Homework #5 Solution

5-1. (a) $\rho_{ps} = \mathbf{P}\cdot\hat{\mathbf{a}}_n = P_0\dfrac{L}{2}$ on all six faces of the cube.

Likewise, $\rho_{pv} = \nabla\cdot\mathbf{P} = -3P_0$

(b) $Q_s = \rho_{ps}S = P_0\dfrac{L}{2}(6L^2) = 3P_0L^3$; $Q_v = \rho_{pv}V = -3P_0(L^3) = -3P_0L^3$.

Thus, $Q = Q_s + Q_v = 0$.

5-2. Let the region 1 denote the $z\geq 0$ region and region 2 denote the $z\leq 0$ region.

(a) Since the interface is normal to the z axis,

$E_{1n} = \mathbf{E}_1\cdot\hat{\mathbf{a}}_n = \mathbf{E}_1\cdot\hat{\mathbf{z}} = 3$.

Using the boundary condition $D_{1n} = D_{2n}$, one obtains

$$D_{2n} = \varepsilon_2 E_{2n} = D_{1n} = \varepsilon_1 E_{1n} \rightarrow E_{2n} = \frac{\varepsilon_1 E_{1n}}{\varepsilon_2} = \frac{\varepsilon_{r1}E_{1n}}{\varepsilon_{r2}} = 4.$$

Since $E_{1t} = E_{2t}$, the tangential component of $\mathbf{E}_2$ equals $\hat{\mathbf{x}}5 - \hat{\mathbf{y}}2$. Thus,

$\mathbf{E}_2 = \hat{\mathbf{x}}5 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}4\,(\text{kV/m})$.

(b) Let β_1,β_2 be the angles $\mathbf{E}_1$, $\mathbf{E}_2$ make with the interface (xy-plane) while α_1,α_2 are the angles they make with the normal to the interface (z-axis). Then,

$$\beta_1 = \frac{\pi}{2} - \alpha_1;\ \beta_2 = \frac{\pi}{2} - \alpha_2$$

Since $E_{1n} = 3;\ E_{1t} = \sqrt{5^2+2^2} = \sqrt{29}$,

$$\tan\alpha_1 = \frac{\sqrt{29}}{3} \rightarrow \alpha_1 = 60.88^\circ \rightarrow \beta_1 = 29.12^\circ.$$

Likewise, since $E_{2n} = 4;\ E_{2t} = \sqrt{5^2+2^2} = \sqrt{29}$,

$$\tan\alpha_2 = \frac{\sqrt{29}}{4} \rightarrow \alpha_2 = 53.40^\circ \rightarrow \beta_2 = 36.60^\circ.$$

(c) The energy density in each region can be found from

$$w_{e1} = \frac{1}{2}\varepsilon_1 E_1^2 = \frac{1}{2}4\varepsilon_0(5^2+2^2+3^2)\times 10^6 = 6.7290\times 10^{-4} = 672.90\,(\mu\text{J}),$$

$$w_{e2} = \frac{1}{2}\varepsilon_2 E_1^2 = \frac{1}{2}3\varepsilon_0(5^2+2^2+4^2)\times 10^6 = 5.9765\times 10^{-4} = 597.65\,(\mu\text{J}).$$

(d) At the center of the cube (3,4,-5) of side 2 m, z = -5 < 0; thus, the cube is in region 2. Hence, using the result from (c) yields

$W_e = 597.65\,(\mu\text{J})\times 2^3\,(\text{m}^3) = 4.78\,(\text{mJ})$.

5-3. Let ρ_s denote the surface charge density on the plate, then

$$E = \frac{D}{\varepsilon} = \frac{D}{\varepsilon_0\varepsilon_r} = \frac{\rho_s}{\varepsilon_0(2+2\times 10^6x^2)}.$$

The voltage between plates is then

$$V = \int_0^{10^{-3}}\frac{\rho_s dx}{\varepsilon_0(2+2\times 10^6x^2)} = \frac{\rho_s}{2\varepsilon_0}\int_0^{10^{-3}}\frac{dx}{1+10^6x^2} = \frac{\rho_s}{2\varepsilon_0\times 10^3}\int_0^1\frac{dt}{1+t^2} = \frac{\rho_s}{2\varepsilon_0\times 10^3}\tan^{-1}t\Big|_0^1 = \frac{\rho_s}{2\varepsilon_0\times 10^3}\frac{\pi}{4}$$

Now, $Q = \rho_s S = .02\rho_s$, hence

$$C=.02\rho_s\left(\frac{\rho_s}{2\varepsilon_0\times10^3}\frac{\pi}{4}\right)^{-1}=451\,\text{pF}.$$

5-4. Let S denote the area of the conducting plate.

In fig. (a), using the boundary condition (continuity of the normal component of **D**) and noticing that **D** in this case has only normal component, one obtains $\mathbf{D}_1=\mathbf{D}_2=\mathbf{D}$. Now, let $+Q$,$-Q$ denote the total charges on the top, bottom faces, respectively, then

$$\mathbf{D}=-\hat{\mathbf{y}}\rho_s=-\hat{\mathbf{y}}\frac{Q}{S}.$$

Hence, $\mathbf{E}_1=-\hat{\mathbf{y}}\dfrac{Q}{\varepsilon_0\varepsilon_{r1}S};\mathbf{E}_2=-\hat{\mathbf{y}}\dfrac{Q}{\varepsilon_0\varepsilon_{r2}S}$, and

$$V_1=-\int_{y=0}^{y=d/2}\boldsymbol{E}_1\cdot d\boldsymbol{l}=\int_{y=0}^{y=d/2}\frac{Q}{\varepsilon_0\varepsilon_{r1}S}dy=\frac{Qd}{2\varepsilon_0\varepsilon_{r1}S};V_2=-\int_{y=d/2}^{y=d}\boldsymbol{E}_2\cdot d\boldsymbol{l}=\int_{y=d/2}^{y=d}\frac{Q}{\varepsilon_0\varepsilon_{r2}S}dy=\frac{Qd}{2\varepsilon_0\varepsilon_{r2}S}$$

It follows that the total voltage $V=V_1+V_2$, and the capacitance is given by

$$C=\frac{Q}{V}=\frac{Q}{V_1+V_2}=\frac{1}{\dfrac{d}{2\varepsilon_0\varepsilon_{r1}S}+\dfrac{d}{2\varepsilon_0\varepsilon_{r2}S}}=\frac{1}{\dfrac{1}{C_1}+\dfrac{1}{C_2}},$$

where

$$C_1=\frac{\varepsilon_0\varepsilon_{r1}S}{d/2};C_2=\frac{\varepsilon_0\varepsilon_{r2}S}{d/2}.$$

This corresponds to the "series" capacitor case. (The calculation of capacitances here is similar to that for finding resistances of "parallel" resistors.)

In fig (b), let V denote the voltage between the bottom and top plates, then

$$\mathbf{E}_1=-\hat{\mathbf{y}}\frac{V}{d}=\mathbf{E}_2\ \text{and}\ \mathbf{D}_1=-\hat{\mathbf{y}}\varepsilon_0\varepsilon_{r1}\frac{V}{d};\mathbf{D}_2=-\hat{\mathbf{y}}\varepsilon_0\varepsilon_{r2}\frac{V}{d},$$

and the surface charge density becomes

$$\rho_{s1}=\varepsilon_0\varepsilon_{r1}\frac{V}{d};\rho_{s2}=\varepsilon_0\varepsilon_{r2}\frac{V}{d}.$$

Hence, the total charge on the conducting plate is given by

$$Q=\frac{S}{2}(\rho_{s1}+\rho_{s2})=\frac{S}{2}\left[\varepsilon_0\varepsilon_{r1}\frac{V}{d}+\varepsilon_0\varepsilon_{r2}\frac{V}{d}\right].$$

Therefore,

$$C=\frac{Q}{V}=\frac{S}{2}\left[\frac{\varepsilon_0\varepsilon_{r1}}{d}+\frac{\varepsilon_0\varepsilon_{r2}}{d}\right]=C_1+C_2$$

where

$$C_1=\frac{\varepsilon_0\varepsilon_{r1}S}{2d};C_2=\frac{\varepsilon_0\varepsilon_{r2}S}{2d}.$$

This corresponds to the "parallel" capacitor. (The calculation of capacitances here is similar to that for finding resistances of "series" resistors.)

Note that the capacitance of each component can be found from $C=\dfrac{\varepsilon S}{d}$.

5-5 (a) Since the dielectric medium is uniform, and neglecting the fringing field (meaning the conductors look like infinitely large sheets of charge), the electric field intensity must be uniform and the direction is from one conductor to the other. Since the electric potential between two conductors is given by V_0, and the separation is d, one obtains

$$\mathbf{E} = -\hat{\mathbf{z}}E_z = -\hat{\mathbf{z}}\frac{V_0}{d} \text{ (V/m)}.$$

(b) It follows that $\mathbf{D} = \varepsilon\mathbf{E} = -\hat{\mathbf{z}}\dfrac{\varepsilon V_0}{d}$ (C/m^2).

(c) From the boundary condition, the surface charge density is given by $\rho_s = D_n = \varepsilon V_0 / d$, thus $Q = \pi a^2 \rho_s = \pi a^2 \varepsilon V_0 / d$.

(d) Hence, the capacitance becomes $C = \dfrac{\varepsilon \pi a^2}{d}$, i.e., $C = \dfrac{\varepsilon S}{d}$ where S denotes the area, as before.

5-2 (a) Same as 4-1 (a), **E** must be uniform and thus $\mathbf{E} = -\hat{\mathbf{z}}E_z = -\hat{\mathbf{z}}V_0 / d$ (V/m).

(b) It follows that $\mathbf{D} = \varepsilon(\rho)\mathbf{E} = -\hat{\mathbf{z}}\dfrac{\varepsilon_0(1+\rho/a)V_0}{d}$ (C/m^2).

(c) Likewise, $\rho_s = D_n = \varepsilon_0(1+\rho/a)V_0/d$, thus

$$Q = \int_S \rho_s ds = 2\pi\frac{\varepsilon_0 V_0}{d}\int_0^a (1+\rho/a)\rho d\rho = \frac{5\varepsilon_0 \pi a^2 V_0}{3d}.$$

(d) The capacitance becomes $C = \dfrac{5\varepsilon_0 \pi a^2}{3d}$.

5-3 (a) Since ε has no variation in the direction of **E**, Laplace's equation can be applied. Thus,

$$\nabla^2 V = \frac{d^2V}{dz^2} = 0 \to V(z) = C_1 z + C_2.$$

Applying the boundary conditions:

$$V(z=0) = 0 \to C_2 = 0; V(z=d) = V_0 \to C_1 = \frac{V_0}{d},$$

one obtains

$$V(z) = \frac{V_0}{d} z \text{ (V)}.$$

Therefore, $\mathbf{E} = -\nabla V = -\hat{\mathbf{z}}\dfrac{dV}{dz} = -\hat{\mathbf{z}}\dfrac{V_0}{d}$ (V/m) as before.

(b)-(d) same as problem 4-2.

5-6. Assume that the point charge is located at the point (0,0,d), then applying the method of images, one can replace the infinite conducting ground plane by the "image" point charge -Q located at the point (0,0,-d). Thus, the electric field intensity on the xy-plane is given by

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R_+^3}(\hat{\boldsymbol{\rho}}\rho - \hat{\mathbf{z}}d) - \frac{Q}{4\pi\varepsilon_0 R_-^3}(\hat{\boldsymbol{\rho}}\rho + \hat{\mathbf{z}}d); \rho = \sqrt{x^2+y^2}; R = \sqrt{\rho^2+d^2}.$$

Since $R_+ = R_- = \sqrt{\rho^2 + d^2} = R$,

$$\mathbf{E} = -\hat{\mathbf{z}}\frac{Qd}{2\pi\varepsilon_0(\rho^2+d^2)^{3/2}}$$

The surface charge density is then given by

$$\mathbf{D} = \varepsilon_0 \mathbf{E}\big|_{y=0} = -\hat{\mathbf{z}} \frac{Qd}{2\pi(\rho^2 + d^2)^{3/2}} = \hat{\mathbf{z}}\rho_s \to \rho_s = -\frac{Qd}{2\pi(\rho^2 + d^2)^{3/2}} .$$

(b) The total charge can be found to be

$$Q = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{Qd}{2\pi(x^2 + y^2 + d^2)^{3/2}} dxdy = -\frac{Qd}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{1}{(\rho^2 + d^2)^{3/2}} \rho d\rho d\phi$$
$$= -\frac{Qd}{2\pi} 2\pi \int_0^{\infty} \frac{1}{(\rho^2 + d^2)^{3/2}} \rho d\rho = -Q ,$$

which has the "magnitude" as the point charge but with opposite sign.