LE 200 Homework

Please show all details of your solutions.

1-1. Explain the "physical" meaning of the following quantities:

a) permittivity b) permeability
c) conductivity d) gradient
e) divergence f) curl
g) vector space h) electric field intensity
i) magnetic field intensity j) electric flux density
k) magnetic flux density l) electric potential

1-2. Use the cross product to prove the law of sines:

$$\frac{\sin\theta_A}{A} = \frac{\sin\theta_B}{B} = \frac{\sin\theta_C}{C}$$

where θ_A, θ_B, θ_C are the angles opposite the three sides with lengths A, B, C, respectively.

1-3. <u>Use vectors to</u> show that if the midpoints of the consecutive sides of any quadrilateral are connected by straight lines, the resulting quadrilateral (PQRS in figure 1) is a parallelogram.

1-4. <u>Use vectors to</u> show that the area of the quadrilateral shown in figure 2 is given by:

$$Area = \frac{1}{2}\left|\overline{\mathbf{P}} \times \overline{\mathbf{Q}}\right|$$

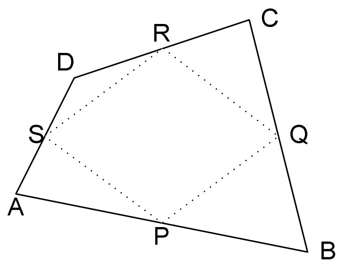


Figure 1

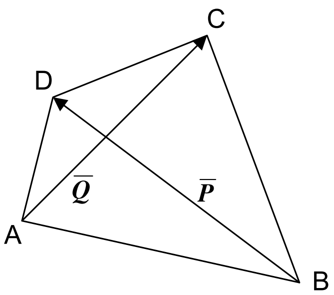


Figure 2

Please show all details of your solutions.

2-1. Given vectors **A** = (4, 0, - 1) , **B** = (1,3, 4), และ **C** = (-5, - 3 , - 3) in Cartesian coordinates,
(a) Prove that vectors **A**, **B**, **C** are on the same plane, then find the equation representing this plane.
(b) Verify the result from (a) by finding the equation of this plane *directly* from the coordinates of three points on the same plane.
(c) Do these three vectors form a triangle? If so, find the area of the triangle.

2-2. If $r = \sqrt{x^2 + y^2 + z^2}$ (distance to a point (x,y,z) in space),
(a) Find ∇r
(b) Find $\nabla \frac{1}{r}$
(c) Show that $\nabla r^n = nr^{n-2}\mathbf{r}$ where $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ (position vector)

2-3. Given a scalar field $\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + z$.
(a) Find the gradient.
(b) Find the equation of the tangential surface at the point (x_0, y_0, z_0).
(c) Given a=2, b=1, ϕ=0, then find two orthogonal vectors of the tangential surface at the point (2,0,-1). What does the ϕ=0 surface look like?

2-4. Verify the divergence theorem for $\mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z^2$ taken over the following regions:
(a) A unit cubic centered at the origin.
(b) The region bounded by $x^2+y^2=4$, z=0 and z=3.

2-5. Given a vector field $\mathbf{F} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2 y$
(a) Find $\nabla \times \mathbf{F}$
(b) Sketch this vector field in the range $|x| \leq 2, |y| \leq 2$
(c) Integrate this vector field from (1,1,0) to (2,2,0) along the following paths:
(i) along y = 1 line to (2,1,0) and along x = 2 line up to (2,2,0)
(ii) along y = x line directly from (1,1,0) to (2,2,0)
(iii) along x = 1 line up to (1,3,0) and go straight down to (2,2,0)

2-6. According to Taylor's theorem, let $f(x)$ be a function that has n+1 continuous derivatives at $x = a$, then Taylor's series of degree n about a is given by

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

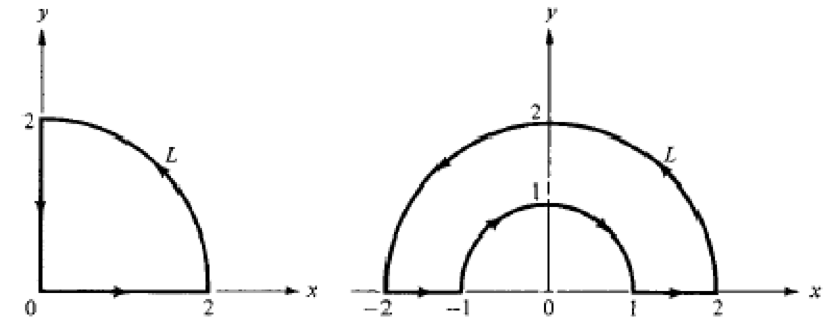
Use this to show that

$$A_x(x_0 + \frac{\Delta x}{2}, y_0, z_0) = A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0, y_0, z_0)} + \text{higher-order terms.}$$

Please show all details of your solutions.

3-1. Verify the Stokes' theorem for $\mathbf{A} = \hat{\mathbf{x}}(y - z + 2) + \hat{\mathbf{y}}(yz + 4) - \hat{\mathbf{z}}xz$ over the following surfaces:
(a) The region bounded by x=0, y=0, x=2, y=2 on x-y plane.
(b) The circle with radius of 2 and centered at (1,2) on x-y plane.

3-2. Verify the Stokes' theorem for $\mathbf{A} = \hat{\boldsymbol{\rho}}\rho \sin\phi + \hat{\boldsymbol{\phi}}\rho^2$ over the following surfaces:



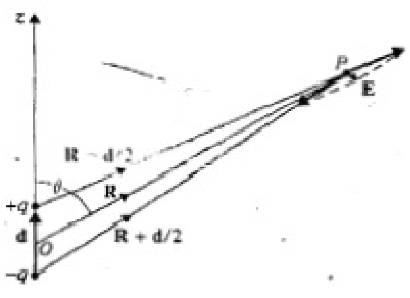
(a) (b)

3-3. Four 10 (nC) positive charges are located in the z = 0 plane at the corner of a square of 8 cm on a side centered at the origin. A fifth 10 (nC) positive charge is located at a point 8 cm distant from all other charges. Find the magnitude and the direction of the total force on this fifth charge for $\varepsilon = \varepsilon_0$.

3-4. Eight identical point charges of q (C) each are located at the corners of a cube of side length a, with one charge at the origin, and with the three nearest charges at (a,0,0), (0,a,0), (0,0,a). Find an expression for the total vector force on the charge at P(a,a,a), assuming free space.

3-5. Given an *electric dipole* consisting of a pair of equal and opposite charges +q and -q, separated by a small distance d, as shown in the figure below. Assuming free space,
(a) Find **E** at P using the formula from Coulomb's law.
(b) Show that the result in (a) when $d \ll R$ where $R = |\mathbf{R}|$ can be simplified to

$$\mathbf{E} \cong \frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right] = \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right] \text{ where } \mathbf{p} = q\mathbf{d}.$$



Please show all details of your solutions.

4-1. Given a circular line charge of radius a with uniform line charge density of ρ_ℓ (C/m). It is located on the x-y plane with its center at the origin.

(a) Find **E** at the point (0,0,h), $h > 0$.
(b) Find the scalar potential V at the point (0,0,h).
(c) Find **E** from V in (b) and verify that they are the same.

4-2. Given a finite straight line charge of length L with uniform line charge density of ρ_ℓ (C/m). It is located along the z axis with its center at the origin.

(a) Find the scalar potential V at a point (x, y, 0), i.e., a point on the x-y plane.
(b) Find **E** at a point (x, y, 0) using Coulomb's law.
(c) Find **E** from V obtained in (a).
(d) Show that if L goes to infinity, **E** approaches that of an infinitely long straight line charge.

4-3. A uniform line charge of density ρ_ℓ (C/m) is arranged in the form of a square with side a and lies in the x-y plane with its center at the origin.

(a) Find the scalar potential V at the point (0,0,h).
(b) Find **E** at the point (0, 0, h) using Coulomb's law.
(c) Find **E** from V obtained in (a).

4-4. A spherical distribution of charge $\rho_v = \rho_0[1-(r^2/b^2)]$ exists in the region $0 \le r \le b$. This charge distribution is concentrically surrounded by a conducting shell with inner radius r_i (>b) and outer radius r_0. Use Gauss's law to determine **E** everywhere.

4-5. In spherical coordinates, $\mathbf{E} = \hat{\mathbf{r}}\dfrac{2r}{\left(r^2+a^2\right)^2}$ (V/m).

(a) Find the potential at any point, using the following references:

(i) $V = 0$ at infinity
(ii) $V = 0$ at $r = 0$
(iii) $V = 0$ at $r = a$

(b) Calculate the work done in moving a 10-nC charge from point A(1, $\pi/6$, $2\pi/3$) to B(4, $\pi/2$, $\pi/3$) using **E**.
(c) Repeat (b) using V. Verify that both give the same result.

You might find the following indefinite integrals useful (especially in problem 2, 3).
Given $R = a + bx + cx^2; \Delta = 4ac - b^2$

1. $\displaystyle\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}}\ln\left(2\sqrt{cR} + 2cx + b\right)$ $\quad [c > 0]$
2. $\displaystyle\int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx+b)}{\Delta\sqrt{R}}$

Please show all details of your solutions.

5-1. The polarization in a dielectric cube of side L centered at the origin is given by $\mathbf{P} = P_0(\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z)$ where P_0 is a constant.

(a) Determine the surface and volume bound-charge densities.

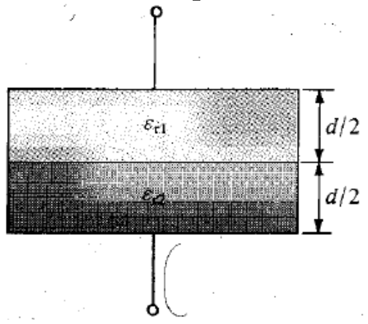
(b) Show that the total bound charge is zero.

5-2. Region 1 ($z \geq 0$) is a dielectric with $\varepsilon_{r1} = 4$, while region 2 ($z < 0$) has $\varepsilon_{r2} = 3$. Let $\mathbf{E}_1 = \hat{\mathbf{x}}5 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3\,(\text{kV/m})$ Find

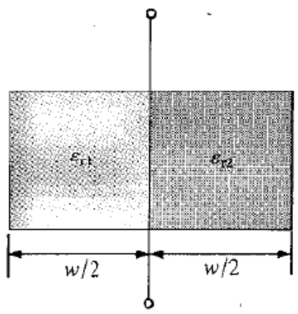
(a) $\mathbf{E}_2$ (b) the angles $\mathbf{E}_1$, $\mathbf{E}_2$ make with the interface (c) Energy density in both regions (d) Energy in the cube of side 2 centered at (3, 4, -5).

5-3. A parallel-plate capacitor is filled with a non-uniform dielectric characterized by $\varepsilon_r = 2 + 2\times10^6 x^2$, where x is the distance from the bottom plate in meters. If $S = 0.02$ m^2 and $d = 1$ mm, find C.

5-4. Find the capacitance of two parallel-plate capacitors shown in figure (a) and (b).



(a)



(b)

5-5. A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. The permittivity of the medium is ε. Find (a) $\mathbf{E}$ (b) $\mathbf{D}$ (c) Q (d) C.

5-2. Repeat problem 6-1 when the permittivity is given by $\varepsilon(\rho) = \varepsilon_0(1 + \rho / a)$.

5-3. Repeat problem 6-2 by starting with Laplace's equation.

5-6. For a positive point charge Q located at a distance d above the infinite conducting ground plane, find

(a) The surface charge density induced on the ground plane.

(b) The total charge induced on the ground plane.

HINT: Use method of images to replace the ground plane with an image charge.

Please show all details of your solutions.

6-1. If $\mathbf{J} = \frac{1}{r^3}\left(\hat{\mathbf{r}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right)$ (A/m^2), calculate the current passing through

(a) A hemispherical shell of radius 20 cm

(b) A spherical shell of radius 10 cm

6-2. A dc voltage of 6 (V) applied to the ends of 1 (km) of a conducting wire of 0.5 (mm) radius results in a current of 1/6 (A). Find

(a) the conductivity of the wire,

(b) the electric field intensity in the wire,

(c) the power dissipated in the wire,

(d) the electron drift velocity, assuming electron mobility in the wire to be 1.4×10^{-3} (m^2/V·s).

6-3. The space between two parallel conducting plates each having an area S is filled with an inhomogeneous Ohmic medium whose conductivity varies linearly from σ_1 at one plate (y=0) to σ_2 at the other plate (y=d). A dc voltage V_0 is applied across the plates as in Fig. 1. Assume that the fringing effects are negligible and the dielectric constant of this medium is 1, determine

(a) the total resistance between the plates,

(b) the surface charge densities on the plates,

(c) the volume charge density and the total amount of charge between the plates.

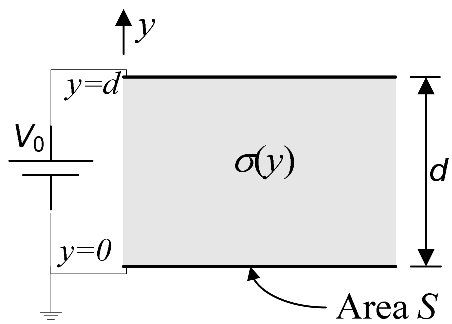


Figure 1: Parallel plate capacitor in problem 3.

Please show all details of your solutions.

7-1. Find the magnetic flux density due to the infinitely long *air-filled* coaxial transmission line. The inner conductor has radius a and carries the current I, while the outer conductor has inner radius b and thickness t and carries the current $-I$. Assume that the current is *uniformly* distributed in both conductors.

7-2. A straight wire of length L carrying a current I is placed along the z axis with its center located at the point $(0,0,h)$, $h>0$.

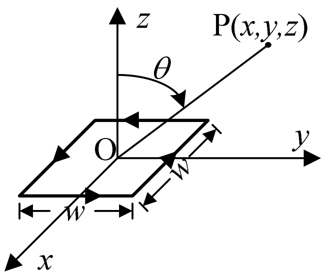
(a) Determine the magnetic flux density at the point $(\rho,0,0)$ in cylindrical coordinates.

(b) Determine the magnetic flux density at the point $(\rho,0,0)$ when $L\to\infty$, an infinitely long line current.

(c) Verify that the result when $h = 0$ is the same as that obtained in example 5-3.

7-3. [Magnetic Dipole] Use the Biot-Savart law to find the magnetic flux density at the point $P(r, \theta, \phi)$ due to a circular current with current I and radius b on the xy-plane and centered at the origin assuming that $r >> b$.

7-4. Given a planar square loop, with side w carrying a direct current I as shown below.



(a) Find the magnetic flux density at the point $(-w/4,0,0)$.

(b) Suppose point $P(r, \theta, \phi)$ (in spherical coordinates) is far away from the loop (i.e., $r>>w$), find the vector potential at the point P and then find the magnetic flux density.

7-5. Given $\mathbf{H}_1 = -\hat{\mathbf{x}}2 + \hat{\mathbf{y}}6 + \hat{\mathbf{z}}4\,(\text{A/m})$ in the region $y - x - 2 \le 0$ where $\mu_1 = 5\mu_0$. Find

(a) $\mathbf{M}_1$ and $\mathbf{B}_1$

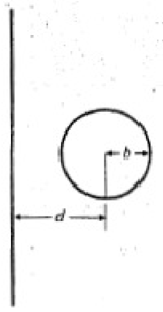
(b) $\mathbf{H}_2$ and $\mathbf{B}_2$ in the region $y - x - 2 \ge 0$ where $\mu_2 = 2\mu_0$.

Please show all details of your solutions.

8-1. Find the inductance per unit length of the coaxial transmission line by starting from finding the magnetic energy.

8-2. Given a toroidal core with a mean radius b = 10 cm and a circular cross section with a = 1 cm. If the core is made of steel (μ = 1000 μ_o) and has a coil with 200 turns which carries the current I = 2 A, calculate the magnetic flux in the core.

8-3. (a) Determine the mutual inductance between a very long, straight wire and a conducting circular loop as shown in the figure on the right. (HINT: Let I be the current on the straight wire, then the magnetic flux density at a point P(r, θ) inside the circle can be given by



$$\mathbf{B}_P = \hat{\phi}\frac{\mu_0 I}{2\pi(d + r\cos\theta)}$$

Also use the following integral formula

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}}\arctan\frac{\sqrt{a^2 - b^2}\tan\frac{x}{2}}{a + b} \quad [a^2 > b^2 \text{ case}].$$

(b) Find the force on the circular loop that is exerted by the magnetic field due to an upward current I_1 in the long straight wire. The circular loop carries a current I_2 in the counterclockwise direction.

(HINT: Show that the force on the circular loop can be given by

$$\mathbf{F} = I_2\int d\mathbf{l} \times \mathbf{B} = \frac{\mu_0 I_1 I_2 b}{2\pi}\int_0^{2\pi}\frac{-\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta}{d + b\cos\theta}d\theta.$$

Then use the following result

$$\int \frac{A + B\cos x}{a + b\cos x}dx = \frac{B}{b}x + \frac{Ab - aB}{b}\int \frac{dx}{a + b\cos x}$$

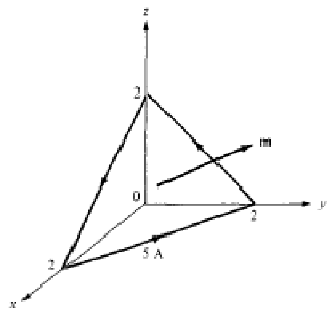
and the integral formula given above.

(c) Assume that the circular loop is rotated about its horizontal axis by an angle α, find the torque exerted on the circular loop.

8-4 Repeat problem 8-3 for a conducting equilateral triangular loop with side length b.

8-5 Repeat problem 8-3 for a conducting square loop with side length b, <u>with one of diagonal lines parallel to the straight wire</u>.

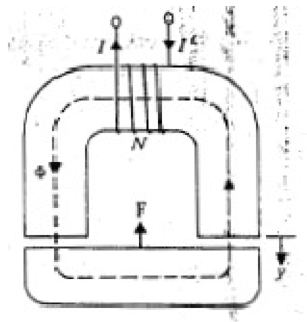
8-6 Determine the magnetic moment of an electric circuit formed by the triangular loop of the right figure.



8-7 A small current loop L_1, with magnetic moment $\hat{\mathbf{z}}5$ A·m^2 is located at the origin while another small loop current L_2 with magnetic moment $\hat{\mathbf{z}}3$ A·m^2 is located at (4, —3, 10). Determine the torque on L_2.

Please show all details of your solutions.

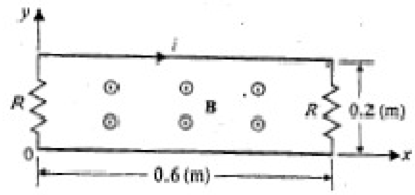
9-1. The electromagnet in the figure below carries a current I in an N-turn coil and produces a magnetic flux Φ in the magnetic circuit. If the cross-sectional area of the iron core is S, the lengths of the core and the air gap are ℓ_i and ℓ_g, respectively, and the relative permeability of the core is μ_r, determine the lifting force on the armature.



9-2. The circuit in the figure below is situated in a magnetic field

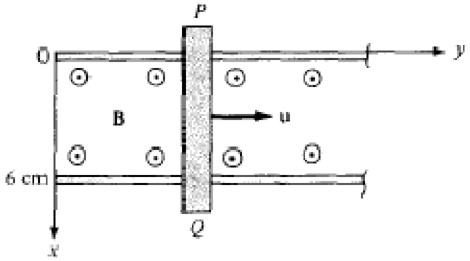
$$\mathbf{B} = \hat{\mathbf{z}} 3\cos(5\pi 10^7 t - \frac{2}{3}\pi x) \ \ (\mu\text{T})$$

Assuming that R = 15 (Ω), find the current i.



9-3. A conducting bar can slide freely over two conducting rails as shown in the figure below. Calculate the induced voltage in the bar if the bar slides at the velocity $\mathbf{u} = \hat{\mathbf{y}}\, 20$ (m/s) and

$$\mathbf{B} = \hat{\mathbf{z}} 4\cos(10^6 t - y) \ \ (\text{mT}).$$



9-4 A conducting equilateral triangular loop of side length b is placed near a very long straight wire with separation $d = b/2$. A current $i(t) = I \sin \omega t$ flows in the straight wire.

(a) Determine the voltage in the loop.

(b) If the triangular loop is rotated by 60° about a perpendicular axis through its center, determine the voltage in the loop.

9-5 A conducting circular loop of a radius 0.1 m. is situated near a very long power line carrying a 60-Hz current, with separation d = 0.15 m. If the 0.3 mA of current flows in the loop and the total impedance of the loop equals 0.01 W,

(a) Find the magnitude of the current in the power line.

(b) In order to reduce the loop current to 0.2 mA, to what angle about the horizontal axis should the loop be rotated?

Please show all details of your solutions.

10-1. Derive the divergence equation in the Maxwell's equation from the curl equation and the equation of continuity.

10-2. The far-zone electric field intensity of an infinitesimal dipole in free space is given by

$$\mathcal{E} = \hat{\boldsymbol{\theta}} \frac{E_0}{r} \sin\theta \cos(\omega t - kr)$$

(a) Find the magnetic field intensity and verify that it is perpendicular to the electric field intensity.

(b) Express the electric field intensity in terms of vector phasor and find the magnetic field intensity.

(c) Verify that $\mathbf{H} = \dfrac{\hat{\mathbf{k}} \times \mathbf{E}}{\eta}$, where $\hat{\mathbf{k}}, \eta$ denote the unit vector representing the propagation direction ($\hat{\mathbf{k}} = \mathbf{k} / |\mathbf{k}|; \mathbf{k}$: wavenumber vector) and the intrinsic impedance, respectively.

(d) Find the Poynting vector and the power density.

10-3. Given $\mathbf{E} = [\hat{\mathbf{x}} + 2(\hat{\mathbf{y}} - \hat{\mathbf{z}})] E_0 e^{j1000\pi(y+z)}$,

(a) Determine the propagation direction, wave vector, and the polarization.

(b) Determine the wavenumber.

(c) If this wave propagates in free space, find the frequency and the magnetic field intensity.

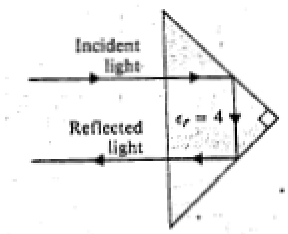
10-4. The wave $\mathbf{E} = [3\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - 4\hat{\mathbf{z}}] E_0 e^{j(4x+3z)}$ in free space (z ≤ 0 region) impinges upon the dielectric medium (z ≥ 0 region, $\mu_r = 1$, $\varepsilon_r = 2.5$).

(a) Find the wavenumber vector, plane of incidence, and angle of incidence.

(b) Determine the reflected wave and the transmitted wave.

(HINT: decompose **E** into perpendicular and parallel polarizations.)

10-5. Glass isosceles triangular prisms shown in the right figure are used in optical instruments. Assuming $\varepsilon_r = 4$ for glass, calculate the percentage of the incident light power reflected back by the prism.



10-6 A glass prism can separate white light into its component colors as shown in the figure below. Suppose white light hits the prism at the 30° angle with respect to the prism surface. Find the angles (with respect to the prism surface) of red, orange, yellow, green, blue lights when they exit the prism. Here, the wavelengths of red, orange, yellow, green, blue lights are 400, 450, 500, 575, 600, 650 nm, respectively, and the refractive index of the prism glass material is shown in the figure below.