Homework #8 Solution

1. The magnetic energy stored in coaxial transmission lines can be divided into 2 parts, namely, the energy inside the conducting wires, and that between two conductors. The energy inside the conducting wires is given by

$$W'_{m1} = \frac{1}{2\mu_0}\int_0^a B_{\phi 1}^2 2\pi r dr = \frac{\mu_0 I^2}{4\pi a^4}\int_0^a r^3 dr = \frac{\mu_0 I^2}{16\pi}\ (\text{J/m}),$$

while the energy between the conductors is given by

$$W'_{m2} = \frac{1}{2\mu_0}\int_a^b B_{\phi 2}^2 2\pi r dr = \frac{\mu_0 I^2}{4\pi}\int_0^a \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}\ (\text{J/m}).$$

Therefore, the per-unit-length inductance of coaxial transmission lines becomes

$$L' = \frac{2}{I^2}(W'_{m1} + W'_{m2}) = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\frac{b}{a}\ (\text{H/m}).$$

2. Since $B = \mu NI/\ell = \mu_0\mu_r NI/2\pi b$,

$$\Phi = BS = \mu_0\mu_r NI\pi a^2/2\pi b = 0.251(\text{mWb}).$$

It can be found using the magnetic circuit approach. First, the reluctance is given by

$$\mathcal{R}^{-1} = \frac{\mu S}{\ell} = \frac{\mu_0\mu_r \pi a^2}{2\pi b}.$$

Then from $NI = \mathcal{R}\Phi$, one obtains $\Phi = NI/\mathcal{R} = \mu_0\mu_r NI\pi a^2/2\pi b = 0.251(\text{mWb})$.

3. (a) Let I be the current on the straight wire, then the magnetic flux density at a point P(r, θ) inside the circle can be given by

$$\mathbf{B}_P = \hat{\phi}\frac{\mu_0 I}{2\pi(d + r\cos\theta)}.$$

Thus, the flux linkage becomes

$$\Lambda_{12} = \frac{\mu_0 I}{2\pi}\int_0^{2\pi}\int_0^b \frac{r dr d\theta}{d + r\cos\theta} = \frac{\mu_0 I}{2\pi}\int_0^b \frac{2\pi r dr}{\sqrt{d^2 - r^2}} = \mu_0 I(d - \sqrt{d^2 - b^2}).$$

$$\left(\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}}\arctan\frac{\sqrt{a^2 - b^2}\tan\frac{x}{2}}{a + b}\quad [a^2 > b^2 \text{ case}], \text{ thus}\right.$$

$$\left.\int_0^{2\pi}\frac{d\theta}{d + r\cos\theta} = 2\int_0^{\pi}\frac{d\theta}{d + r\cos\theta} = 2\frac{2}{\sqrt{d^2 - r^2}}\arctan\frac{\sqrt{d^2 - r^2}\tan\frac{\theta}{2}}{d + r}\Bigg|_0^{\pi} = \frac{2\pi}{\sqrt{d^2 - r^2}}\right)$$

Hence, $L_{12} = \dfrac{\Lambda_{12}}{I} = \mu_0(d - \sqrt{d^2 - b^2})$.

(b) Let the straight wire aligned along the z-axis and the circular loop is placed in the xz plane. **B** due to the current I_1 at the segment $bd\theta$ on the loop is given by

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I_1}{2\pi(d + b\cos\theta)} = (-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi)\frac{\mu_0 I_1}{2\pi(d + b\cos\theta)} = \hat{\mathbf{y}}\frac{\mu_0 I_1}{2\pi(d + b\cos\theta)}\ \text{on xz-plane}.$$

Since $d\boldsymbol{\ell} = (-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta)bd\theta; d\boldsymbol{\ell}\times\mathbf{B} = (-\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta)bd\theta$, the force on the loop becomes

$$\mathbf{F} = I_2\int d\boldsymbol{\ell}\times\mathbf{B} = \frac{\mu_0 I_1 I_2 b}{2\pi}\int_0^{2\pi}\frac{-\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta}{d + b\cos\theta}d\theta.$$

$$F_x = -\frac{\mu_0 I_1 I_2 b}{2\pi}\int_0^{2\pi}\frac{\cos\theta}{d+b\cos\theta}d\theta = -\frac{\mu_0 I_1 I_2 b}{\pi}\int_0^{\pi}\frac{\cos\theta}{d+b\cos\theta}d\theta .$$

Using $\int \frac{A+B\cos x}{a+b\cos x}dx = \frac{B}{b}x + \frac{Ab-aB}{b}\int\frac{dx}{a+b\cos x}$ and the integral formula given in (a) yield

$$F_x = -\frac{\mu_0 I_1 I_2 b}{\pi}\left(\frac{1}{b}\pi - \frac{d}{b}\frac{\pi}{\sqrt{d^2-b^2}}\right) = -\mu_0 I_1 I_2\left(1-\frac{1}{\sqrt{1-(b/d)^2}}\right).$$

Likewise, since $\int_\pi^{2\pi}\frac{\sin\theta}{d+b\cos\theta}d\theta = \int_\pi^0\frac{-\sin\theta'}{d+b\cos\theta'}(-d\theta') = -\int_0^\pi\frac{\sin\theta}{d+b\cos\theta}d\theta$,

$$\int_0^{2\pi}\frac{\sin\theta}{d+b\cos\theta}d\theta = \int_0^{\pi}\frac{\sin\theta}{d+b\cos\theta}d\theta + \int_\pi^{2\pi}\frac{\sin\theta}{d+b\cos\theta}d\theta = 0 .$$

Thus, $F_z = 0$.

Therefore, $\mathbf{F} = \hat{\mathbf{x}}\mu_0 I_1 I_2\left(\frac{1}{\sqrt{1-(b/d)^2}}-1\right)$.

(c) Let the x-axis parallel to the ground. Dividing the loop into numeral magnetic dipole moments where each dipole moment is given by $d\mathbf{m} = I_2\, d\mathbf{s}$, then the torque due to each dipole moment and the total torque become

$$d\mathbf{T} = d\mathbf{m}\times\mathbf{B}; \mathbf{T} = \int d\mathbf{T} = I_2\int d\mathbf{s}\times\mathbf{B} = -\hat{\mathbf{x}}I_2\sin\alpha\int B ds = -\hat{\mathbf{x}}\mu_0 I_1 I_2\sin\alpha(d-\sqrt{d^2-b^2}) .$$

(refer to part (a))