LE 200 Homework #2 Solution

2-1. Given **A** = (4, 0, -1) , **B** = (1,3, 4), and **C** = (-5, -3 , -3) in Cartesian coordinates.
(a) Let **D** be a vector perpendicular to the plane containing vectors **A** and **B** (denoted by AB plane afterwards), then

$$\mathbf{D} = \mathbf{A}\times\mathbf{B} = 3\hat{\mathbf{x}} - 17\hat{\mathbf{y}} + 12\hat{\mathbf{z}}\,.$$

Since $\mathbf{C}\cdot\mathbf{D} = (3)(-5)+(-3)(-17)+(-3)(12) = 0$, **C** is perpendicular to **D** and is thus perpendicular to AB plane. Consequently, **A**, **B**, **C** are on the same plane.
Next, let **P** be an arbitrary vector on the AB plane given by $\mathbf{P} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$, then all **P** must be perpendicular to **D**, thus

$$\mathbf{P}\cdot\mathbf{D} = 3x - 17y + 12z = 0\,,$$

which is the equation of this plane.
(b) An equation of a plane is generally given by $\alpha x + \beta y + \gamma z = C$ **.** The coefficients can be determined from the points on the plane. Since the origin is on the plane, then C = 0. Next, substituting (4, 0, -1) and (1,3,4) into the plane equation yields

$$4\alpha - \gamma = 0, \alpha + 3\beta + 4\gamma = 0$$

Thus, $\gamma = 4\alpha, \alpha + 3\beta + 12\alpha = 0 \rightarrow 3\beta = -17\alpha$ **.** Choosing $\alpha = 3$, one obtains

$$3x - 17y + 12z = 0\,.$$

It can be easily verified that (-5, -3, -3) is also on the plane.
(c) It can be shown that $\mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0}$, thus **A**, **B**, **C** forms a triangle. Using the result from Problem 3, one obtains

$$S = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|(-3\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 5\hat{\mathbf{z}})\times(-9\hat{\mathbf{x}} - 3\hat{\mathbf{y}} - 2\hat{\mathbf{z}})\right| = \frac{1}{2}\left|-9\hat{\mathbf{x}} - 51\hat{\mathbf{y}} + 36\hat{\mathbf{z}}\right| = 31.54$$

In fact, one can verify it using Helon's formula as follows:

$$a = \left|\overrightarrow{AB}\right| = \sqrt{43}; b = \left|\overrightarrow{BC}\right| = 11; c = \left|\overrightarrow{CA}\right| = \sqrt{94}; s = \frac{a+b+c}{2};$$

$$Area = \sqrt{s(s-a)(s-b)(s-c)} = 31.54$$

2-2. Given $r = \sqrt{x^2 + y^2 + z^2}$.

(a) Since $\frac{\partial r}{\partial x} = \frac{x}{r}; \frac{\partial r}{\partial y} = \frac{y}{r}; \frac{\partial r}{\partial z} = \frac{z}{r}$, $\nabla r = \frac{1}{r}\left(\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z\right)$.

(b) Likewise, since $\frac{\partial}{\partial x}\frac{1}{r} = -\frac{x}{r^3}; \frac{\partial}{\partial y}\frac{1}{r} = -\frac{y}{r^3}; \frac{\partial}{\partial z}\frac{1}{r} = -\frac{z}{r^3}$, $\nabla\frac{1}{r} = -\frac{1}{r^3}\left(\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z\right)$.

(c) In general, since $\frac{\partial}{\partial x}r^n = nxr^{n-2}; \frac{\partial}{\partial y}r^n = nyr^{n-2}; \frac{\partial}{\partial z}r^n = nzr^{n-2}$,

$\nabla r^n = nr^{n-2}\left(\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z\right) = nr^{n-2}\mathbf{r}$ where $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$.

2-3. Given a scalar field $\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + z$.

(a) $\nabla\phi = \hat{\mathbf{x}}\frac{2x}{a^2} + \hat{\mathbf{y}}\frac{2y}{b^2} + \hat{\mathbf{z}}$

(b) Since the gradient at the point (x_0, y_0, z_0) is given by $\nabla\phi = \hat{\mathbf{x}}\frac{2x_0}{a^2} + \hat{\mathbf{y}}\frac{2y_0}{b^2} + \hat{\mathbf{z}}$, the equation of the tangential surface at the point (x_0, y_0, z_0) is given by

$$\nabla\phi\cdot(\mathbf{r}-\mathbf{r}_0) = \left(\hat{\mathbf{x}}\frac{2x_0}{a^2} + \hat{\mathbf{y}}\frac{2y_0}{b^2} + \hat{\mathbf{z}}\right)\cdot\left(\hat{\mathbf{x}}(x-x_0) + \hat{\mathbf{y}}(y-y_0) + \hat{\mathbf{z}}(z-z_0)\right)$$

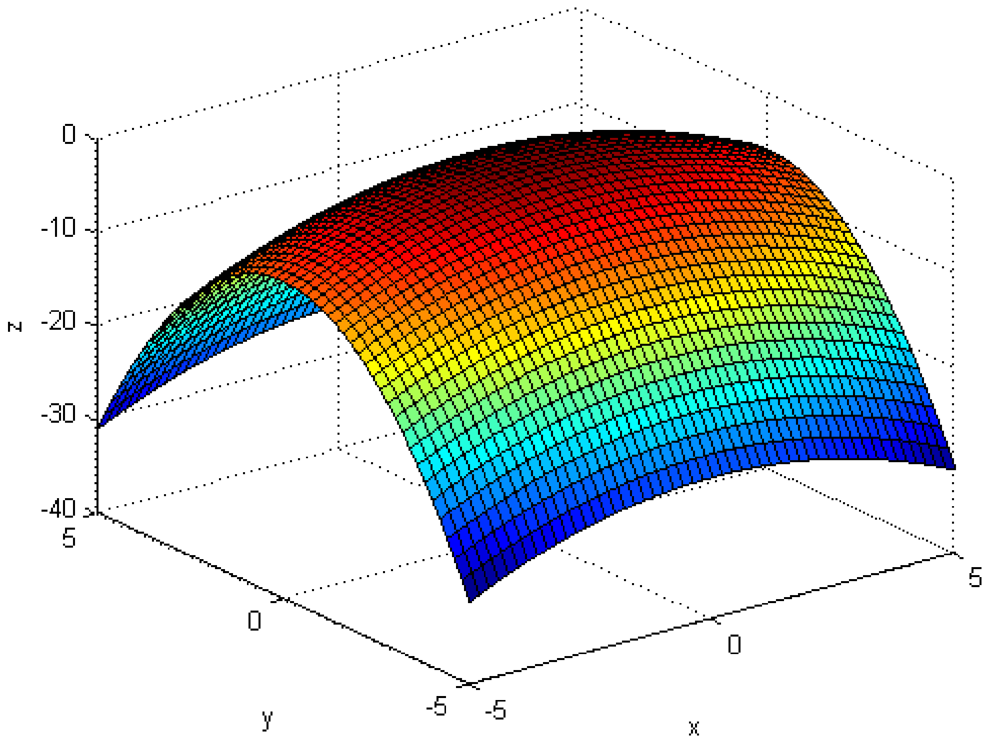
$$= \frac{2x_0}{a^2}(x-x_0) + \frac{2y_0}{b^2}(y-y_0) + z - z_0 = 0$$

(c) Using the result from (a) and substituting a=2, b=1 yields $\nabla\phi = \hat{\mathbf{x}}\frac{x}{2} + \hat{\mathbf{y}}2y + \hat{\mathbf{z}}$, which at at the point (2,0,-1) becomes $\nabla\phi = \hat{\mathbf{x}} + \hat{\mathbf{z}}$. It follows that two orthogonal vectors of this tangential surface can be given for instance by $\hat{\mathbf{y}}, \hat{\mathbf{x}} - \hat{\mathbf{z}}$.

Using the following commands in MATLAB:

```
[x,y] = meshgrid(-5:.2:5, -5:.2:5);
z = -x.^2/4 - y.^2 ;
surf(x,y,z); % surface plot
```

one can generate the ϕ=0 surface plot when a=2, b=1 as shown in the figure below.



2-4. Given $\mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z^2$.

(a) A unit cubic centered at the origin.

<u>Surface Integral</u>

On x = 1/2 surface: $\mathbf{A} = \hat{\mathbf{x}}\frac{1}{2} + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z^2; d\mathbf{s} = \hat{\mathbf{x}}dydz; \mathbf{A}\cdot d\mathbf{s} = \frac{1}{2}dydz$

$$\therefore \int \mathbf{A}\cdot d\mathbf{s} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{1}{2}dydz = \frac{1}{2}$$

Likewise, on x = -1/2 surface: $\mathbf{A}=-\hat{\mathbf{x}}\frac{1}{2}+\hat{\mathbf{y}}y+\hat{\mathbf{z}}z^2;d\mathbf{s}=-\hat{\mathbf{x}}dydz;\mathbf{A}\cdot d\mathbf{s}=\frac{1}{2}dydz$

$\therefore \int \mathbf{A}\cdot d\mathbf{s}=\frac{1}{2}$

It can be found that the same results can be obtained for y=1/2 and y=-1/2 surfaces.

However, on z = 1/2 surface: $\mathbf{A}=\hat{\mathbf{x}}x+\hat{\mathbf{y}}y+\hat{\mathbf{z}}\frac{1}{4};d\mathbf{s}=\hat{\mathbf{z}}dxdy;\mathbf{A}\cdot d\mathbf{s}=\frac{1}{4}dxdy$, but on z = -1/2 surface : $\mathbf{A}=\hat{\mathbf{x}}x+\hat{\mathbf{y}}y+\hat{\mathbf{z}}\frac{1}{4};d\mathbf{s}=-\hat{\mathbf{z}}dxdy;\mathbf{A}\cdot d\mathbf{s}=-\frac{1}{4}dxdy$ and thus the integrals will cancel each other. Summing all contributions yields $\oint_S \mathbf{A}\cdot d\mathbf{s}=2$.

Volume Integral

Since, $\nabla\cdot\mathbf{A}=2z+2$, $\int_V \nabla\cdot\mathbf{A}dv=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(2z+2)dxdydz=2$.

(b) The region bounded by $x^2+y^2=4$, z=0 and z=3.

Surface Integral

On $x^2+y^2=4$ surface:

$\mathbf{A}=\hat{\mathbf{x}}2\cos\phi+\hat{\mathbf{y}}2\sin\phi+\hat{\mathbf{z}}z^2;d\mathbf{s}=\hat{\boldsymbol{\rho}}2d\phi dz;\mathbf{A}\cdot d\mathbf{s}=4\cos^2\phi+4\sin^2\phi=4$

$\therefore \int \mathbf{A}\cdot d\mathbf{s}=\int_0^3\int_0^{2\pi}4d\phi dz=24\pi$

Likewise, on z = 0 surface : $\mathbf{A}=\hat{\mathbf{x}}x+\hat{\mathbf{y}}y;d\mathbf{s}=\hat{\mathbf{z}}dxdy;\mathbf{A}\cdot d\mathbf{s}=0$

On z = 3 surface : $\mathbf{A}=\hat{\mathbf{x}}x+\hat{\mathbf{y}}y+\hat{\mathbf{z}}9;d\mathbf{s}=\hat{\mathbf{z}}dxdy;\mathbf{A}\cdot d\mathbf{s}=9dxdy=9(2d\phi)$,

$\therefore \int \mathbf{A}\cdot d\mathbf{s}=\int_0^{2\pi}18d\phi=36\pi$

Therefore, $\oint_S \mathbf{A}\cdot d\mathbf{s}=60\pi$.

Volume Integral

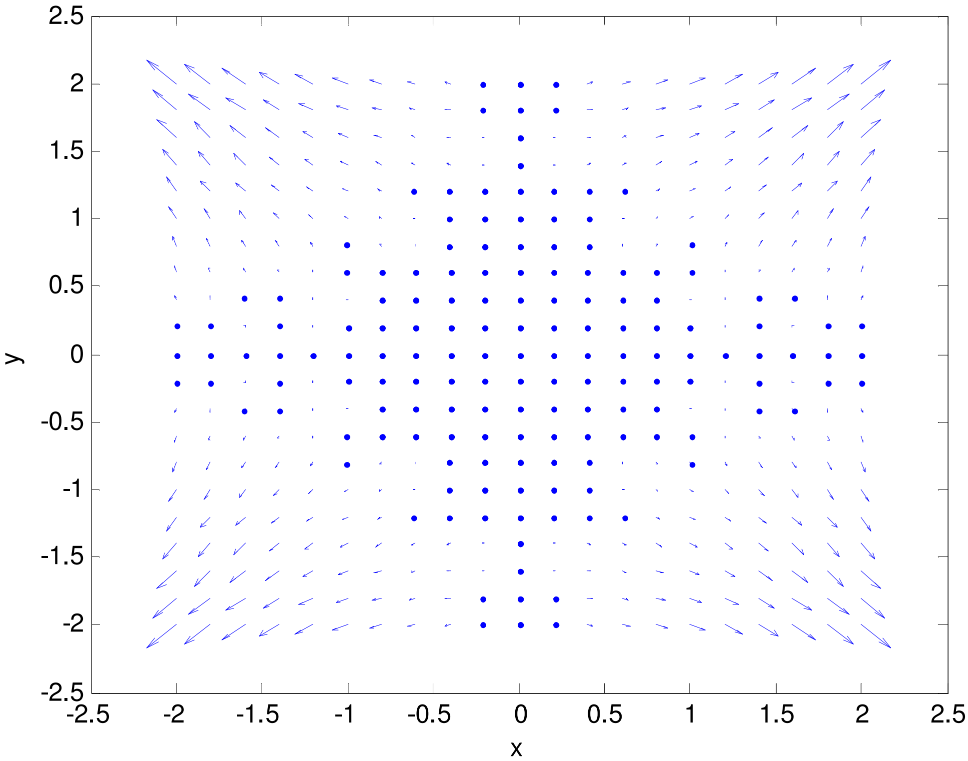
$\int_V \nabla\cdot\mathbf{A}dv=\int_0^3\int_0^{2\pi}\int_0^2(2z+2)\rho d\rho d\phi dz=60\pi$.

2-5. Given a vector field $\mathbf{F}=\hat{\mathbf{x}}xy^2+\hat{\mathbf{y}}x^2y$

(a) Since $(\nabla\times\mathbf{F})_x=(\nabla\times\mathbf{F})_y=0;(\nabla\times\mathbf{F})_z=\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}=0,\ \nabla\times\mathbf{F}=\mathbf{0}$

(b) This vector field in the range $|x|\le 2, |y|\le 2$ can be shown in the figure below using the following MATLAB commands:

```
[x,y] = meshgrid(-2:.2:2, -2:.2:2);
px = x.*y.^2 ;
py = x.^2.*y ;
quiver(x,y,px,py) ;
```



Note that there is no vortex source in this vector field.

(c) Integrate this vector field from (1,1,0) to (2,2,0) along the following paths:

(i) along y = 1 line to (2,1,0) : $\mathbf{F} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2 y\Big|_{y=1} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}x^2; d\mathbf{l} = \hat{\mathbf{x}}dx; \mathbf{F}\cdot d\mathbf{l} = xdx$.

$$\therefore \int \mathbf{F}\cdot d\mathbf{l} = \int_1^2 xdx = \frac{3}{2}.$$

along x = 2 line up to (2,2,0) : $\mathbf{F} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2 y\Big|_{x=2} = \hat{\mathbf{x}}2y^2 + \hat{\mathbf{y}}4y\Big|; d\mathbf{l} = \hat{\mathbf{y}}dy; \mathbf{F}\cdot d\mathbf{l} = 4ydy$.

$$\therefore \int \mathbf{F}\cdot d\mathbf{l} = \int_1^2 4ydy = 6.$$

$$\therefore \int_{(1,1,0)}^{(2,2,0)} \mathbf{F}\cdot d\mathbf{l} = \frac{3}{2} + 6 = \frac{15}{2}.$$

(ii) along y = x line directly from (1,1,0) to (2,2,0):

$\mathbf{F} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2 y\Big|_{x=y} = \hat{\mathbf{x}}x^3 + \hat{\mathbf{y}}x^3; d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dx; \mathbf{F}\cdot d\mathbf{l} = 2x^3dx$.

$$\therefore \int_{(1,1,0)}^{(2,2,0)} \mathbf{F}\cdot d\mathbf{l} = \int_1^2 2x^3dx = \frac{15}{2}.$$

(iii) along x = 1 line up to (1,3,0): $\mathbf{F} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2 y\Big|_{x=1} = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}y\Big|; d\mathbf{l} = \hat{\mathbf{y}}dy; \mathbf{F}\cdot d\mathbf{l} = ydy$

$$\therefore \int \mathbf{F}\cdot d\mathbf{l} = \int_1^3 ydy = 4.$$

go straight down from (1,3,0) to (2,2,0) : it can be shown that the equation of this straight line is given by $y = -x + 4$, thus

$\mathbf{F} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2 y\Big|_{y=-x+4} = \hat{\mathbf{x}}x(-x+4)^2 + \hat{\mathbf{y}}x^2(-x+4); d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy = \hat{\mathbf{x}}dx - \hat{\mathbf{y}}dx;$

$$\mathbf{F}\cdot d\mathbf{l} = \left(x(-x+4)^2 - x^2(-x+4)\right)dx = (-x+4)(-x^2+4x-x^2)dx = 2x(x^2-6x+8)dx$$

$$\therefore \int \mathbf{F}\cdot d\mathbf{l} = \int_1^2 2x(x^2-6x+8)dx = \frac{7}{2}.$$

$$\therefore \int_{(1,1,0)}^{(2,2,0)} \mathbf{F}\cdot d\mathbf{l} = 4+\frac{7}{2} = \frac{15}{2}.$$

It can be noticed that the line integration does not depend on the path because this vector field is conservative (irrotational).

2-6. Taylor's series of degree n about a is given by

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!}.$$

Now, let $f(x)=A_x(x,y_0,z_0)$, $x = x_0+\Delta x/2$, $a = x_0$, ($x-a = \Delta x/2$), and use partial derivatives instead of ordinary derivatives, then one obtains

$$A_x(x_0+\frac{\Delta x}{2}, y_0, z_0) = A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\left.\frac{\partial A_x}{\partial x}\right|_{(x_0,y_0,z_0)} + \text{higher-order terms.}$$