Homework #3 Solution

3-1. Given $\mathbf{A} = \hat{\mathbf{x}}(y - z + 2) + \hat{\mathbf{y}}(yz + 4) - \hat{\mathbf{z}}xz$; $\nabla \times \mathbf{A} = -\hat{\mathbf{x}}y - \hat{\mathbf{y}} - \hat{\mathbf{z}}$

(a) The region bounded by x=0, y=0, x=2, y=2 on x-y plane.

Surface Integral: $d\mathbf{s} = \hat{\mathbf{z}}dxdy; \nabla \times \mathbf{A} \cdot d\mathbf{s} = -dxdy$

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{s} = -\int_0^2 \int_0^2 dxdy = -4$$

Contour Integral: The contour can be divided into the following paths:

(0,0,0)->(2,0,0): $\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}4; d\boldsymbol{\ell} = \hat{\mathbf{x}}dx; \mathbf{A} \cdot d\boldsymbol{\ell} = 2dx; \int_{(0,0,0)}^{(2,0,0)} \mathbf{A} \cdot d\boldsymbol{\ell} = \int_0^2 2dx = 4.$

(2,0,0)->(2,2,0): $\mathbf{A} = \hat{\mathbf{x}}(y+2) + \hat{\mathbf{y}}4; d\boldsymbol{\ell} = \hat{\mathbf{y}}dy; \mathbf{A} \cdot d\boldsymbol{\ell} = 4dy; \int_{(2,0,0)}^{(2,2,0)} \mathbf{A} \cdot d\boldsymbol{\ell} = \int_0^2 4dy = 8.$

(2,2,0)->(0,2,0): $\mathbf{A} = \hat{\mathbf{x}}4 + \hat{\mathbf{y}}4; d\boldsymbol{\ell} = \hat{\mathbf{x}}dx; \mathbf{A} \cdot d\boldsymbol{\ell} = 4dx; \int_{(2,2,0)}^{(0,2,0)} \mathbf{A} \cdot d\boldsymbol{\ell} = \int_2^0 4dx = -8.$

(0,2,0)->(0,0,0): $\mathbf{A} = \hat{\mathbf{x}}(y+2) + \hat{\mathbf{y}}4; d\boldsymbol{\ell} = \hat{\mathbf{y}}dy; \mathbf{A} \cdot d\boldsymbol{\ell} = 4dy; \int_{(0,2,0)}^{(0,0,0)} \mathbf{A} \cdot d\boldsymbol{\ell} = \int_2^0 4dy = -8.$

Therefore, $\oint_C \mathbf{A} \cdot d\boldsymbol{\ell} = -4.$

(b) The circle with radius of 2 and centered at (1,2) on x-y plane.

Surface Integral: $d\mathbf{s} = \hat{\mathbf{z}}dxdy; \nabla \times \mathbf{A} \cdot d\mathbf{s} = -dxdy = -\rho d\rho d\phi$

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{s} = -\int_0^{2\pi} \int_0^2 \rho d\rho d\phi = -4\pi$$

Contour Integral: $d\boldsymbol{\ell} = \hat{\phi}2d\phi$; $\mathbf{A} \cdot d\boldsymbol{\ell} = 2d\phi[-(y+2)\sin\phi + 4\cos\phi]$

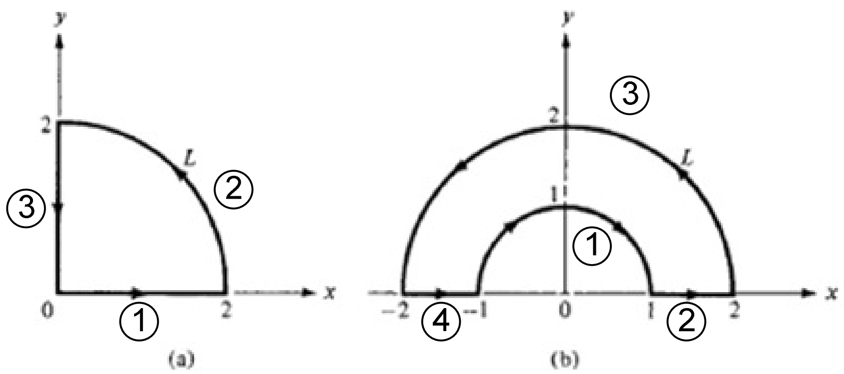
Using $y - 2 = 2\sin\phi$, then $\mathbf{A} \cdot d\boldsymbol{\ell} = 2d\phi[-(2\sin\phi + 4)\sin\phi + 4\cos\phi]$. Hence

$$\oint_C \mathbf{A} \cdot d\boldsymbol{\ell} = \int_0^{2\pi} 2[-(2\sin\phi + 4)\sin\phi + 4\cos\phi]d\phi = -4\pi.$$

3-2. Since $\mathbf{A} = \hat{\mathbf{r}}r\sin\phi + \hat{\boldsymbol{\phi}}r^2$, and $\nabla \times \mathbf{A}$ in cylindrical coordinates is given by

$$\nabla \times \mathbf{F} = \frac{1}{r}\left[\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right]\hat{\mathbf{r}} + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\hat{\boldsymbol{\phi}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\phi) - \frac{\partial F_r}{\partial \phi}\right]\hat{\mathbf{z}}.$$

Hence, $\nabla \times \mathbf{A} = \hat{\mathbf{z}}(3r - \cos\phi)$.



(a) Specify the path as shown in figure (a) above, then

Along Path 1: $\phi = 0, \mathbf{A} = \hat{\boldsymbol{\phi}}r^2, d\boldsymbol{\ell} = \hat{\mathbf{x}}dx$ but $A_x = A_r\cos\phi - A_\phi\sin\phi = 0$, $\int_{Path1} \mathbf{A} \cdot d\boldsymbol{\ell} = 0$

Along Path 2: $r = 2, \mathbf{A} = \hat{\mathbf{r}}2\sin\phi + \hat{\boldsymbol{\phi}}4, d\mathbf{l} = \hat{\boldsymbol{\phi}}rd\phi = \hat{\boldsymbol{\phi}}2d\phi \rightarrow \mathbf{A} \cdot d\mathbf{l} = 8d\phi$, thus, $\int_{Path2} \mathbf{A}\cdot d\boldsymbol{\ell} = \int_0^{\pi/2} 8d\phi = 4\pi$.

Along Path 3: $\phi = \pi/2, \mathbf{A} = \hat{\mathbf{r}}r + \hat{\boldsymbol{\phi}}r^2, d\mathbf{l} = -\hat{\mathbf{y}}dy$ but $A_y = A_r \sin\phi + A_\phi \cos\phi = r = y$,

$\int_{Path3} \mathbf{A}\cdot d\boldsymbol{\ell} = -\int_0^2 ydy = -2$

Thus, $\oint \mathbf{A}\cdot d\boldsymbol{\ell} = 4\pi - 2$.

On the other hand, $\nabla\times\mathbf{A} = \hat{\mathbf{z}}(3r - \cos\phi); d\mathbf{s} = \hat{\mathbf{z}}rdrd\phi$,

$$\oint_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \int_0^{\pi/2}\int_0^2 (3r - \cos\phi)rdrd\phi = \int_0^{\pi/2}(8 - 2\cos\phi)d\phi = 4\pi - 2$$

(b) Specify the path as shown in figure (b) above, then

Along Path 1: $r = 1, \mathbf{A} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}, d\mathbf{l} = -\hat{\boldsymbol{\phi}}rd\phi = -\hat{\boldsymbol{\phi}}d\phi \to \mathbf{A}\cdot d\mathbf{l} = -d\phi$,

thus $\int_{Path1} \mathbf{A}\cdot d\boldsymbol{\ell} = -\int_0^{\pi} d\phi = -\pi$.

Along Path 2: $\phi = 0, \mathbf{A} = \hat{\boldsymbol{\phi}}r^2, d\mathbf{l} = \hat{\mathbf{x}}dx$ but $A_x = A_r\cos\phi - A_\phi\sin\phi = 0$, $\int_{Path2} \mathbf{A}\cdot d\boldsymbol{\ell} = 0$

Along Path 3: $r = 2, \mathbf{A} = \hat{\mathbf{r}}2\sin\phi + \hat{\boldsymbol{\phi}}4, d\mathbf{l} = \hat{\boldsymbol{\phi}}rd\phi = \hat{\boldsymbol{\phi}}2d\phi \to \mathbf{A}\cdot d\boldsymbol{\ell} = 8d\phi$,

thus $\int_{Path3} \mathbf{A}\cdot d\boldsymbol{\ell} = \int_0^{\pi} 8d\phi = 8\pi$.

Along Path 4: $\phi = \pi, \mathbf{A} = \hat{\boldsymbol{\phi}}r^2, d\mathbf{l} = \hat{\mathbf{x}}dx$ but $A_x = A_r\cos\phi - A_\phi\sin\phi = 0$, $\int_{Path4} \mathbf{A}\cdot d\boldsymbol{\ell} = 0$

Therefore, $\oint \mathbf{A}\cdot d\boldsymbol{\ell} = 7\pi$.

On the other hand, $\nabla\times\mathbf{A} = \hat{\mathbf{z}}(3r - \cos\phi); d\mathbf{s} = \hat{\mathbf{z}}rdrd\phi$,

$$\oint_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \int_0^{\pi}\int_1^2 (3r - \cos\phi)rdrd\phi = \int_0^{\pi}(7 - \frac{3}{2}\cos\phi)d\phi = 7\pi .$$

3-3. Arrange the charges in the *xy* plane at locations (4,4), (4,-4), (-4,4), and (-4,-4). Then the fifth charge will be on the *z* axis at location $z = 4\sqrt{2}$, which puts it at 8cm distance from the other four. By symmetry, the force on the fifth charge will be *z*-directed, and will be four times the *z* component of force produced by each of the four other charges.

$$F = \frac{4}{\sqrt{2}}\frac{q^2}{4\pi\varepsilon_0 d^2} = \frac{4}{\sqrt{2}}\times\frac{(10^{-8})^2}{4\pi(8.85\times10^{-12})(0.08)^2} = 4.0\times10^{-4}\,(\mathrm{N})$$

3-4. The total electric field at P(*a*, *a*, *a*) that produces a force on the charge there will be the sum of the fields from the other seven charges, i.e.,

$$\mathbf{E}_{total}(a,a,a) = \frac{q}{4\pi\varepsilon_0 a^2}\left[\underbrace{\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}}{3\sqrt{3}}}_{(0,0,0)} + \underbrace{\frac{\hat{\mathbf{y}}+\hat{\mathbf{z}}}{2\sqrt{2}}}_{(a,0,0)} + \underbrace{\frac{\hat{\mathbf{x}}+\hat{\mathbf{z}}}{2\sqrt{2}}}_{(0,a,0)} + \underbrace{\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{2\sqrt{2}}}_{(0,0,a)} + \underbrace{\hat{\mathbf{x}}}_{(0,a,a)} + \underbrace{\hat{\mathbf{y}}}_{(a,0,a)} + \underbrace{\hat{\mathbf{z}}}_{(0,a,a)}\right].$$

The force is now the product of this field and the charge at (*a*,*a*,*a*). Simplifying, one obtains

$$\mathbf{F}(a,a,a) = q\mathbf{E}_{total}(a,a,a) = \frac{q}{4\pi\varepsilon_0 a^2}\left[\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{2}} + 1\right](\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}) = \frac{1.90q^2}{4\pi\varepsilon_0 a^2}(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}).$$

3-5. a) The **E** field at the point P is the sum of contributions due to +*q* and –*q*. Thus,

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{r}-\mathbf{d}/2}{|\mathbf{r}-\mathbf{d}/2|^3} - \frac{\mathbf{r}+\mathbf{d}/2}{|\mathbf{r}+\mathbf{d}/2|^3}\right].$$

b) If $d << r$, then the following approximation can be applied

$$|\mathbf{r}-\mathbf{d}/2|^{-3} = \left[R^2 - \mathbf{r}\cdot\mathbf{d} + d^2/4\right]^{-3/2}$$

$$\cong R^{-3}\left[1 - \frac{\mathbf{r}\cdot\mathbf{d}}{R^2}\right]^{-3/2} \quad \text{(Ignore the term } d^2/4\text{)}$$

$$\cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{r}\cdot\mathbf{d}}{R^2}\right] \quad \text{(Apply the binomial expansion and ignore higher - order terms)}$$

Likewise, $|\mathbf{r}+\mathbf{d}/2|^{-3} \cong R^{-3}\left[1 - \frac{3}{2}\frac{\mathbf{r}\cdot\mathbf{d}}{R^2}\right]$.

Using these two results in a) yields

$$\mathbf{E} \cong \frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{r}\cdot\mathbf{d}}{R^2}\mathbf{r} - \mathbf{d}\right] = \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{r}\cdot\mathbf{p}}{R^2}\mathbf{r} - \mathbf{p}\right] \text{ where } \mathbf{p} = q\mathbf{d}.$$