Homework #4 Solution

4-1. (a) Let the position vector be $\mathbf{r}'=\hat{\boldsymbol{\rho}}a=\hat{\mathbf{x}}a\cos\phi'+\hat{\mathbf{y}}a\sin\phi'$, then $\mathbf{R}=\hat{\mathbf{z}}h-\hat{\mathbf{x}}a\cos\phi'-\hat{\mathbf{y}}a\sin\phi'$ and the electric field intensity due to differential charge segment $\rho_\ell a d\phi'$ is given by

$$d\mathbf{E}=\frac{\rho_\ell a d\phi'}{4\pi\varepsilon_0}\frac{\hat{\mathbf{z}}h-\hat{\mathbf{x}}a\cos\phi'-\hat{\mathbf{y}}a\sin\phi'}{(h^2+a^2)^{3/2}},$$

and the total electric field intensity is given by $\mathbf{E}=\int d\mathbf{E}=\int_0^{2\pi}\frac{\rho_\ell a d\phi'}{4\pi\varepsilon_0}\frac{\hat{\mathbf{z}}h-\hat{\mathbf{x}}a\cos\phi'-\hat{\mathbf{y}}a\sin\phi'}{(h^2+a^2)^{3/2}}$.

Since $\int_0^{2\pi}\cos\phi' d\phi'=\int_0^{2\pi}\sin\phi' d\phi'=0$, it follows that

$$\mathbf{E}=\hat{\mathbf{z}}\frac{\rho_\ell a h}{2\varepsilon_0(h^2+a^2)^{3/2}}.$$

(b) Likewise, V can be found to be

$$V=\int_0^{2\pi}\frac{\rho_\ell a d\phi'}{4\pi\varepsilon_0(h^2+a^2)^{1/2}}=\frac{\rho_\ell a}{2\varepsilon_0(h^2+a^2)^{1/2}}.$$

(c) Replacing h with the variable z, then calculating $\mathbf{E}=-\nabla V$ yields

$$\mathbf{E}=\hat{\mathbf{z}}\frac{\rho_\ell a z}{2\varepsilon_0(z^2+a^2)^{3/2}}.$$

Then substituting $z=h$, one obtains the result given in (a).

4-2. (a) Given a continuous line charge distribution, V can be found to be

$$V=\int_{-L/2}^{L/2}\frac{\rho_\ell dz'}{4\pi\varepsilon_0 R}=\frac{2\rho_\ell}{4\pi\varepsilon_0}\int_0^{L/2}\frac{dz'}{\sqrt{\rho^2+z'^2}}=\frac{\rho_\ell}{2\pi\varepsilon_0}\ln\left[z'+\sqrt{\rho^2+z'^2}\right]_0^{L/2}$$

$$=\frac{\rho_\ell}{2\pi\varepsilon_0}\left\{\ln\left[\frac{L}{2}+\sqrt{\rho^2+\left(\frac{L}{2}\right)^2}\right]-\ln\rho\right\}$$

(b) Likewise, using Coulomb's Law, one obtains

$$\mathbf{E}=\hat{\boldsymbol{\rho}}E_\rho=\int_{-L/2}^{L/2}\hat{\boldsymbol{\rho}}\frac{\rho_\ell \rho dz'}{4\pi\varepsilon_0 R^3}=\hat{\boldsymbol{\rho}}\frac{2\rho_\ell}{4\pi\varepsilon_0}\int_0^{L/2}\frac{\rho dz'}{\left(\rho^2+z'^2\right)^{3/2}}=\hat{\boldsymbol{\rho}}\frac{\rho_\ell\rho}{2\pi\varepsilon_0}\frac{z'}{\rho^2\left(\rho^2+z'^2\right)^{1/2}}\Bigg|_0^{L/2}$$

$$=\hat{\boldsymbol{\rho}}\frac{\rho_\ell}{2\pi\varepsilon_0\rho}\frac{L/2}{\sqrt{\rho^2+(L/2)^2}}$$

(c) Using $\mathbf{E}=-\nabla V$, one can obtain the same result as follows:

$$\mathbf{E}=-\frac{\partial}{\partial\rho}\left(\frac{\rho_\ell}{2\pi\varepsilon_0}\left\{\ln\left[\frac{L}{2}+\sqrt{\rho^2+\left(\frac{L}{2}\right)^2}\right]-\ln\rho\right\}\right)$$

$$=-\frac{\rho_\ell}{2\pi\varepsilon_0}\left[\frac{\rho}{\sqrt{\rho^2+(L/2)^2}}\frac{1}{L/2+\sqrt{\rho^2+(L/2)^2}}-\frac{1}{\rho}\right]$$

$$=-\frac{\rho_\ell}{2\pi\varepsilon_0\sqrt{\rho^2+(L/2)^2}}\left[\frac{\rho^2-\sqrt{\rho^2+(L/2)^2}\left(L/2+\sqrt{\rho^2+(L/2)^2}\right)}{\rho\left(L/2+\sqrt{\rho^2+(L/2)^2}\right)}\right]$$

$$=-\frac{\rho_\ell}{2\pi\varepsilon_0\rho\sqrt{\rho^2+(L/2)^2}}\left[\frac{-(L/2)\sqrt{\rho^2+(L/2)^2}-(L/2)^4}{L/2+\sqrt{\rho^2+(L/2)^2}}\right]$$

$$=\frac{\rho_\ell(L/2)}{2\pi\varepsilon_0\rho\sqrt{\rho^2+(L/2)^2}}$$

(d) Take the limit as $L\to\infty$, then

$\lim_{L\to\infty}\frac{L/2}{\sqrt{\rho^2+(L/2)^2}}\to 1,\ \therefore \mathbf{E}\to\hat{\boldsymbol{\rho}}\frac{\rho_\ell}{2\pi\varepsilon_0\rho}$, which is the electric field intensity due to an infinitely long straight line with charge density ρ_ℓ (C/m).

4-3. (a) Using the result from problem 2 (a), with $\rho^2=(a/2)^2+h^2$, and $L=a$, then the contribution from each side of the square is given by

$$V_{side}=\frac{\rho_\ell}{2\pi\varepsilon_0}\left\{\ln\left[\frac{a}{2}+\sqrt{(a/2)^2+h^2+(a/2)^2}\right]-\frac{1}{2}\ln[(a/2)^2+h^2]\right\}$$

Summing up contributions from all sides yield

$$V=\frac{2\rho_\ell}{\pi\varepsilon_0}\left\{\ln\left[\frac{a}{2}+\sqrt{2(a/2)^2+h^2}\right]-\frac{1}{2}\ln[(a/2)^2+h^2]\right\}$$

(b) Again, using the result from problem 2(b), the contribution from the side whose center is located at (a/2,0,0) can be given by

$$\mathbf{E}_1=\left(-\hat{\mathbf{x}}\frac{a}{2}+\hat{\mathbf{z}}h\right)\frac{\rho_\ell}{2\pi\varepsilon_0 d^2}\frac{a/2}{\sqrt{d^2+(a/2)^2}};d^2=\left(\frac{a}{2}\right)^2+h^2.$$

Likewise, the contribution from the side whose center is located at (-a/2,0,0) is given by

$$\mathbf{E}_2=\left(\hat{\mathbf{x}}\frac{a}{2}+\hat{\mathbf{z}}h\right)\frac{\rho_\ell}{2\pi\varepsilon_0 d^2}\frac{a/2}{\sqrt{d^2+(L/2)^2}};d^2=\left(\frac{a}{2}\right)^2+h^2.$$

Hence, the contributions from two sides parallel to y-axis are given by

$$\mathbf{E}_y=\hat{\mathbf{z}}\frac{\rho_\ell h}{\pi\varepsilon_0 d^2}\frac{a/2}{\sqrt{d^2+(a/2)^2}};d^2=\left(\frac{a}{2}\right)^2+h^2.$$

Using the same approach, the contributions from two sides parallel to x-axis are given by

$$\mathbf{E}_x=\hat{\mathbf{z}}\frac{\rho_\ell h}{\pi\varepsilon_0 d^2}\frac{a/2}{\sqrt{d^2+(a/2)^2}};d^2=\left(\frac{a}{2}\right)^2+h^2$$

Hence, the electric field intensity due to this square line charge is given by

$$\mathbf{E} = \hat{\mathbf{z}}\frac{2\rho_\ell h}{\pi\varepsilon_0 d^2}\frac{a/2}{\sqrt{d^2+(a/2)^2}};d^2 = \left(\frac{a}{2}\right)^2 + h^2$$

(c) Replacing h with the variable z, i.e.,

$$V = \frac{2\rho_\ell}{\pi\varepsilon_0}\left\{\ln\left[\frac{a}{2}+\sqrt{2(a/2)^2+z^2}\right]-\frac{1}{2}\ln[(a/2)^2+z^2]\right\}$$

then calculating $\mathbf{E} = -\nabla V$ yields

$$\mathbf{E} = \hat{\mathbf{z}}\frac{2\rho_\ell z}{\pi\varepsilon_0\left[(a/2)^2+z^2\right]}\frac{a/2}{\sqrt{2(a/2)^2+z^2}}$$

Then substituting $z = h$, one obtains the result given in (b).

4-4. Due to the spherical symmetry, $\mathbf{E} = \hat{\mathbf{r}}E_r$ and the Gauss's Law can be applied.

$$\boxed{\oint_S \boldsymbol{E}\cdot d\boldsymbol{s} = \frac{Q}{\varepsilon_0}}$$

a) $0 \le r \le b$

$$4\pi r^2 E_{r1} = \frac{1}{\varepsilon_0}\int_V \rho_v dv = \frac{\rho_0}{\varepsilon_0}\int_0^r\left(1-\frac{t^2}{b^2}\right)4\pi t^2 dt = \frac{4\pi\rho_0}{\varepsilon_0}\left(\frac{r^3}{3}-\frac{r^5}{5b^2}\right)$$

Thus,

$$E_{r1} = \frac{\rho_0}{\varepsilon_0}r\left(\frac{1}{3}-\frac{r^2}{5b^2}\right)$$

b) $b \le r \le R_i$

$$4\pi r^2 E_{r2} = \frac{\rho_0}{\varepsilon_0}\int_0^b\left(1-\frac{t^2}{b^2}\right)4\pi t^2 dt = \frac{4\pi\rho_0}{\varepsilon_0}\left(\frac{b^3}{3}-\frac{b^5}{5b^2}\right) = \frac{8\pi\rho_0}{15\varepsilon_0}b^3$$

Thus,

$$E_{r2} = \frac{2\rho_0 b^3}{15\varepsilon_0 r^2}$$

c) $R_i \le r \le R_o$; $E_{r3} = 0$

d) $R_o < r$; $E_{r4} = \dfrac{2\rho_0 b^3}{15\varepsilon_0 r^2}$

4-5. (a) (i) $V = 0$ at infinity

$$V(r) = -\int \mathbf{E}\cdot d\boldsymbol{\ell} = -\int_\infty^r \frac{2tdt}{\left(t^2+a^2\right)^2} = \left.\frac{1}{t^2+a^2}\right|_\infty^r = \frac{1}{r^2+a^2}.$$

(ii) $V = 0$ at $r = 0$

$$V(r) = -\int_0^r \frac{2tdt}{\left(t^2+a^2\right)^2} = \left.\frac{1}{t^2+a^2}\right|_0^r = \frac{1}{r^2+a^2}-\frac{1}{a^2}.$$

(iii) $V = 0$ at $r = a$

$$V(r) = -\int_a^r \frac{2tdt}{\left(t^2+a^2\right)^2} = \left.\frac{1}{t^2+a^2}\right|_a^r = \frac{1}{r^2+a^2}-\frac{1}{2a^2}.$$

(b) $W = -\int q\mathbf{E}\cdot d\mathbf{l} = -\int_{(1,\pi/6,2\pi/3)}^{(4,\pi/2,\pi/3)} 10\times10^{-9}\,\frac{2rdr}{\left(r^2+a^2\right)^2} = \left.\frac{10\times10^{-9}}{r^2+a^2}\right|_1^4 = -\frac{150\times10^{-9}}{(16+a^2)(1+a^2)}\text{(J)}$

(c) $W = qV_2 - qV_1 = 10\times10^{-9}\left(\frac{1}{16+a^2}-\frac{1}{1+a^2}\right) = -\frac{150\times10^{-9}}{(16+a^2)(1+a^2)}\text{(J)}$