Homework #7 Solution

1. Aligning the coaxial transmission line with the z-axis and applying the Ampere's law $\oint_C \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{s} = \mu_0 I_{enc}$ yields

$$B_\phi = \frac{\mu_0 I_{enc}}{2\pi\rho}$$

where ρ denotes the distance from the center of the cross section and I_{enc} is the current enclosed in the loop. Since the current distribution is symmetrical, one can separate into four regions as follows:

(a) $0 \le \rho \le a$ region: Since the current enclosed here is the current in the inner conductor,

$$\mathbf{J} = \mathbf{J}_1 = \hat{\mathbf{z}}\frac{I}{\pi a^2}; d\mathbf{s} = \hat{\mathbf{z}}\rho d\rho d\phi\,,\ I_{enc} = \frac{I\rho^2}{a^2}; B_\phi = \frac{\mu_0 I_{enc}}{2\pi\rho} = \frac{\mu_0 I\rho}{2\pi a^2}.$$

(b) $a \le \rho \le b$ region: Like case (a),

$$I_{enc} = I; B_\phi = \frac{\mu_0 I_{enc}}{2\pi\rho} = \frac{\mu_0 I}{2\pi\rho}.$$

(c) $b \le \rho \le b+t$ region: The current enclosed here is the sum of currents in the inner and outer conductors. The current in the outer conductor is given by

$$\mathbf{J}_2 = -\hat{\mathbf{z}}\frac{I}{\pi\left[(b+t)^2 - b^2\right]}, \text{ thus}$$

$$I_{enc} = I + \int_S \mathbf{J}_2 \cdot d\mathbf{s} = I - \frac{I}{\pi\left[(b+t)^2 - b^2\right]}\int_b^\rho 2\pi\rho d\rho = I - I\frac{\rho^2 - b^2}{\left[(b+t)^2 - b^2\right]} = I\left(1 - \frac{\rho^2 - b^2}{2bt + t^2}\right);$$

$$B_\phi = \frac{\mu_0 I_{enc}}{2\pi\rho} = \frac{\mu_0 I}{2\pi\rho}\left(1 - \frac{\rho^2 - b^2}{2bt + t^2}\right)$$

(d) $\rho \ge b+t$ region: Since the current enclosed here is zero,

$B_\phi = 0$.

2. (a) $d\mathbf{B}$ due to the current $Id\ell$ at the point $(0,0,z)$ is given by

$$d\mathbf{B} = \mu_0 \frac{Id\boldsymbol{\ell}\times\mathbf{R}}{4\pi R^3}; d\boldsymbol{\ell} = \hat{\mathbf{z}}dz; \mathbf{R} = \hat{\boldsymbol{\rho}}\rho - \hat{\mathbf{z}}z; d\boldsymbol{\ell}\times\mathbf{R} = \hat{\phi}\rho dz.$$

Thus,

$$\mathbf{B} = \hat{\phi}\mu_0\int_{h-L/2}^{h+L/2}\frac{I\rho dz}{4\pi R^3} = \hat{\phi}\frac{\mu_0 I\rho}{4\pi}\int_{h-L/2}^{h+L/2}\frac{dz}{(z^2+\rho^2)^{3/2}}.$$

Let $z = \rho\tan t; dz = \dfrac{\rho dt}{\cos^2 t}$, then

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{4\pi\rho}\int_{\tan^{-1}\frac{h-L/2}{\rho}}^{\tan^{-1}\frac{h+L/2}{\rho}}\cos t\,dt = \hat{\phi}\frac{\mu_0 I}{4\pi\rho}\left[\frac{h+L/2}{\sqrt{\rho^2+(h+L/2)^2}} - \frac{h-L/2}{\sqrt{\rho^2+(h-L/2)^2}}\right] \qquad (3.1)$$

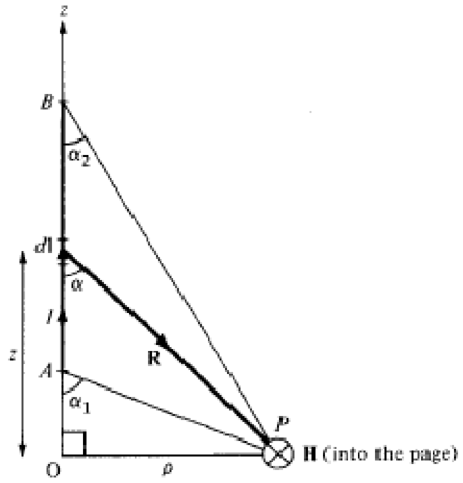
Define the angles α_1, α_2 as shown in the figure, then $\mathbf{B}$ can be rewritten as

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1) \qquad (3.2)$$

(b) When $L\to\infty$, the terms inside the parenthesis approach 2, thus

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{2\pi\rho}.$$

Clearly, it can also be calculated by letting $\alpha_1\to\pi$, $\alpha_2\to 0$. This represents the magnetic field due to an infinitely long straight current.



(c) Substituting h=0 in (3.1) yields

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{2\pi\rho}\frac{L/2}{\sqrt{\rho^2 + (L/2)^2}}.$$

3. The Biot-Savart law is given by

$$\mathbf{B} = \oint_{C'}\frac{\mu_0 I}{4\pi}\frac{d\ell'\times\mathbf{R}}{R^3} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\ell'\times\mathbf{R}}{R^3}.$$

$$d\ell' = (-\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi')bd\phi' = \hat{\boldsymbol{\phi}}'bd\phi';$$

$$\mathbf{R} = \mathbf{r} - \hat{\boldsymbol{\rho}}'b = \mathbf{r} - b(\hat{\mathbf{x}}\cos\phi' + \hat{\mathbf{y}}\sin\phi') = \hat{\mathbf{x}}(r\sin\theta\cos\phi - b\cos\phi') + \hat{\mathbf{y}}(r\sin\theta\sin\phi - b\sin\phi') + \hat{\mathbf{z}}r\cos\theta$$

$$\begin{aligned} d\ell'\times\mathbf{R} &= \hat{\mathbf{x}}rb\cos\theta\cos\phi' d\phi' + \hat{\mathbf{y}}rb\cos\theta\sin\phi' d\phi' \\ &\quad - \hat{\mathbf{z}}\left[(r\sin\theta\sin\phi - b\sin\phi')b\sin\phi' d\phi' + (r\sin\theta\cos\phi - b\cos\phi')b\cos\phi' d\phi'\right] \\ &= \hat{\mathbf{x}}rb\cos\theta\cos\phi' d\phi' + \hat{\mathbf{y}}rb\cos\theta\sin\phi' d\phi' \\ &\quad + \hat{\mathbf{z}}\left[bd\phi' - rb\sin\theta\sin\phi\sin\phi' d\phi' - rb\sin\theta\cos\phi\cos\phi' d\phi'\right] \end{aligned}$$

Since $r >> b$, the following approximation can be applied:

$$R^{-3} = \left[r^2 + b^2 - 2rb\sin\theta\cos(\phi-\phi')\right]^{-3/2} \approx r^{-3}\left[1 + \frac{3b}{r}\sin\theta\cos(\phi-\phi')\right].$$

Note that

$$\cos(\phi-\phi')\cos\phi' = \frac{1}{2}\left[\cos\phi + \cos(\phi - 2\phi')\right];\ \cos(\phi-\phi')\sin\phi' = \frac{1}{2}\left[\sin\phi - \sin(\phi-2\phi')\right].$$

Carrying out the integration for each component yields

x-component : $\frac{\mu_0 I}{4\pi}\frac{3b}{2r^4}2\pi rb\sin\theta\cos\theta\cos\phi = \mu_0 I\frac{3b^2}{4r^3}\sin\theta\cos\theta\cos\phi$

y-component: $\frac{\mu_0 I}{4\pi}\frac{3b}{2r^4}2\pi rb\sin\theta\cos\theta\sin\phi = \mu_0 I\frac{3b^2}{4r^3}\sin\theta\cos\theta\sin\phi$

z-component: $\frac{\mu_0 I}{4\pi}\left[\frac{2\pi b^2}{r^3} - \frac{3b}{2r^4}2\pi rb\sin^2\theta\right] = \mu_0 I\left(\frac{b^2}{2r^3} - \frac{3b^2}{4r^3}\sin^2\theta\right)$

Combining all components yields

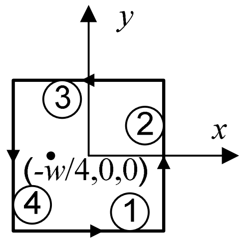
$$\mathbf{B} = \frac{\mu_0 I b^2}{4r^3}\left[\hat{\mathbf{x}}3\sin\theta\cos\theta\cos\phi + \hat{\mathbf{y}}3\sin\theta\cos\theta\sin\phi + \hat{\mathbf{z}}(2 - 3\sin^2\theta)\right]$$

Converting to spherical coordinates yields

$$\mathbf{B} = \frac{\mu_0 I b^2}{4r^3}\left[3(\hat{\mathbf{r}}\sin^2\theta\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\cos^2\theta) + (\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)(2 - 3\sin^2\theta)\right]$$

$$= \frac{\mu_0 I b^2}{4r^3}(\hat{\mathbf{r}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta)$$

which is the same as the result obtained before.



4. (a) Notice that the direction of **B** is +z. Assigning a number to each side of the loop as shown in the figure, then using the result from problem 3 of homework #7, one obtains **B** due to the sides 1 and 3 as follows:

$$B_1 = B_3 = \frac{\mu_0 I}{4\pi w/2}(\cos\alpha_2 - \cos\alpha_1) = \frac{\mu_0 I}{2\pi w}(\cos\alpha_2 - \cos\alpha_1)$$

where $\cos\alpha_2 = \frac{3w/4}{\sqrt{(3w/4)^2 + (w/2)^2}} = \frac{3}{\sqrt{13}}; \cos\alpha_1 = -\frac{w/4}{\sqrt{(w/4)^2 + (w/2)^2}} = \frac{1}{\sqrt{5}}$.

Thus, $B_1 = B_3 = \frac{\mu_0 I}{2\pi w}\left(\frac{3}{\sqrt{13}} + \frac{1}{\sqrt{5}}\right)$.

Likewise, **B** due to the side 2 is given by

$$B_2 = \frac{\mu_0 I}{4\pi(3w/4)}(\cos\alpha_4 - \cos\alpha_3) = \frac{\mu_0 I}{3\pi w}2\frac{w/2}{\sqrt{(w/2)^2 + (3w/4)^2}} = \frac{\mu_0 I}{3\pi w}\frac{4}{\sqrt{13}}$$

since $\cos\alpha_4 = \frac{w/2}{\sqrt{(w/2)^2 + (3w/4)^2}} = \frac{2}{\sqrt{13}} = -\cos\alpha_3$.

Finally, **B** due to the side 4 is given by

$$B_4 = \frac{\mu_0 I}{4\pi(w/4)}(\cos\alpha_6 - \cos\alpha_5) = \frac{\mu_0 I}{\pi w}2\frac{w/2}{\sqrt{(w/2)^2 + (w/4)^2}} = \frac{\mu_0 I}{\pi w}\frac{4}{\sqrt{5}}$$

since $\cos\alpha_6 = \frac{w/2}{\sqrt{(w/2)^2 + (w/4)^2}} = \frac{2}{\sqrt{5}}w = -\cos\alpha_5$.

Thus, the total magnetic flux density becomes

$$\mathbf{B} = \hat{\mathbf{z}}(B_1 + B_2 + B_3 + B_4) = \hat{\mathbf{z}}\frac{\mu_0 I}{\pi w}\left[2\times\frac{1}{2}\left(\frac{3}{\sqrt{13}} + \frac{1}{\sqrt{5}}\right) + \frac{4}{3\sqrt{13}} + \frac{4}{\sqrt{5}}\right] = \hat{\mathbf{z}}3.4379\frac{\mu_0 I}{\pi w}.$$

(b) Using the same procedure used to determine **B** due to a circular loop, i.e., first finding **A** and then determining **B** from **A**.

First, finding **A** due to the currents on sides 1 and 3:

$$\mathbf{A}_1 = \hat{\mathbf{x}}\frac{\mu_0 I}{4\pi}\int_{-w/2}^{w/2}\frac{dx'}{R_1}; R_1 = |\mathbf{R}_1| = |\mathbf{r}-\mathbf{r}_1'| = |\mathbf{r}-(\hat{\mathbf{x}}x' - \hat{\mathbf{y}}\frac{w}{2})|$$

$$\mathbf{A}_3 = -\hat{\mathbf{x}}\frac{\mu_0 I}{4\pi}\int_{-w/2}^{w/2}\frac{dx'}{R_3}; R_3 = |\mathbf{R}_3| = |\mathbf{r}-\mathbf{r}_3'| = |\mathbf{r}-(\hat{\mathbf{x}}x' + \hat{\mathbf{y}}\frac{w}{2})| \quad (1)$$

Since $\mathbf{r} = \hat{\mathbf{x}}r\sin\theta\cos\phi + \hat{\mathbf{y}}r\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$

$$\mathbf{R}_1 = \hat{\mathbf{x}}(r\sin\theta\cos\phi - x') + \hat{\mathbf{y}}(r\sin\theta\sin\phi + \frac{w}{2}) + \hat{\mathbf{z}}r\cos\theta .$$

If the point P is far away from the loop (R>>w), the following approximation can be applied:

$$\begin{aligned}\frac{1}{R_1} &= \left[(r\sin\theta\cos\phi - x')^2 + (r\sin\theta\sin\phi + \frac{w}{2})^2 + r^2\cos^2\theta\right]^{-1/2} \\ &\approx r^{-1}\left[\sin^2\theta\cos^2\phi - \frac{2x'}{r}\sin\theta\cos\phi + \sin^2\theta\sin^2\phi + \frac{w}{r}\sin\theta\sin\phi + \cos^2\theta\right]^{-1/2} \\ &= r^{-1}\left[1 - \frac{2x'}{r}\sin\theta\cos\phi + \frac{w}{r}\sin\theta\sin\phi\right]^{-1/2} \\ &\approx r^{-1}\left(1 + \frac{x'}{r}\sin\theta\cos\phi - \frac{w}{2r}\sin\theta\sin\phi\right)\end{aligned}$$

Likewise, $1/R_3$ can be approximated as

$$\frac{1}{R_3} \approx r^{-1}\left(1 + \frac{x'}{r}\sin\theta\cos\phi + \frac{w}{2r}\sin\theta\sin\phi\right).$$

Using the approximations given above yields

$$\begin{aligned}\mathbf{A}_1 + \mathbf{A}_3 &= \hat{\mathbf{x}}\frac{\mu_0 I}{4\pi}\int_{-w/2}^{w/2}\left(\frac{dx'}{R_1} - \frac{dx'}{R_3}\right) \\ &\approx -\hat{\mathbf{x}}\frac{\mu_0 I}{4\pi r}\int_{-w/2}^{w/2}\frac{w}{r}\sin\theta\sin\phi dx' = -\hat{\mathbf{x}}\frac{\mu_0 I w^2}{4\pi r^2}\sin\theta\sin\phi\end{aligned}$$

Using the same procedure to find **A** due to the currents on sides 2 and 4 yields

$$\mathbf{A}_2 = \hat{\mathbf{y}}\frac{\mu_0 I}{4\pi}\int_{-w/2}^{w/2}\frac{dy}{R_2}; R_1 = |\mathbf{R}_2| = |\mathbf{r}-\mathbf{r}_2'| = |\mathbf{r}-(\hat{\mathbf{x}}\frac{w}{2} + \hat{\mathbf{y}}y')|$$

$$\mathbf{A}_4 = -\hat{\mathbf{y}}\frac{\mu_0 I}{4\pi}\int_{-w/2}^{w/2}\frac{dy}{R_4}; R_3 = |\mathbf{R}_4| = |\mathbf{r}-\mathbf{r}_4'| = |\mathbf{r}-(-\hat{\mathbf{x}}\frac{w}{2} + \hat{\mathbf{y}}y')|$$

Then applying the same approximations, one obtains

$$\frac{1}{R_2} \approx r^{-1}\left(1 + \frac{w}{2r}\sin\theta\cos\phi + \frac{y'}{r}\sin\theta\sin\phi\right);$$

$$\frac{1}{R_4} \approx r^{-1}\left(1 - \frac{w}{2r}\sin\theta\cos\phi + \frac{y'}{r}\sin\theta\sin\phi\right)$$

Hence,

$$\mathbf{A}_2 + \mathbf{A}_4 = \hat{\mathbf{y}}\frac{\mu_0 I}{4\pi}\int_{-w/2}^{w/2}\left(\frac{dy'}{R_2} - \frac{dy'}{R_4}\right)$$

$$\approx \hat{\mathbf{y}}\frac{\mu_0 I}{4\pi r}\int_{-w/2}^{w/2}\frac{w}{r}\sin\theta\cos\phi dx' = \hat{\mathbf{y}}\frac{\mu_0 I w^2}{4\pi r^2}\sin\theta\cos\phi$$

Therefore,

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 = \frac{\mu_0 I w^2}{4\pi r^2}(-\hat{\mathbf{x}}\sin\theta\sin\phi + \hat{\mathbf{y}}\sin\theta\cos\phi) = \hat{\phi}\frac{\mu_0 I w^2}{4\pi r^2}\sin\theta .$$

Notice that if one defines a magnetic dipole $\mathbf{m} = \hat{\mathbf{z}}Iw^2 = \hat{\mathbf{z}}IS; S = w^2$, $\mathbf{A}$ can be rewritten as

$$\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \hat{\mathbf{r}}}{4\pi R^2},$$

which is the same as the *circular loop* case. Finally, determining **B** from **A** yields

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 I w^2}{4\pi r^3}(\hat{\mathbf{r}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta) = \frac{\mu_0 m}{4\pi r^3}(\hat{\mathbf{r}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta).$$

Note that if the distance is far away from the loop (or the loop is very small), the magnetic field due to the square loop is the same as that due to the circular loop.

5. Given $\mathbf{H}_1 = -\hat{\mathbf{x}}2 + \hat{\mathbf{y}}6 + \hat{\mathbf{z}}4$ (A/m) in the region $y - x - 2 \le 0$ where $\mu_1 = 5\mu_0, \mu_2 = 2\mu_0$.

Since $y - x - 2 = 0$ represents a *plane*, region 1 can be represented by $y - x \le 2$ or $y \le x + 2$. The vector normal to this plane is then given by $\hat{\mathbf{a}}_n = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}$, which can be found by defining the scalar field $f(x,y) = y - x - 2$ and then finding the normal vector from

$$\hat{\boldsymbol{a}}_n = \frac{\nabla f}{|\nabla f|} = \frac{-\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}}{\sqrt{2}}$$

since the gradient of a scalar field is in the normal direction.

(a) $\mathbf{M}_1 = \chi_{m1}\mathbf{H}_1 = (\mu_{r1} - 1)\mathbf{H}_1 = -\hat{\mathbf{x}}8 + \hat{\mathbf{y}}24 + \hat{\mathbf{z}}16$ (A/m)

$\mathbf{B}_1 = \mu_1\mathbf{H}_1 = 5\mu_0\mathbf{H}_1 = -\hat{\mathbf{x}}12.57 + \hat{\mathbf{y}}37.7 + \hat{\mathbf{z}}25.13$ (μT)

(b) $H_{1n} = \mathbf{H}_1 \cdot \hat{\mathbf{a}}_n = (-2,6,4)\cdot(-1,1,0)/\sqrt{2} = 4\sqrt{2}$, thus

$$\mathbf{H}_{1n} = \hat{\mathbf{a}}_n H_{1n} = 4\sqrt{2}(-1,1,0)/\sqrt{2} = -\hat{\mathbf{x}}4 + \hat{\mathbf{y}}4 .$$

Therefore, $\mathbf{H}_{1t} = \mathbf{H}_1 - \mathbf{H}_{1n} = (-2,6,4) - (-4,4) = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}4$.

Applying the boundary condition $\mathbf{H}_{1t} = \mathbf{H}_{2t}$ yields $\mathbf{H}_{2t} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}4$, and the boundary condition $\mathbf{B}_{1n} = \mathbf{B}_{2n}$ yields $\mathbf{H}_{2n} = \frac{\mu_1}{\mu_2}\mathbf{H}_{1n} = \frac{5}{2}(-\hat{\mathbf{x}}4 + \hat{\mathbf{y}}4) = -\hat{\mathbf{x}}10 + \hat{\mathbf{y}}10$.

Hence, $\mathbf{H}_2 = -\hat{\mathbf{x}}8 + \hat{\mathbf{y}}12 + \hat{\mathbf{z}}4$ (A/m) and

$$\mathbf{B}_2 = \mu_2\mathbf{H}_2 = 2\mu_0\mathbf{H}_2 = -\hat{\mathbf{x}}20.11 + \hat{\mathbf{y}}30.16 + \hat{\mathbf{z}}10.05\,(\mu\text{T}).$$