permit") an electric field.

which asserts that "the scalar line integral of the static electric field intensity around any closed path vanishes". (3-6), (3-7) are of *integral* forms. Since the scalar product $\mathbf{E} \cdot d\mathbf{l}$ integrated over any path is the voltage along that path, i.e.,

$$V = \int_C \mathbf{E} \cdot d\mathbf{l} \quad \text{(V)},$$

thus (3-7) is equivalent to Kirchhoff's voltage law, i.e., the algebraic sum of voltage drops around any closed circuit is zero.

3-3 Coulomb's Law

Consider a single point charge q at rest in boundless free space. In order to find the electric field intensity due to q, a spherical surface of an arbitrary radius r centered at q—a hypothetical enclosed surface (*a Gaussian surface*) around the source— is drawn, upon which Gauss's law is applied to determine the field. Since a point charge has no preferred directions, its electric field must be everywhere radial and has the same intensity at all points on the spherical surface. Applying (3-6) to Fig. 1 (a) yields

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \oint_S (\hat{\mathbf{r}} E_r) \cdot \hat{\mathbf{r}} ds = \frac{q}{\varepsilon_0} \text{ or } E_r \oint_S ds = E_r 4\pi r^2 = \frac{q}{\varepsilon_0}.$$

Therefore,

$$\boxed{\mathbf{E} = \hat{\mathbf{r}} E_r = \hat{\mathbf{r}} \frac{q}{4\pi\varepsilon_0 r^2}} \quad \text{(V/m)} \qquad \text{(3-8)}$$

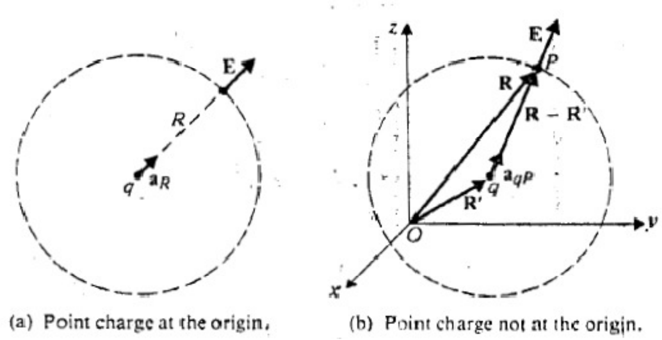


Fig. 1

From (3-8), *the electric field intensity of a point charge is in the outward radial direction and has a magnitude proportional to the charge and inversely proportional to the square of the distance from the charge*. If the charge q is not located at the origin, referring to Fig. 1(b), one obtains the electric field intensity at point P to be

$$\mathbf{E}_P = \hat{a}_{qP} \frac{q}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|^2}.$$

But $\hat{a}_{qP} = \dfrac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$

$$\boxed{\mathbf{E}_P = \frac{q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|^3}} \quad \text{(V/m)} \qquad \text{(3-11)}$$

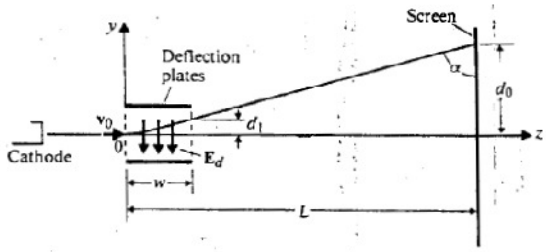
Example 3-1 Determine the electric field intensity at $P(-0.2,0,-2.3)$ due to a point charge of $+5$ (nC) at $Q(0.2,0.1,-2.5)$ in air. All dimensions are in meters.

When a point charge q_2 is placed in the electric field of another point charge q_1, a force $\mathbf{F}_{12}$ is experienced by q_2 due to $\mathbf{E}_{12}$ of q_1 at q_2, which is given by

$$\boxed{\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \hat{a}_{12}\frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2}} \text{ (N) ; } R_{12} = |\mathbf{R}_{12}| = |\mathbf{r}_2 - \mathbf{r}_1| \qquad \text{(3-13)}$$

(3-13) is a mathematical form of **Coulomb's law** : *the force between two point charges is proportional to the product of the charges and inversely proportional to the square of the distance of separation.*

Example 3-2 The electrostatic deflection system of a cathode-ray oscillograph is depicted in the right figure. Electrons from a heated cathode are given an initial velocity $\mathbf{u}_0 = \hat{\mathbf{z}}u_0$ by a positively charged anode. The electrons enter at z=0 into a region of deflection



plates where a uniform electric field $\mathbf{E}_d = -\hat{\mathbf{y}}E_d$ is maintained over a width w. Ignoring gravitational effects, find the vertical deflection of the electrons on the fluorescent screen at z=L.

3-3.1 Electric field due to a system of discrete charges

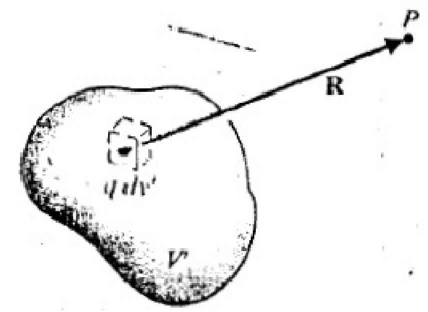
Suppose an electrostatic field is created by a group of n discrete point charges, q_1, q_2, …, q_n, located at different positions, the principle of superposition can be applied to find the total electric field due to this system of discrete charges, which is given by

$$\mathbf{E} = \frac{q_1(\mathbf{r}-\mathbf{r}_1')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_1'|^3} + \frac{q_2(\mathbf{r}-\mathbf{r}_2')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_2'|^3} + \cdots + \frac{q_2(\mathbf{r}-\mathbf{r}_n')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_n'|^3} \text{ or}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k(\mathbf{r}-\mathbf{r}_k')}{|\mathbf{r}-\mathbf{r}_k'|^3} \text{ (V/m)} \qquad \text{(3-14)}$$

3-3.2 Electric field due to a continuous distribution charges

The electric field caused by a continuous distribution of charge as shown in the figure on the right can be obtained by integrating the contribution of an element of charge over the charge distribution. Let ρ_v be the volume charge density (C/m^3), then the electric field intensity due to qdv' at P is given by



$$d\mathbf{E} = \hat{a}_R\frac{\rho_v dv'}{4\pi\varepsilon_0 R^2}$$

Therefore,

$$\mathbf{E} = \int_{V'} d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{V'}\hat{a}_R\frac{\rho_v dv'}{R^2} = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\mathbf{R}}{R^3}\rho_v dv' \text{ (V/m) ;} \qquad \hat{a}_R = \frac{\mathbf{R}}{R} \qquad \text{(3-16)}$$

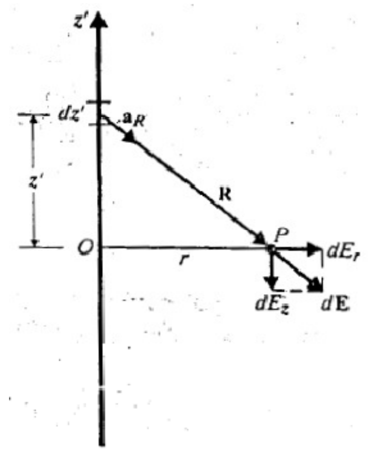
For the charge distributed on a surface with a surface charge density ρ_s (C/m^2) (3-16) becomes

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{S'} \hat{a}_R \frac{\rho_s ds'}{R^2} \quad \text{(V/m)} \tag{3-17}$$

For a line charge with a line charge density ρ_ℓ (C/m), (3-16) becomes

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{L'} \hat{a}_R \frac{\rho_\ell d\ell'}{R^2} \quad \text{(V/m)} \tag{3-18}$$

Example 3-3 Determine the electric field of an infinitely long, straight, line charge of uniform density ρ_ℓ (C/m) in air



3-4 Gauss's Law and Applications

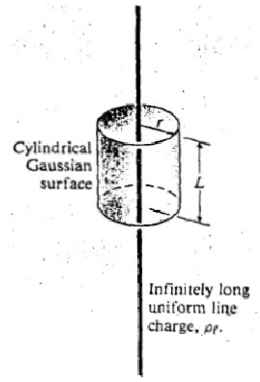
Gauss's law follows directly from (3-3) and is given by

$$\boxed{\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{Q}{\varepsilon_0}} \tag{3-6}$$

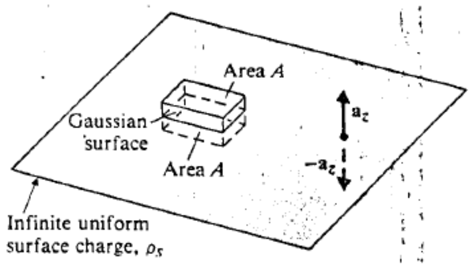
Gauss's law asserts that *the total outward flux of the* **E**-*field over any closed surface in free space is equal to the total charge enclosed in the surface divided by* ε_0. The surface S can be hypothetical closed surface chosen for convenience, not necessarily be a physical surface.

Gauss's law is useful in determining **E** when *the normal component of the electric field intensity is constant over an enclosed surface*. The first step to apply Gauss's law is to choose such surface, referred to as a ***Gaussian surface***, and then evaluate both sides of (3-6) in order to determine **E**.

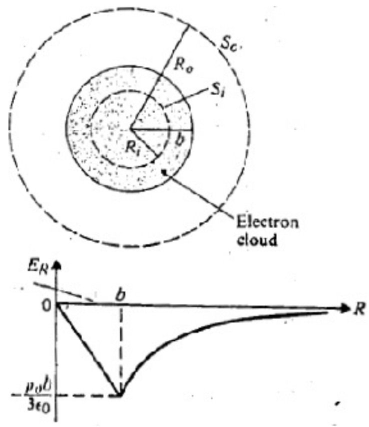
Example 3-4 Use Gauss's law for Example 3-3



Example 3-5 Determine the electric field intensity due to an infinite planar charge with a uniform surface charge density ρ_s.



Example 3-6 Determine the **E** field due to a spherical cloud of electrons with a volume charge density ρ_0 inside and 0 outside.



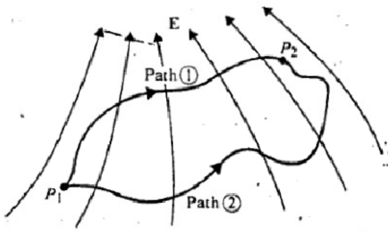
3-5 Electric Potential

Since $\nabla\times(\nabla V)\equiv\mathbf{0}$ and $\nabla\times\mathbf{E}=\mathbf{0}$ in electrostatics, one can define a scalar ***electric potential*** *V* from (3-4) such that

$$\mathbf{E}=-\nabla V \qquad (3\text{-}26)$$

Electric potential is related to the work in carrying a charge from one point to another. Since the electric field intensity is the force acting on a unit test charge, the work required to move a unit charge from point P_1 to P_2 is given by

$$\frac{W}{q}=-\int_{P_1}^{P_2}\mathbf{E}\cdot d\boldsymbol{\ell} \quad \text{(J/C or V)} \qquad (3\text{-}27)$$



Since the static electric field is "conservative", the line integral on the right does not depend on the integration path, for instance integrations along path 1 and path 2 give the same result.

Analogous to the concept of potential energy in mechanics, (3-27) represents the difference in electric potential energy of a unit charge between point P_2 and point P_1. Let *V* denote the electric potential energy per unit charge, *the electric potential*, then

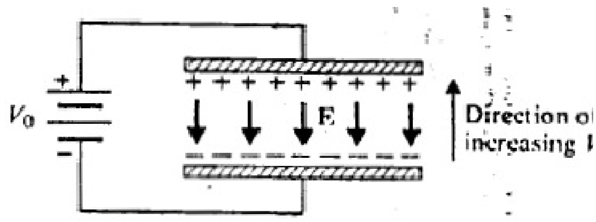
$$V_2-V_1=-\int_{P_1}^{P_2}\mathbf{E}\cdot d\boldsymbol{\ell} \quad \text{(V)} \qquad (3\text{-}28)$$

since $-\int_{P_1}^{P_2}\mathbf{E}\cdot d\boldsymbol{\ell}=\int_{P_1}^{P_2}\nabla V\cdot\hat{\mathbf{a}}_\ell d\ell=\int_{P_1}^{P_2}dV=V_2-V_1$. Thus, a ***potential difference*** (***electrostatic voltage***) is equivalent to the electric potential energy per unit charge. Note that point P_1 here is the reference *zero-potential* point. In most cases, the reference point is taken at infinity; this convention normally applies when the reference point is not specified explicitly.

Observations regarding electric potential

1. Because of the negative sign, the direction of **E** is opposite to the direction of increasing *V*.

2. The direction of ∇V is normal to surfaces of constant V, thus **E** is perpendicular to *equipotential lines* or *equipotential surfaces*.



3-5.1 Electric Potential due to a charge distribution

Let infinity be the reference point, then the electric potential of a point at a distance R from a point charge q is given by

$$V = -\int_{\infty}^{R} \hat{\mathbf{r}} \frac{q}{4\pi\varepsilon_0 r^2} \cdot \hat{\mathbf{r}} dr = \frac{q}{4\pi\varepsilon_0 R} \text{ (V)} \qquad (3\text{-}29)$$

The potential difference between 2 points, P_2 and P_1, at distances R_2 and R_1, respectively, is given by

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \qquad (3\text{-}30)$$

The electric potential due to a system of n discrete charges, $q_1, \ldots, q_n$, is given by

$$V = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\mathbf{r} - \mathbf{r}'_k|} \text{ (V)} \qquad (3\text{-}31)$$

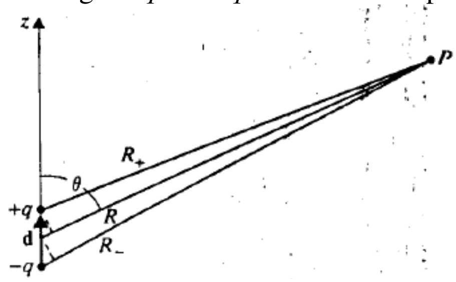
For continuous charge distributions in confined regions, electric potentials are given by

$$V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho_v dv'}{R} \text{ (V/m) (volume charge)} \qquad (3\text{-}38)$$

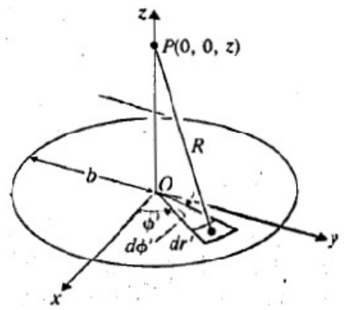
$$V = \frac{1}{4\pi\varepsilon_0} \int_{S'} \frac{\rho_s ds'}{R} \text{ (V) (surface charge)} \qquad (3\text{-}39)$$

$$V = \frac{1}{4\pi\varepsilon_0} \int_{L'} \frac{\rho_\ell d\ell'}{R} \text{ (V) (line charge)} \qquad (3\text{-}40)$$

Example 3-7 [Electric dipole moment] Electric potential due to an electric dipole consisting of charges $+q$ and $-q$ with a small separation of d (assume $R >> d$)



Example 3-8 Obtain a formula for the electric field intensity on the axis of a circular disk of radius b that carries a uniform surface charge density ρ_s.



3-6 Material Media in Static Electric Field

Consider energy band theory of solids based on solid state physics as shown in Fig. 2, electrical materials can be classified into 3 types, namely, conductors, dielectrics[2] (or insulators), and semiconductors.

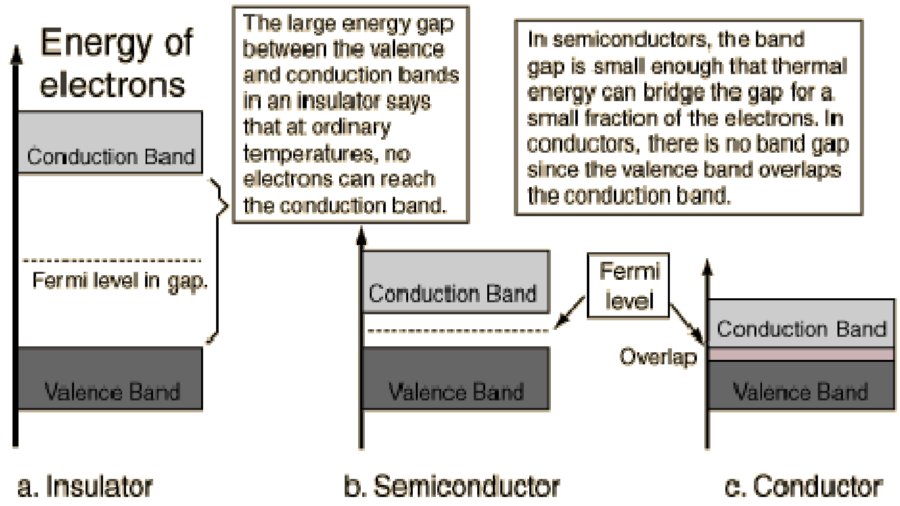


Figure 2: Energy band structure

<u>3-6.1 Conductors in Static Electric Field</u>

Assume that some electric charges are introduced in the interior of a good conductor. An electric field will be set up and create a force that causes the movement of charges. This movement will continue until *all* charges reach the conductor surface and redistribute in such a way that both the charge and the field inside vanish. Hence,

$\rho_v = 0; \mathbf{E} = \mathbf{0}$

When there are no *free* charges in the interior of a conductor ($\rho_v = 0$), $\mathbf{E}$ must be zero according to Gauss's law. Furthermore, *under static conditions the* $\mathbf{E}$ *field on a conductor surface is everywhere <u>normal</u> to the surface*, otherwise there exists a tangential force that moves the charges.

Consider the boundary conditions at the interface between a conductor and free space as shown in Fig. 3.

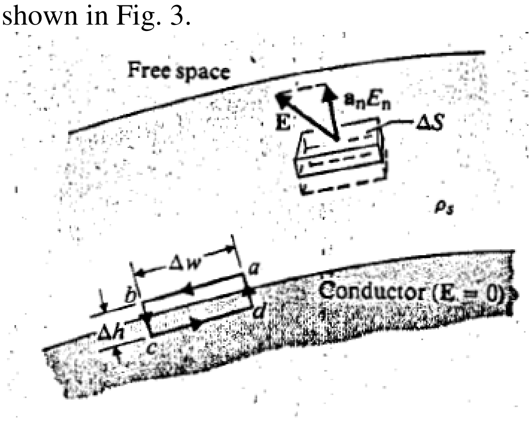


Figure 3: A conductor-free space interface

Integrating $\mathbf{E}$ along the contour *abcda* and taking the limit as $\Delta h \to 0$ yield

$\lim_{\Delta h \to 0} \oint_{abcda} \mathbf{E} \cdot d\boldsymbol{\ell} = E_t \Delta w = 0$ or

$E_t = 0$

Which says that *the tangential component of the* $\mathbf{E}$ *field on a conductor surface is zero under static conditions*. In other words, the surface of a conductor is an ***equipotential surface***.

Next, integrating $\mathbf{E}$ on the Gaussian surface in the figure and taking the limit as $\Delta h \to 0$:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = E_n \Delta S = \frac{\rho_s}{\varepsilon_0} \Delta S \qquad \text{or} \qquad E_n = \frac{\rho_s}{\varepsilon_0}$$

[2] A **dielectric** is a <u>nonconducting</u> substance, i.e. an <u>insulator</u>. The term was coined by <u>William Whewell</u> in response to a request from <u>Michael Faraday</u>. Although "dielectric" and "insulator" are generally considered synonymous, the term "dielectric" is more often used to describe materials where the <u>dielectric polarization</u> is important, such as the insulating material between the metallic plates of a <u>capacitor</u>, while "insulator" is more often used when the material is being used to prevent a current flow across it.

Example 3-9 A positive point charge Q is at the center of a spherical conducting shell of an inner radius R_i and an outer radius R_o. Determine **E** and V as functions of the radial distance r.

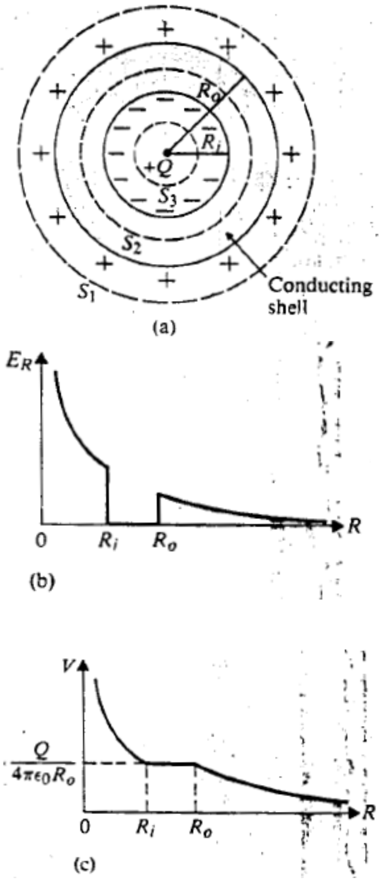


Figure 3: Example 3-9

3-6.2 Dielectrics in Static Electric Field

All material media are composed of atoms with a positively charged nucleus surrounded by negatively charged electrons. In the absence of an external electric field, the molecules of dielectrics are macroscopically neutral. The presence of an electric field causes a force on each charged particle and results in small displacements of positive and negative charges in opposite directions. These are ***bound charges***. The displacements *polarize* a dielectric material and create electric dipoles (i.e., *polarization*). The molecules of some dielectrics possess permanent dipole moments, even in the absence of an external electric field. Such molecules are called ***polar molecules***, in contrast to ***nonpolar molecules***. An example is the water molecule H_2O. Generally, dielectric materials consist of both polar and nonpolar molecules (Fig. 5).

When there is no external field, dipoles in polar dielectrics are randomly oriented (Fig. 6 (a)), producing no net dipole moment *macroscopically*. An applied electric field will tend to align the dipoles with the field as shown in Fig. 6 (b). producing the nonzero net dipole moment (Fig. 7).

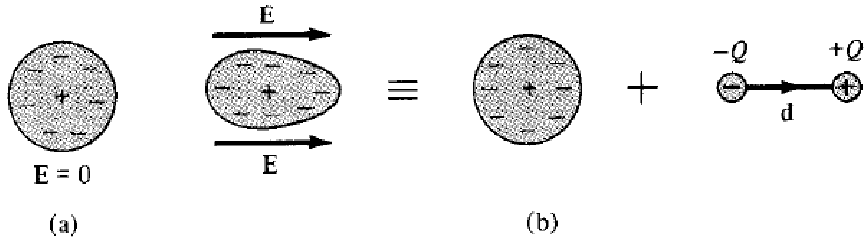
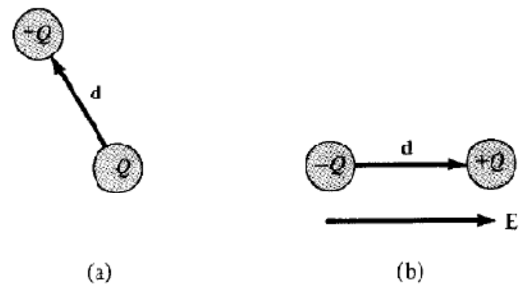


Fig. 5: Molecules in dielectrics



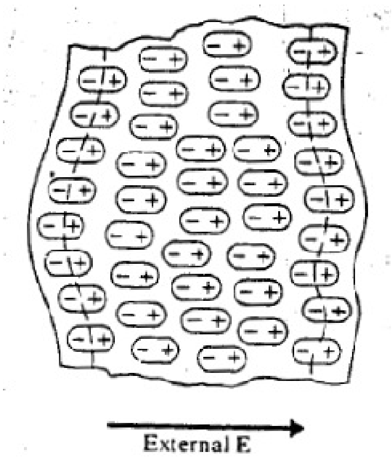


Figure 6: Polar molecule

Figure 7: Interior of a dielectric medium

A polarization vector **P** is defined as

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N\Delta v} \mathbf{p}_k}{\Delta v} \quad (\text{C/m}^2)$$

where N is the number of molecules per unit volume and the numerator represents the vectopr sum of the induced dipole moments contained in a very small volume Δv. The vector **P** is the *volume density of electric dipole moment*. The dipole moment d**p** produces an electric potential

$$dV = \frac{\mathbf{P} \cdot \hat{\mathbf{r}}}{4\pi\varepsilon_0 R^2} dv'$$

Thus, the potential due to the polarized dielectric is given by

$$V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\mathbf{P} \cdot \hat{\mathbf{r}}}{R^2} dv'$$

Interpretation of the effects of the induced electric dipoles:

1. <u>Equivalent polarization surface charge density</u>

$$\rho_{ps} = \mathbf{P} \cdot \hat{\mathbf{a}}_n \qquad (\text{C/m}^2)$$

2. <u>Equivalent polarization volume charge density</u> Since

$$Q = -\oint_S \mathbf{P} \cdot \hat{\mathbf{a}}_n ds = \int_V (-\nabla \cdot \mathbf{P}) dv = \int_V \rho_{pv} dv,$$

Thus, one can define the polarization volume charge density as

$$\rho_{pv} = -\nabla \cdot \mathbf{P} \quad (\text{C/m}^3)$$

It follows that

$$\text{total charge} = \oint_S \rho_{ps} ds + \int_V \rho_{pv} dv = \oint_S \mathbf{P} \cdot \hat{\mathbf{a}}_n ds - \int_V (\nabla \cdot \mathbf{P}) dv = 0,$$

i.e., the total "free" charge of the dielectric body after polarization must remain zero.

<u>Example 3-10</u> The polarization vector in a dielectric sphere of radius R_0 is $\mathbf{P} = \hat{\mathbf{x}} P_0$. Determine

a) the equivalent polarization surface and volume charge densities and
b) the total equivalent charge on the surface and inside of the sphere

3-7 Electric Flux Density and Dielectric Constant

In dielectrics,

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_v + \rho_{pv}\right),$$

but since $\rho_{pv} = -\nabla \cdot \mathbf{P}$, $\nabla \cdot \left(\varepsilon_0 \mathbf{E} + \mathbf{P}\right) = \rho_v$.

Here, one can define a new fundamental field quantity, Electric Flux Density (or electric displacement) **D** to be

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (\text{C/m}^2)$$

It follows that

$$\nabla \cdot \mathbf{D} = \rho_v$$

Applying the divergence theorem yields

$$\int_V \nabla \cdot \mathbf{D} dv = \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho_v dv = Q .$$

Hence, $\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ (C)

In *linear* and *isotropic* media, **P** can be given in terms of **E** as

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

where χ_e is called *electric susceptibility* (dimensionless). Here, **D** can be rewritten as

$$\mathbf{D} = \varepsilon_0 \left(1 + \chi_e\right)\mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E} \qquad (\text{C/m}^2)$$

where

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}$$

is called *relative permittivity* or *dielectric constant* (dimensionless).

In general, dielectric materials can be classified based on the property of dielectric constants into

Linear : dielectric constant doesn't change with applied electric field ↔ *non-linear*

Isotropic: dielectric constant doesn't change with direction ↔ *anisotropic*

Homogeneous: dielectric constant doesn't change from point to point ↔ *inhomogeneous*

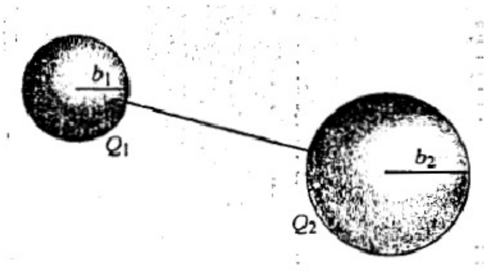
<u>3-7.1 Dielectric Strength</u>

If the electric field is very strong, it will pull electrons completely out of molecules. The electrons will accelerate under the influence of the electric field, collide violently with the molecular structure and avalanche effect of ionization due to collisions may occur. The material will become *conducting* and large currents may result; this phenomenon is called a ***dielectric breakdown***. The maximum electric field intensity that a dielectric material can withstand without breakdown is the ***dielectric strength***. For instance, the dielectric strength of air at the atmospheric pressure is 3 (kV/mm).

<u>Example 3-11</u> Consider two spherical conductors with radii b_1 and b_2 ($b_1 > b_2$) that are connected by a conducting wire. The distance of separation between the conductors is assumed to be very large in comparison to b_2 so that the charges on the spherical conductors may be considered as uniformly distributed.

a) the charges on the two spheres, and

b) the electric field intensities at the sphere surfaces.



Example 3-9* A positive point charge Q is at the center of a spherical dielectric shell, with a dielectric constant of ε_r and (inner, outer) radii, (R_i, R_o), respectively. Determine **E**, V, **D**, **P** as functions of the radial distance r.

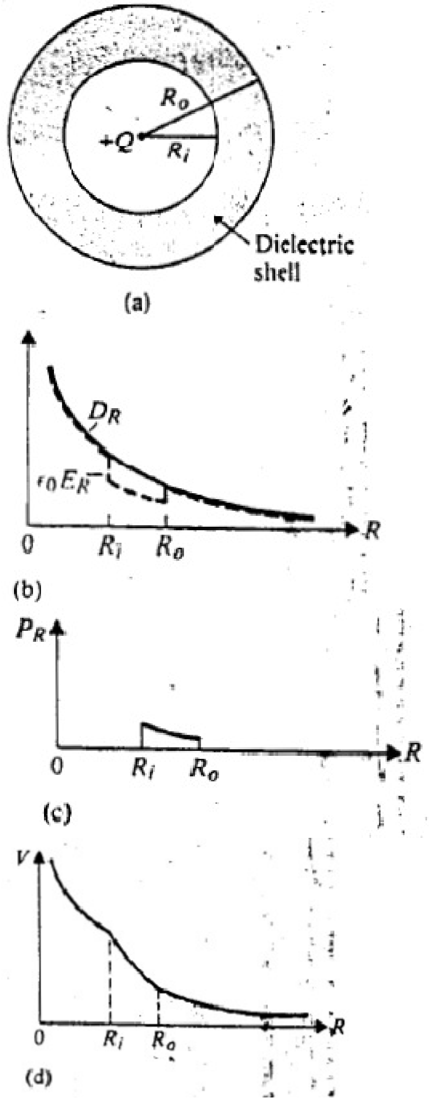


Figure 8 : Example 3-9*

3-8 Boundary Conditions for Electrostatic Fields

Consider the boundary conditions at the interface between two dielectric media as shown in Fig. 9.

Integrating **E** along the contour *abcda* and taking the limit as $\Delta h \to 0$ yield

$$\lim_{\Delta h \to 0} \oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = E_1 \Delta \mathbf{w} + E_2(-\Delta \mathbf{w}) = 0 \quad \text{or } E_{1t} = E_{2t}$$

which says that *the tangential component of the* **E** *field is continuous across the interface.* If ε_1, ε_2 denote the permittivities of media 1, 2, respectively, then

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Next, integrating **E** on the Gaussian surface in the figure and taking the limit as $\Delta h \to 0$:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \left(\mathbf{D}_1 \cdot \hat{\mathbf{a}}_{n2} + \mathbf{D}_2 \cdot \hat{\mathbf{a}}_{n1}\right)\Delta S$$

$$= \hat{\mathbf{a}}_{n2} \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right)\Delta S = \rho_s \Delta S$$

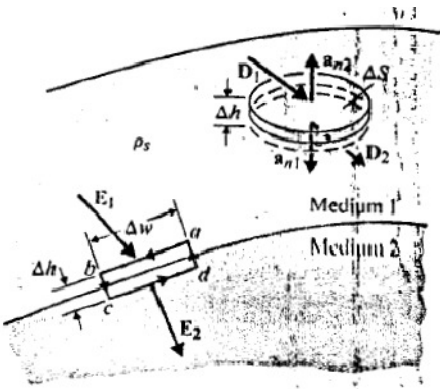


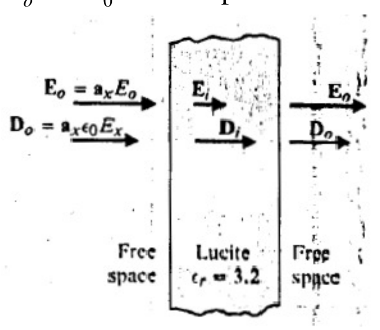
Figure 9 : An interface between two media

Thus,

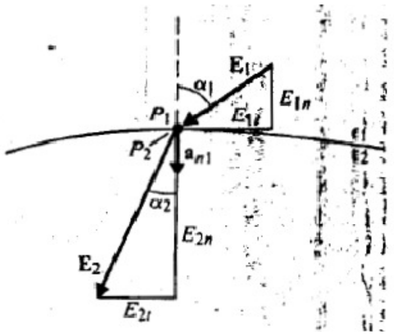
$$\hat{\mathbf{a}}_{n2} \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right) = \rho_s \quad \text{or} \quad D_{1n} - D_{2n} = \rho_s \quad (\text{C/m}^2)$$

where ρ_s denotes the surface charge density on the interface.

<u>Example 3-13</u> A lucite sheet (ε_r=3.2) is introduced perpendicularly in a uniform electric field $\mathbf{E}_o = \hat{\mathbf{x}}E_0$ in free space. Determine $\mathbf{E_i}$,$\mathbf{D_i}$,$\mathbf{P_i}$ inside the lucite.



<u>Example 3-14</u> Two dielectric media with permittivities ε_1 and ε_2 are separated by a charge free boundary. The electric field intensity in medium 1 at the point P_1 has a magnitude E_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of **E** at point P_2 in medium 2.



3-9 CAPACITANCES AND CAPACITORS

It is known from 3-6 that a conductor in a static electric field is an *equipotential* body and that charges on a conductor will distribute themselves in such a way that the electric field inside vanishes. Suppose the potential due to a charge Q is V, then increasing the total charge by a factor k only increases the surface charge density without changing the charge distribution. It is also noted that increasing Q also leads to increasing **E** and thus V also increases. Reciprocally, increasing V by a factor of k leads to increase in Q.

$$V = -\int \mathbf{E}\cdot d\ell = -\int \frac{\rho_s}{\varepsilon_0} dn \text{ and } kV = -\int k\mathbf{E}\cdot d\ell = -\int \frac{k\rho_s}{\varepsilon_0} dn,$$

Thus, one can conclude that the Q/V ratio remains unchanged. This ratio is called the **capacitance** of the isolated conducting body, which has the unit **Farad (F)**, or C/V. Using C, one can write

$$\boxed{Q = CV}$$

Of considerable importance in practice is the **Capacitor (or Condenser)** as shown in Fig. 10.

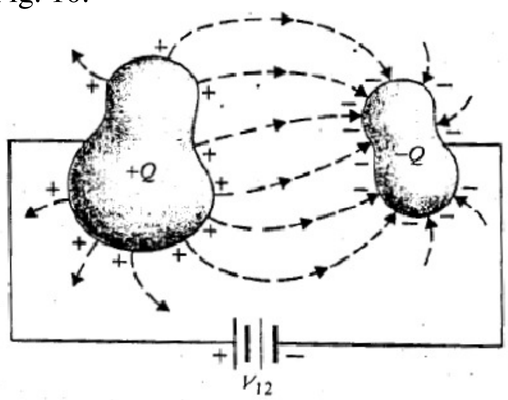


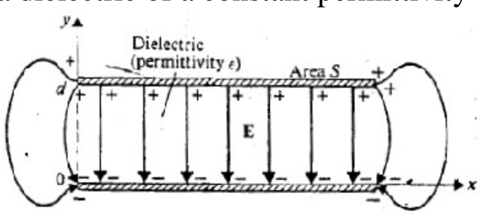
Figure 10 : A two-conductor capacitor

Here, the capacitor consists of two conductors separated by free space or a dielectric medium. When a dc voltage source is applied between conductors, a charge transfer occurs, resulting in $+Q$ on one conductor and $-Q$ on the other. Note that the field lines are perpendicular to the conductor surfaces. Let V_{12} be the potential difference between two conductors, then the capacitance C is given by

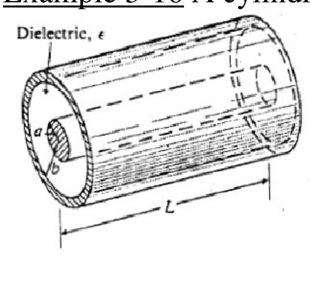
$$C = \frac{Q}{V_{12}} \text{ (F)}$$

The capacitance of a capacitor depends on the geometry and the permittivity of the medium.

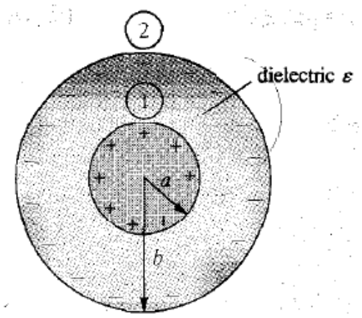
<u>Example 3-15</u> A parallel-plate capacitor consists of two parallel conducting plates of areas S separated by a uniform distance d. The space between the plates is filled with a dielectric of a constant permittivity ε. Determine the capacitance.



<u>Example 3-16</u> A cylindrical capacitor



Example 3-16* A spherical capacitor



3-10 ELECTROSTATIC ENERGY AND FORCES

Since electric potential at a point in an electric field is the work required to bring a unit charge from infinity (the reference point) to that point, to bring a charge Q_2 from infinity *against* the field of a charge Q_1 in free space to a distance R_{12} requires the work of amount

$$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\varepsilon_0 R_{12}} = Q_1 \frac{Q_2}{4\pi\varepsilon_0 R_{12}} = Q_1 V_1,$$

which is path-independent. The work is stored in the assembly of two charges as potential energy,

$$W_2 = \frac{1}{2}\left(Q_1 V_1 + Q_2 V_2\right)$$

Here, if a charge Q_3 is brought from infinity to a point that is R_{13} from Q_1 and R_{23} from Q_2, then an additional amount of work is required that equals

$$\Delta W = Q_3 V_3 = Q_3 \left(\frac{Q_1}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}} \right)$$

The potential energy stored in 3 charges is given by

$$\begin{aligned} W_3 = W_2 + \Delta W &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1 Q_2}{R_{12}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_2 Q_3}{R_{23}} \right) \\ &= \frac{1}{2}\left[Q_1\left(\frac{Q_2}{4\pi\varepsilon_0 R_{12}} + \frac{Q_3}{4\pi\varepsilon_0 R_{13}} \right) + Q_2\left(\frac{Q_1}{4\pi\varepsilon_0 R_{12}} + \frac{Q_3}{4\pi\varepsilon_0 R_{23}} \right) + Q_3\left(\frac{Q_1}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}} \right) \right] \\ &= \frac{1}{2}\left(Q_1 V_1 + Q_2 V_2 + Q_3 V_3\right) \end{aligned}$$

Note that V_1, the potential at the position of Q_1, is caused by charges Q_2, Q_3, and it is different from the V_1 in the two-charge case. Using the same procedure, the potential energy (electrostatic energy) of a group of N discrete point charges can be given by

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \qquad \text{(J)}$$

Likewise, the potential energy due to continuous charges can be given by

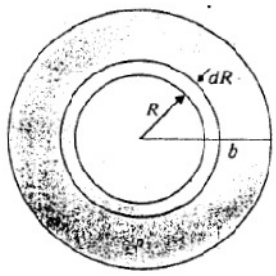
$$W_e = \frac{1}{2}\int_{V'} \rho_v V dv' \qquad \text{(J)}$$

Since the SI unit for energy, Joule (J), is too large, a more convenient unit, *electron-volt* (eV), which is the energy or work required to move an electron against a potential difference of one volt, i.e.,

$1\,(\mathrm{eV}) = 1.60\times10^{-19}\,(\mathrm{J})$

is used instead.

<u>Example 3-17</u> Find the energy required to assemble a uniform sphere of charge of radius *b* and volume charge density ρ_v.



3-10.1 Electrostatic Energy in Terms of Field Quantities

Since $\nabla\cdot\mathbf{D} = \rho_v$, $W_e = \frac{1}{2}\int_{V'}(\nabla\cdot\mathbf{D})V dv'$,

using $\nabla\cdot(V\mathbf{D}) = V\nabla\cdot\mathbf{D} + \mathbf{D}\nabla V$ yields

$$W_e = \frac{1}{2}\int_{V'}\nabla\cdot(V\mathbf{D})dv' - \frac{1}{2}\int_{V'}\mathbf{D}\nabla V dv' = \frac{1}{2}\oint_{S'}V\mathbf{D}\cdot d\mathbf{s}' + \frac{1}{2}\int_{V'}\mathbf{D}\cdot\mathbf{E}dv'.$$

Since at least **D** is proportional to $1/r^2$ and *V* is proportional to $1/r$, let *V'* be the sphere of radius *r*, and taking $r\to\infty$, the first term of the right hand side vanishes. Hence,

$$W_e = \frac{1}{2}\int_{V'}\mathbf{D}\cdot\mathbf{E}dv' \qquad (\mathrm{J})$$

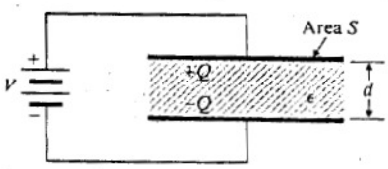
For a *linear, isotropic medium*,

$$W_e = \frac{1}{2}\int_{V'}\varepsilon E^2 dv' \qquad (\mathrm{J})$$

Here, one can define ***electrostatic energy density*** w_e as

$$w_e = \varepsilon E^2\ (\mathrm{J/m^3}); W_e = \frac{1}{2}\int_{V'}w_e dv'$$

<u>Example 3-18</u> A parallel-plate capacitor



<u>Example 3-19</u> A cylindrical capacitor (Figure 3)

3-10.2 Electrostatic Forces

Recall that Coulomb's law governs the force between two point charges, but it might be hard to determine the force using Coulomb's law in a more complex system of charged bodies. In such cases, the following principle is useful.

Principle of Virtual Displacement : calculate the force on an object in a charged system from the electrostatic energy of the system

Consider an *isolated system* of charged conducting, as well as dielectric, bodies separated from one another with no connection to the outside world. The mechanical work done by *the system* to displace one of the bodies by a differential distance *dl* (***a virtual displacement***) is given by

$$dW = \mathbf{F}_Q \cdot d\mathbf{l} \tag{3-112}$$

where $\mathbf{F}_Q$ denotes the total electric force. Since it is an isolated system with no external supply of energy, the mechanical work must be done at the expense of the stored electrostatic energy, i.e.,

$$dW = -dW_e = \mathbf{F}_Q \cdot d\ell\ . \tag{3-113}$$

Writing the force in terms of the gradient of the work, one can write

$$dW_e = (\nabla W_e) \cdot d\ell \tag{3-114}$$

Since *dl* is arbitrary, comparison of (3-113) and (3-114) yields

$$\mathbf{F}_Q = -\nabla W_e \qquad \text{(N)}$$

Example 3-20 the force on conducting plates of a parallel-plate capacitor

3-11 SOLUTION OF ELECTROSTATIC BOUNDARY-VALUE PROBLEMS

So far, techniques for determining **E, D,** *V, etc* for a given charge distribution have been discussed. In many practical problems, the charge distribution is not known everywhere. In such cases, differential equations that govern the electric potential in an electrostatic situation are formulated, and the boundary conditions are applied to obtain what are called ***boundary-value problems*****.**

3-11.1 Poisson's and Laplace's Equations

In Electrostatics,

$$\nabla \cdot \mathbf{D} = \rho_v; \nabla \times \mathbf{E} = \mathbf{0}; \mathbf{E} = -\nabla V$$

In *linear, isotropic* medium, since $\mathbf{D} = \varepsilon\mathbf{E}$,

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\varepsilon\mathbf{E}) = \nabla \cdot (-\varepsilon\nabla V) = \rho_v$$

Hence, one obtains ***Poisson's equation*** and ***Laplace's equation*** (ρ_v =0 case) as follows:

$$\boxed{(Poisson's): \nabla^2 V = -\frac{\rho_v}{\varepsilon}}\ ;\ \boxed{(Laplace's): \nabla^2 V = 0}$$

where ∇^2(del square) is called Laplacian operator.

3-11.1* Uniqueness Theorem

Uniqueness theorem asserts that a solution of an electrostatic problem satisfying its boundary conditions (Poisson's equation or Laplace's equation) is *the only possible solution*, irrespective of the method by which the solution is obtained.

Proof Suppose a volume τ is bounded outside by a surface S_o which may be a surface at infinity. Inside the closed surface S_o there are a number of charged conducting bodies with surfaces S_1, S_2, …, S_n at specified potentials, as depicted in Fig. 11.

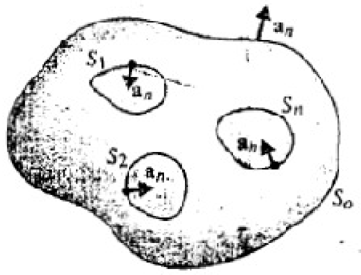


Figure 11 proof of Uniqueness theorem

Now assume that, contrary to the uniqueness theorem, there are two solutions, V_1 and V_2, to Poisson's equation in τ:

$$\nabla^2 V_1 = -\frac{\rho_v}{\varepsilon}; \nabla^2 V_2 = -\frac{\rho_v}{\varepsilon}$$

Also assume that both V_1 and V_2 satisfy the *same* boundary conditions on S_1, …, S_n and S_o. Let $V_d = V_1 - V_2$, then

$$\nabla^2 V_d = 0\big|_{\text{in } \tau}; V_d = 0\big|_{\text{in } S_1,\ldots,S_n}$$

Using $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A}\nabla f$ and letting $f = V_d$, $\mathbf{A} = \nabla V_d$ yields

$$\nabla \cdot (V_d \nabla V_d) = V_d \nabla \cdot (\nabla V_d) + (\nabla V_d) \cdot (\nabla V_d) = V_d \nabla^2 V_d + |\nabla V_d|^2 = |\nabla V_d|^2$$

Integrating both sides of the equation above yields

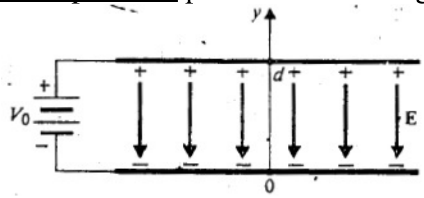
$$\int_\tau \nabla \cdot (V_d \nabla V_d) dv = \oint_S (V_d \nabla V_d) \cdot d\mathbf{s} = \int_\tau |\nabla V_d|^2 dv$$

Since $V_d = 0$ on S_1, S_2, …, S_n and on S_0

$r \to \infty; V_d \propto 1/r; \nabla V_d \propto 1/r^2; ds \propto r^2$.

Thus, the integral on the left hand side vanishes. Since $|\nabla V_d|^2$ is nonnegative, $|\nabla V_d|$ must be identically 0, which means V_d has the same value at all points in τ as it has on the bounding surfaces, S_1,…,S_n, where V_d=0. Thus, V_d=0 everywhere, and therefore $V_1 = V_2$, i.e., only one solution exists.

<u>Example 3-21</u> parallel conducting plates separated by d with $\rho_v = -\rho_0 y/d$



<u>Example 3-22</u> Two infinite insulated conducting plates maintained at potentials 0 and V_0 (Figure 12) Find the potential distribution for $0 < \phi < \alpha$ and $\alpha < \phi < 2\pi$.

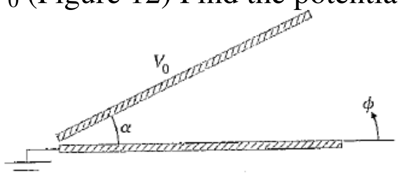
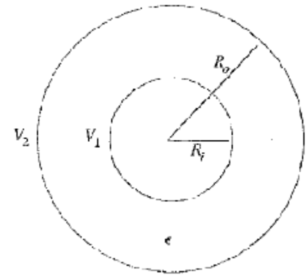


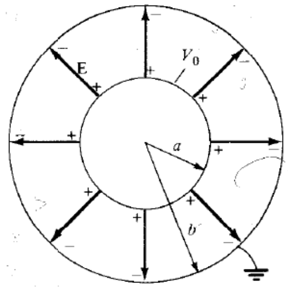
Figure 12 Example 3-22

<u>Example 3-23</u> Given the inner and outer radii of two concentric, thin, conducting, spherical shells (R_i, R_o),



Respectively, and the space between the shells is filled with a dielectric. Determine the potential distribution in the dielectric material by solving Laplace's equation.

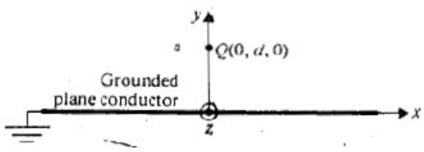
Example 3-16** A spherical capacitor



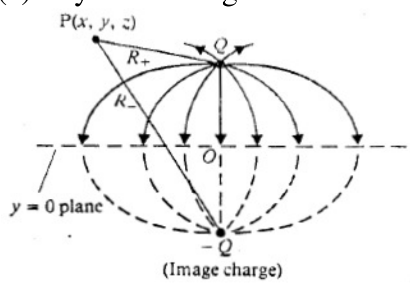
3-11.5 Method of Images

The method of images is the technique applied to boundary value problems by replacing boundary surfaces with appropriate ***image charges***, instead of attempting to solve a Poisson's or Laplace's equation.

Example 3-24 Point Charges Near Conducting Planes as shown in Fig. 13 (a)



(a) Physical arrangement



(b) Image charge and field lines

Fig. 13

A formal procedure would require the solution of Poisson's equation in the $y > 0$ region with boundary conditions $V = 0$ at $y = 0$ and at infinity. Here, if an *appropriate* image charge can be used to replace the conducting plane such that all boundary conditions are satisfied, then the solution would be obtained in a straightforward manner. Suppose one replaces the conductor with the charge $-Q$ at $(0,-d,0)$, then the potential at a point P(x, y, z) is given by

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

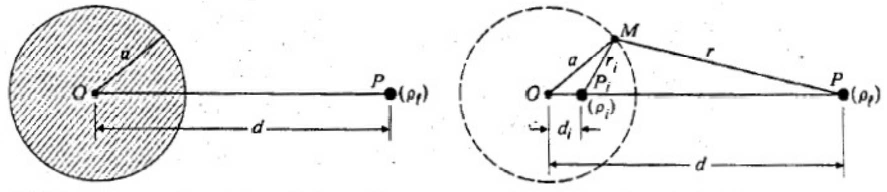
$$= \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\left[\dfrac{1}{\sqrt{x^2+(y-d)^2+z^2}} - \dfrac{1}{\sqrt{x^2+(y+d)^2+z^2}}\right] & y \geq 0 \\ 0 & y < 0 \end{cases}.$$

Note that the condition $V=0$ at $y=0$ is satisfied. Then, $\mathbf{E}$ for $y \geq 0$ is given by

$$\mathbf{E} = -\nabla V = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}z}{\left[x^2+(y-d)^2+z^2\right]^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}z}{\left[x^2+(y+d)^2+z^2\right]^{3/2}}\right\}$$

Hence, the surface charge density becomes $\rho_s = \varepsilon_0 E_y\big|_{y=0} = -\dfrac{Qd}{2\pi\left(x^2+d^2+z^2\right)^{3/2}}$.

Line charge near a parallel conducting cylinder



(a) Line charge and parallel conducting cylinder. (b) Line charge and its image.

Consider the problem of a line charge ρ_ℓ located at a distance d from the axis of a parallel, conducting, circular cylinder of radius a. Both are assumed to be infinitely long. Fig. (a) shows a cross section of this arrangement. To apply the method of images, first observe that (1) The image must be a parallel line charge inside the cylinder in order to make the cylindrical surface at $r = a$ an *equipotential surface*. Let call this image line charge ρ_i (2) Because of the symmetry with respect to the line OP, the image line charge must lie somewhere along OP, say at a point P_i, which is a distance d_i from the axis (Fig. (b)). The unknowns needed to be determined here are ρ_i and d_i. First, let $\rho_i = -\rho_\ell$, then the potential at a distance r from a line charge of density ρ_ℓ is given by

$$V = -\int_{r_0}^{r} E_r dr = -\frac{\rho_\ell}{2\pi\varepsilon_0}\int_{r_0}^{r}\frac{1}{r}dr = \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$

Thus, the potential at point M can be found by adding contributions of ρ_ℓ and ρ_i, i.e.,

$$V_M = \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\frac{r_0}{r} - \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\frac{r_0}{r_i} = \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\frac{r_i}{r}$$

In order for an equipotential surface to coincide with the surface $r=a$, r_i/r must be a constant. The point P_i must be located such that ΔOMP_i is similar to ΔOPM, i.e., $\angle OMP_i = \angle OPM$. Hence,

$$\frac{\overline{P_iM}}{\overline{PM}} = \frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}};\text{or}\ \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{constant}$$

Therefore, $d_i=a^2/d$. The point P_i is called ***inverse point*** of P with respect to a circle of radius a.

<u>Example 3-25</u> Capacitance per unit length between two long parallel circular conducting wires of radius a