Homework #10 Solution

1. First, taking the divergence of $\nabla \times \mathcal{E}$ yields

$$\nabla \cdot (\nabla \times \mathcal{E}) = \nabla \cdot (-\frac{\partial \mathcal{B}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla \cdot \mathcal{B})$$

Since $\nabla \cdot (\nabla \times \mathbf{A}) = 0, \forall \mathbf{A}$, $\nabla \cdot \mathcal{B} = 0$.

Likewise, taking the divergence of $\nabla \times \mathcal{H}$ yields

$$\nabla \cdot (\nabla \times \mathcal{H}) = \nabla \cdot (\mathcal{J} + \frac{\partial \mathcal{D}}{\partial t}) = \nabla \cdot \mathcal{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathcal{D}) = 0.$$

From the equation of continuity, $\nabla \cdot \mathcal{J} = -\frac{\partial \rho_v}{\partial t}$, thus $\nabla \cdot \mathcal{D} = \rho_v$.

It can be seen that the divergence equations and the curl equations in Maxwell's equations are not *independent*.

2. $\mathcal{E} = \hat{\boldsymbol{\theta}} \frac{E_0}{r} \sin\theta \cos(\omega t - kr)$

(a) In free space, which is linear and isotropic,

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} = -\mu_0 \frac{\partial \mathcal{H}}{\partial t}.$$

Since $\nabla \times \mathcal{E} = \hat{\boldsymbol{\phi}} \frac{E_0}{r} k \sin\theta \sin(\omega t - kr)$,

$$\frac{\partial \mathcal{H}}{\partial t} = -\hat{\boldsymbol{\phi}} \frac{E_0}{r} \frac{k}{\mu_0} \sin\theta \sin(\omega t - kr) \rightarrow \mathcal{H} = -\hat{\boldsymbol{\phi}} \frac{E_0}{r} \frac{k}{\mu_0} \sin\theta \int \sin(\omega t - kr)dt = \hat{\boldsymbol{\phi}} \frac{E_0}{r} \frac{k}{\omega\mu_0} \sin\theta \cos(\omega t - kr)$$

Notice that $\frac{k}{\omega\mu_0} = \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\mu_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}} = \frac{1}{\eta}$, thus

$\mathcal{H} = \hat{\phi} \frac{E_0}{\eta r} \sin\theta \cos(\omega t - kr)$, which is perpendicular to the electric field intensity.

(b) Consider the phase term of $\mathbf{E}$, one obtains the vector phasor as $\mathbf{E} = \hat{\boldsymbol{\theta}} \frac{E_0}{r} \sin\theta e^{-jkr}$, which is a "non-uniform" plane wave. Obtaining the magnetic field intensity from Maxwell's equation yields

$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -j\omega\mu_0 \mathbf{H} = \hat{\boldsymbol{\phi}}(-jk) \frac{E_0}{r} \sin\theta e^{-jkr}$, thus

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{k}{\omega\mu_0} \frac{E_0}{r} \sin\theta e^{-jkr} = \hat{\boldsymbol{\phi}} \frac{E_0}{\eta r} \sin\theta e^{-jkr} \rightarrow \mathcal{H} = \hat{\boldsymbol{\phi}} \frac{E_0}{\eta r} \sin\theta \cos(\omega t - kr).$$

(c) From part (b), $\frac{\hat{\mathbf{k}} \times \mathbf{E}}{\eta} = \frac{\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}}{\eta} \frac{E_0}{r} \sin\theta e^{-jkr} = \hat{\boldsymbol{\phi}} \frac{E_0}{\eta r} \sin\theta e^{-jkr}$, which is equal to the result obtained in part (b). Hence, $\mathbf{H} = \frac{\hat{\mathbf{k}} \times \mathbf{E}}{\eta}$.

(d) The Poynting Vector is given by

$$\mathcal{P} = \mathcal{E} \times \mathcal{H} = \hat{\mathbf{r}} \frac{E_0^2}{\eta r^2} \sin^2\theta \cos^2(\omega t - kr).$$

The power density then becomes

$$|\mathcal{P}| = \frac{E_0^2}{\eta r^2} \sin^2\theta \cos^2(\omega t - kr) \text{ (W/m}^2\text{)}.$$

3. $\mathbf{E} = [\hat{\mathbf{x}} + 2(\hat{\mathbf{y}} - \hat{\mathbf{z}})] E_0 e^{j1000\pi(y+z)}$

(a) From $\mathbf{k}\cdot\mathbf{r}=1000\pi(y+z)$,

$k_y=k_z=1000\pi\rightarrow\hat{\mathbf{k}}=(\hat{\mathbf{y}}+\hat{\mathbf{z}})/\sqrt{2};k=1000\sqrt{2}\pi;\mathbf{k}=\hat{\mathbf{k}}k$.

Furthermore, since the phase difference between two orthogonal field components $(\hat{\mathbf{x}},(\hat{\mathbf{y}}-\hat{\mathbf{z}})/\sqrt{2})$ is 0 (i.e., same phase), this is a linear polarization.

(b) $k=1000\sqrt{2}\pi$

(c) $k=1000\sqrt{2}\pi=\omega\sqrt{\mu_0\varepsilon_0}=2\pi f/c\rightarrow f=ck/2\pi=1.5\times10^{11}\sqrt{2}=212.13(\text{GHz})$

$$\mathbf{H}=\frac{\hat{\mathbf{k}}\times\mathbf{E}}{\eta}=\frac{1}{\eta}\left(\frac{\hat{\mathbf{y}}+\hat{\mathbf{z}}}{\sqrt{2}}\right)\times[\hat{\mathbf{x}}+2(\hat{\mathbf{y}}-\hat{\mathbf{z}})]E_0e^{j1000\pi(y+z)}=\frac{1}{\eta}E_0e^{j1000\pi(y+z)}\left[\left(\frac{\hat{\mathbf{y}}-\hat{\mathbf{z}}}{\sqrt{2}}\right)-2\sqrt{2}\hat{\mathbf{x}}\right]$$

$$=\frac{1}{\sqrt{2}\eta}E_0e^{j1000\pi(y+z)}[(\hat{\mathbf{y}}-\hat{\mathbf{z}})-4\hat{\mathbf{x}}]$$

4. $\mathbf{E}=[3\hat{\mathbf{x}}+2\hat{\mathbf{y}}-4\hat{\mathbf{z}}]E_0e^{j(4x+3z)}$

(a) Since $\mathbf{k}\cdot\mathbf{r}=-(4x+3z)$,

$$\mathbf{k}=-\hat{\mathbf{x}}4-\hat{\mathbf{z}}3=\hat{\mathbf{k}}k;\hat{\mathbf{k}}=-\frac{\hat{\mathbf{x}}4+\hat{\mathbf{z}}3}{5},k=5.$$

Also, since the direction normal to the boundary plane is *z*, the xz-plane is the plane of incidence. It follows that

$\cos\theta_i=|\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}|=3/5\rightarrow\theta_i=0.9273=53.13°$.

Using the Snell's law of refraction yields

$\sin\theta_t=(n_1/n_2)\sin\theta_i=\sqrt{1/2.5}(4/5)=0.5060\rightarrow\theta_t=0.5305=30.40°;\cos\theta_t=0.8626$

$\mathbf{k}_t=k_2(-\hat{\mathbf{x}}\sin\theta_t-\hat{\mathbf{z}}\cos\theta_t)=5\sqrt{2.5}(-\hat{\mathbf{x}}0.5060-\hat{\mathbf{z}}0.8626)=-\hat{\mathbf{x}}4-\hat{\mathbf{z}}6.8195$.

(b) Can decompose the field into perpendicular polarization and parallel polarization as

$\mathbf{E}_\perp=\hat{\mathbf{y}}2E_0e^{j(4x+3z)}$; $\mathbf{E}_\parallel=[3\hat{\mathbf{x}}-4\hat{\mathbf{z}}]E_0e^{j(4x+3z)}$.

i) Perpendicular polarization

$$\Gamma_\perp=\frac{\eta_2\cos\theta_i-\eta_1\cos\theta_t}{\eta_2\cos\theta_i+\eta_1\cos\theta_t}=\frac{(3/5)/\sqrt{2.5}-0.8626}{(3/5)/\sqrt{2.5}+0.8626}=-0.389;\tau_\perp=\frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i+\eta_1\cos\theta_t}=0.611.$$

Therefore, the reflected wave and the transmitted wave become

$\mathbf{E}_{\perp,r}=\hat{\mathbf{y}}\Gamma_\perp 2E_0e^{j(4x-3z)}=-\hat{\mathbf{y}}0.778E_0e^{j(4x-3z)}$; $\mathbf{E}_{\perp,t}=\hat{\mathbf{y}}\tau_\perp 2E_0e^{j(4x+6.8195z)}=\hat{\mathbf{y}}1.222E_0e^{j(4x+6.8195z)}$

ii) Parallel polarization

$$\Gamma_\parallel=\frac{\eta_2\cos\theta_t-\eta_1\cos\theta_i}{\eta_2\cos\theta_t+\eta_1\cos\theta_i}=\frac{0.8626/\sqrt{2.5}-(3/5)}{0.8626/\sqrt{2.5}+(3/5)}=-0.0475;\tau_\parallel=\frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t+\eta_1\cos\theta_i}=0.6625$$

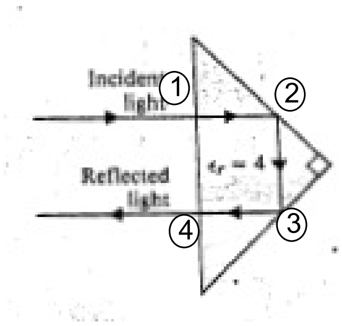
Therefore, the reflected wave becomes

$\mathbf{E}_{\parallel,r}=[3\hat{\mathbf{x}}+4\hat{\mathbf{z}}]\Gamma_\parallel E_0e^{j(4x-3z)}=-[0.1425\hat{\mathbf{x}}+0.190\hat{\mathbf{z}}]E_0e^{j(4x-3z)}$.

Note that $\mathbf{E}_\parallel=[3\hat{\mathbf{x}}-4\hat{\mathbf{z}}]E_0e^{j(4x+3z)}=[\frac{3}{5}\hat{\mathbf{x}}-\frac{4}{5}\hat{\mathbf{z}}]5E_0e^{j(4x+3z)}$. The transmitted wave becomes

$\mathbf{E}_{\parallel,t}=[0.8625\hat{\mathbf{x}}-0.5060\hat{\mathbf{z}}]\tau_\parallel 5E_0e^{j(4x+6.8195z)}=[2.857\hat{\mathbf{x}}+1.6761\hat{\mathbf{z}}]E_0e^{j(4x+6.8195z)}$.

5. Assign points 1, 2, 3, 4 as shown in the figure below.



Let E_{i0} be the magnitude of the electric field intensity impinging upon the prism (i.e., the incident field) and η_0, η_1 denote the intrinsic impedances of the air and the prism, respectively. Since ε_r inside the prism is 4, $\eta_1 = \eta_0/2$.

Now, let E_{t0} denote the magnitude of the light transmitted into the prism at point 1 (transmitted field), then

$$E_{t0} = \tau E_{i0} = \frac{2\eta_1}{\eta_1 + \eta_0} E_{i0} = \frac{2}{3} E_{i0}.$$

The critical angle between the medium inside the prism and the air is given by

$$\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_0}{\varepsilon_1}} = \sin^{-1}\sqrt{\frac{1}{\varepsilon_r}} = \sin^{-1}\frac{1}{2} = 30°.$$

Since $\theta_i = 45° > \theta_c$, the reflections at points 2, 3 inside the prism are total reflections, resulting in the light reaching point 4 has the magnitude E_{t0}.

Finally, let E_{t1} be the magnitude of the light transmitted out of the prism at point 4, then

$$E_{t1} = \tau' E_{t0} = \frac{2\eta_0}{\eta_1 + \eta_0} E_{t0} = \frac{4}{3} E_{t0} = \frac{4}{3}\frac{2}{3} E_{i0} = \frac{8}{9} E_{i0}.$$

Since the time-average power densities of incoming light and outgoing light are given by

$$\left(P_{av}\right)_i = \frac{E_{i0}^2}{2\eta_0}; \left(P_{av}\right)_r = \frac{E_{t1}^2}{2\eta_0} = \frac{64}{81}\frac{E_{i0}^2}{2\eta_0},$$

the ratio of the reflected light power to the input light power becomes 64/81 = 0.79.