# Magnetostatics

## 5-1 Overview

When a small test charge $q$ is placed in an electric field **E**, it experiences an *electric force* $\mathbf{F}_e$, which is a function of the position of $q$.

$\mathbf{F}_e = q\mathbf{E}$ (N)

When the test charge is in motion in a magnetic field characterized by a *magnetic flux density* **B**, experiments show that charge $q$ also experiences a *magnetic force* $\mathbf{F}_m$ given by

$\mathbf{F}_m = q\mathbf{u}\times\mathbf{B}$ (N),

where **u** (m/s) is the velocity of the moving charge, and **B** is measured in webers per square meter ($Wb/m^2$) or Teslas (T). The total *electromagnetic force* on a charge $q$ is then,

$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$ (N),

which is called ***Lorentz's force equation***. Recall that **E** can be defined as $\mathbf{F}_e/q$, one can define **B** as $\mathbf{u}\times\mathbf{B} = \mathbf{F}_m/q$.

Charges in motion produce a current that creates a magnetic field, i.e., *static magnetic fields* are created by *steady currents*, which can be regarded as *magnetic charges* (as analogous to *electric charges*).

## 5-2 Fundamental Postulates of Magnetostatics in Free Space

The two fundamental postulates specifying the divergence and the curl of **B** in *nonmagnetic media* (e.g., free space) are

$\nabla\cdot\mathbf{B} = 0$ (1) ; $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ (2),

where $\mu_0 = 4\pi\times10^{-7}$ (H/m). Note that

$\nabla\cdot(\nabla\times\mathbf{B}) = \mu_0\nabla\cdot\mathbf{J} = 0$, i.e., $\nabla\cdot\mathbf{J} = 0$,

which is consistent with the condition for steady currents. Taking the volume integral of (1) and applying the divergence theorem yield

$\oint_S \mathbf{B}\cdot d\mathbf{s} = 0$,

which implies that *there are no magnetic flow sources and the magnetic flux lines always close upon themselves*. This is also referred to as *the law of conservation of magnetic flux*.

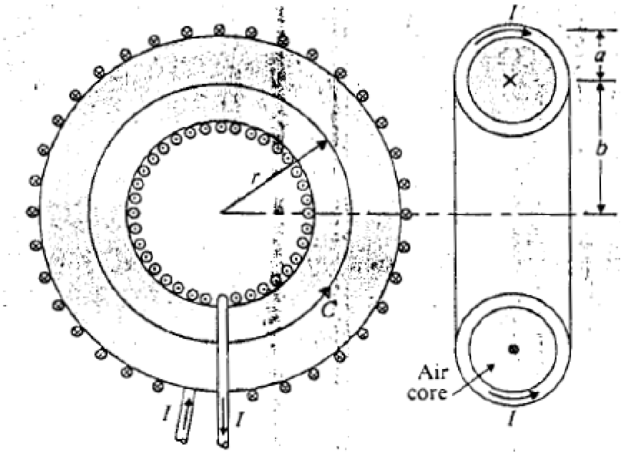
Taking the surface integral of (2) and applying Stokes' theorem yield

$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0\int_S \mathbf{J}\cdot d\mathbf{s} = \mu_0 I$ ,

which states that *the circulation of the magnetic flux density in a nonmagnetic medium around any closed path is equal to* $\mu_0$ *times the total current flowing through the surface bounded by the path.* This is a form of Ampere's circuital law.

<u>EX 5-1</u> An infinitely long, straight, solid, nonmagnetic conductor with a circular cross section of radius $b$ carries a steady current $I$. Determine the magnetic flux density both inside and outside the conductor.

EX 5-2 Determine the magnetic flux density inside a closely wound toroidal coil with an air core having $N$ turns of coil and carrying a current $I$. The toroid has a mean radius $b$, and the radius of each turn is $a$.




**5-3 Vector Magnetic Potential**

The divergence-free postulate of **B** assures that **B** is solenoidal. Thus, **B** can be rewritten as

$\mathbf{B} = \nabla \times \mathbf{A}$,

where **A** is called the *vector magnetic potential*. Its SI unit is Wb/m. Using **A** yields

$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}$.

Since $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$.

Choosing $\nabla \cdot \mathbf{A} = 0$, which is called *the Coulomb condition for* $\nabla \cdot \mathbf{A}$, yields

$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ : vector Poisson's equation

Recall that the solution to the Poisson's equation $\nabla^2 V = -\rho_v / \varepsilon_0$ is given by

$$V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho_v}{R} dv'.$$

Using this result, the solution to the vector Poisson's equation can be obtained as

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R} dv' \text{ (Wb/m)}.$$

Vector potential **A** is related to the magnetic flux $\Phi$ as

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l} \text{ (Wb)}.$$

**5-4 The Biot-Savart Law and Applications**

Here, the magnetic field due to a current-carrying circuit is of interest. For a thin wire with cross-section area $S$, $dv'$ equals $Sd\ell'$, and the current flow is entirely along the wire. It follows that $\mathbf{J}dv' = JSd\boldsymbol{\ell}' = Id\boldsymbol{\ell}'$, and the vector potential is given by

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R} \text{ (Wb/m)}.$$

The magnetic flux density **B** is then

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R} = \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \frac{d\boldsymbol{\ell}'}{R}.$$

Applying the vector identity $\nabla \times f\mathbf{A} = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A}$ yields

$$\nabla \times \frac{d\boldsymbol{\ell}'}{R} = \frac{1}{R} \nabla \times d\boldsymbol{\ell}' + \nabla \frac{1}{R} \times d\boldsymbol{\ell}' = \nabla \frac{1}{R} \times d\boldsymbol{\ell}'.$$

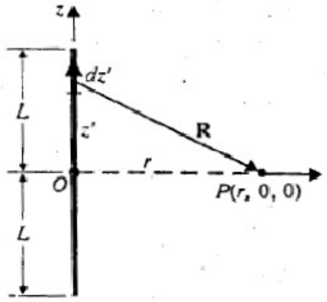
Since

$$\nabla\frac{1}{R} = -\frac{\mathbf{R}}{R^3} = -\frac{\hat{\mathbf{a}}_R}{R^2}, \mathbf{B} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\boldsymbol{\ell}'\times\mathbf{R}}{R^3} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\boldsymbol{\ell}'\times\hat{\mathbf{a}}_R}{R^2}.$$

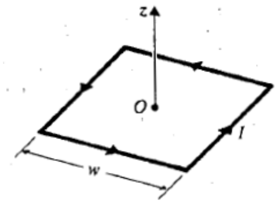
It can also be written as

$$\mathbf{B} = \oint_{C'} d\mathbf{B}; d\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{d\boldsymbol{\ell}'\times\mathbf{R}}{R^3} = \frac{\mu_0 I}{4\pi}\frac{d\boldsymbol{\ell}'\times\hat{\mathbf{a}}_R}{R^2}.$$

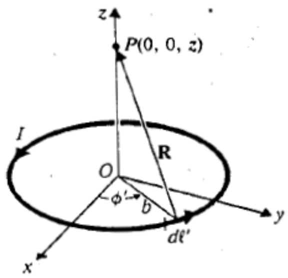
<u>EX 5-3</u> A direct current $I$ flows in a straight wire of length $2L$. Find the magnetic flux density **B** at a point located at a distance $r$ from the wire in the bisecting plane: (a) by determining the vector magnetic potential **A** first, and (b) by applying the Biot-Savart law.




<u>EX 5-4</u> Find the magnetic flux density at the center of a planar square loop, with side $w$ carrying a direct current $I$.

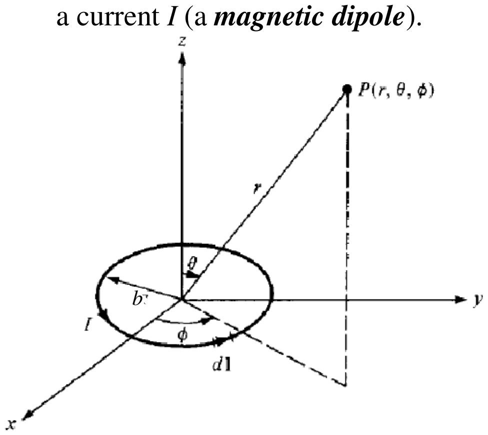



<u>EX 5-5</u> Find the magnetic flux density at a point on the axis of a circular loop of radius $b$ that carries a direct current $I$.




**5-5 The Magnetic Dipole**

<u>EX 5-6</u> Find the magnetic flux density at a distant point of a small circular loop of radius $b$ that carries a current $I$ (a ***magnetic dipole***).




The vector potential is given by

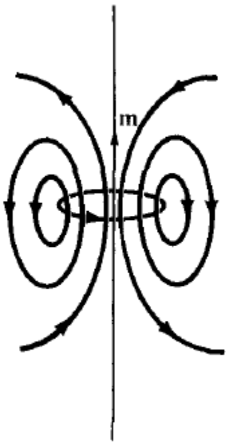
$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\int_C \frac{d\boldsymbol{\ell}}{R},$$

where $d\boldsymbol{\ell}' = (-\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi')bd\phi' = \hat{\boldsymbol{\phi}}' bd\phi'$, and

$R^2 = r^2 + b^2 - 2rb\sin\theta\cos(\phi-\phi') = r^2 + b^2 - 2rb\cos\psi$ since

$$b\hat{\boldsymbol{\rho}}'\cdot\mathbf{r} = br\hat{\boldsymbol{\rho}}'\cdot\hat{\mathbf{r}}$$
$$= br(\hat{\mathbf{x}}\cos\phi' + \hat{\mathbf{y}}\sin\phi')\cdot(\hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta)$$
$$= br\sin\theta\cos(\phi-\phi') = br\cos\psi$$

Thus, $\mathbf{A} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{(-\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi')bd\phi'}{\left[r^2 + b^2 - 2rb\sin\theta\cos(\phi-\phi')\right]^{1/2}}$.




For $r^2 >> b^2$,

$$\left[r^2 + b^2 - 2rb\cos\psi\right]^{-1/2} = \frac{1}{r}\left[1 + \frac{b^2}{r^2} - \frac{2b}{r}\cos\psi\right]^{-1/2} \approx \frac{1}{r}\left[1 - \frac{2b}{r}\cos\psi\right]^{-1/2}.$$

Now, using $(1-x)^{-1/2} \overset{x<<1}{\approx} 1 + x/2$ yields

$$\left[r^2 + b^2 - 2rb\cos\psi\right]^{-1/2} \approx \frac{1}{r}\left[1 + \frac{b}{r}\cos\psi\right].$$ Thus,

$$\mathbf{A} = \frac{\mu_0 Ib}{4\pi r}\int_0^{2\pi}(-\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi')\left[1 + b\sin\theta\cos(\phi - \phi')/r\right]d\phi'$$

$$= \frac{\mu_0 Ib^2\sin\theta}{4\pi r^2}\int_0^{2\pi}(-\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi')\cos(\phi - \phi')d\phi'$$

Since $\int_0^{2\pi}\sin\phi'\cos(\phi - \phi')d\phi' = \frac{1}{2}\int_0^{2\pi}[\sin\phi + \sin(2\phi' - \phi)]d\phi' = \pi\sin\phi$,

$\int_0^{2\pi}\cos\phi'\cos(\phi - \phi')d\phi' = \frac{1}{2}\int_0^{2\pi}[\cos\phi + \cos(2\phi' - \phi)]d\phi' = \pi\cos\phi$, and $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$,

$\mathbf{A} = \hat{\boldsymbol{\phi}}\frac{\mu_0 Ib^2\sin\theta}{4r^2} = \frac{\mu_0\mathbf{m}\times\hat{\mathbf{r}}}{4\pi r^2}$; $\mathbf{m} = \hat{\mathbf{z}}I\pi b^2 = \hat{\mathbf{z}}IS == \hat{\mathbf{z}}m$: magnetic dipole moment

Thus, $\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(\hat{\mathbf{r}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta)$. Note that $\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}(\hat{\mathbf{r}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta); \mathbf{p} = q\mathbf{d}$.

**5-6 Magnetization and Equivalent Current Densities**

Magnetic materials are those that exhibit magnetic polarization when they are subjected to an applied magnetic field. The magnetization phenomenon is represented by the alignment of the magnetic dipoles of the material with the applied magnetic field, similar to the alignment of the electric dipoles of the dielectric material with the applied electric field.

The commonly used atomic models represent the electrons as negative charges orbiting around the positive charged nucleus, as shown in the figure below. Each orbiting electron can be modeled by an equivalent small electric current loop, i.e., a magnetic dipole moment.

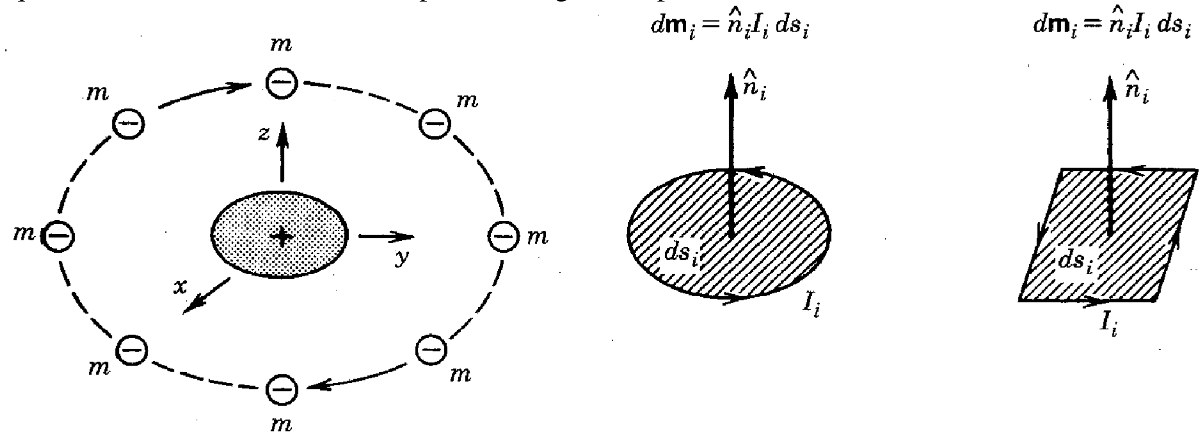



Figure Atomic models and their equivalents (Left) Orbiting electrons (Center) equivalent circular electric loop (Right) equivalent square electric loop

In the absence of external magnetic field, the directions of magnetic dipole moments are random resulting in no net magnetic moment, as shown below. The application of external magnetic field causes alignment of magnetic dipole moments into the same direction.

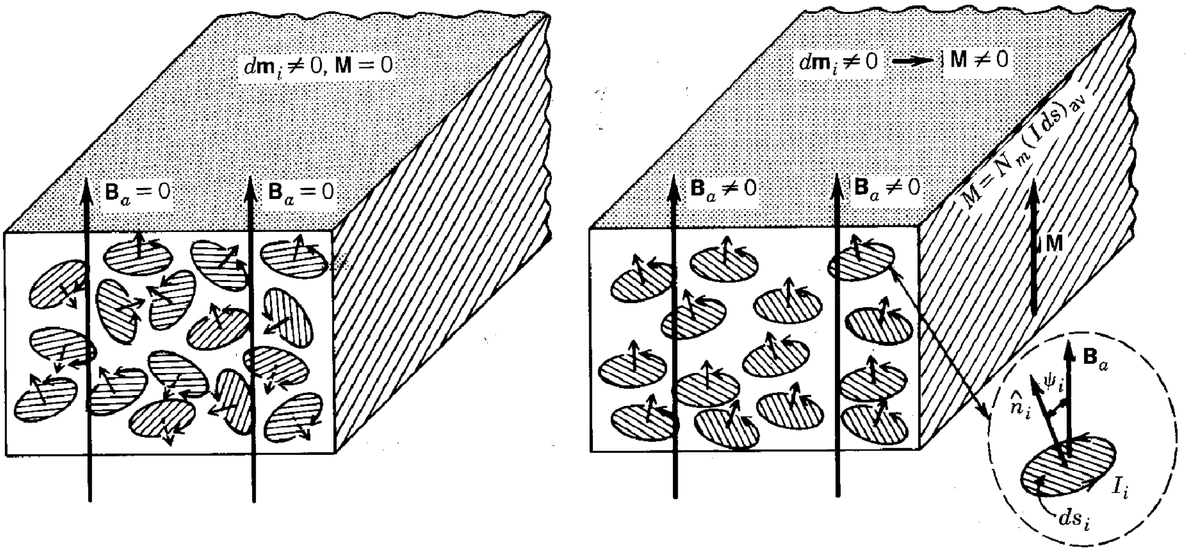



Figure Random orientation of magnetic dipoles and their alignment (Left) in the absence of and (Right) under an applied field.

Similar to the polarization vector, the magnetization vector is defined as

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N\Delta v} \mathbf{m}_k}{\Delta v} \qquad \text{(A/m)}$$

where $N$ is the number of atoms per unit volume and $\mathbf{m}_k$ is the dipole moment of the k[th] atom. The magnetization vector represents the volume density of magnetic dipole moments. Let $d\mathbf{m} = \mathbf{M}dv'$, then

$$d\mathbf{A} = \frac{\mu_0 \mathbf{M} \times \hat{\mathbf{R}}}{4\pi R^2} dv' = \frac{\mu_0}{4\pi} \mathbf{M} \times \nabla'\left(\frac{1}{R}\right) dv'$$

Thus,

$$\mathbf{A} = \int_{V'} d\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \mathbf{M} \times \nabla'\left(\frac{1}{R}\right) dv'.$$

Using $\nabla \times (V\mathbf{A}) = \nabla V \times \mathbf{A} + V\nabla \times \mathbf{A}$,

$$\mathbf{M} \times \nabla'\left(\frac{1}{R}\right) = \frac{1}{R}\nabla' \times \mathbf{M} - \nabla' \times \left(\frac{\mathbf{M}}{R}\right).$$

Therefore,

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{1}{R} \nabla' \times \mathbf{M} dv' - \frac{\mu_0}{4\pi} \int_{V'} \nabla' \times \left(\frac{\mathbf{M}}{R}\right) dv'.$$

Since $\int_{V'} \nabla' \times \mathbf{F} dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{s}' = -\oint_{S'} \mathbf{F} \times \hat{\mathbf{a}}_n ds'$,

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{1}{R} \nabla' \times \mathbf{M} dv' + \frac{\mu_0}{4\pi} \int_{S'} \frac{\mathbf{M}}{R} \times \hat{\mathbf{a}}_n ds'.$$

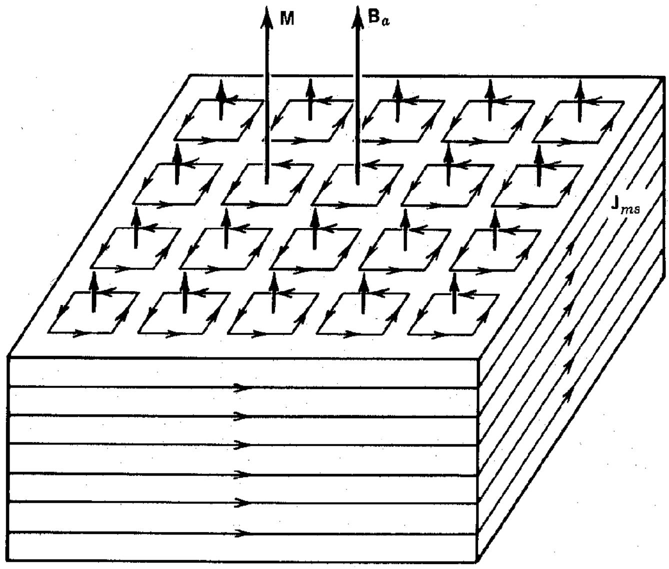
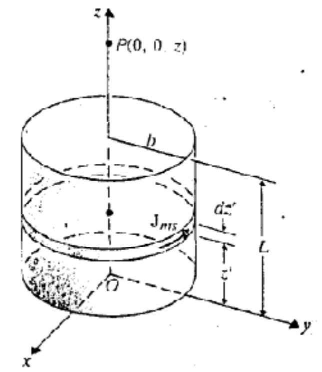



Figure The formation of magnetization current density

By comparison, the magnetization surface current density and magnetization volume current density are defined as

$\mathbf{J}_{ms} = \mathbf{M} \times \hat{\mathbf{a}}_n$ ; $\mathbf{J}_{mv} = \nabla' \times \mathbf{M}$, respectively. (The prime symbol can be omitted.)

<u>Ex 5-7</u> Determine the magnetic flux density on the z-axis of a uniformly magnetized circular cylinder of a magnetic material. The cylinder has a radius $b$, length $L$, and axial magnetization $\mathbf{M} = \hat{\mathbf{z}} M_0$.




**5-7 Magnetic Field Intensity and Relative Permeability**

Accounting for the existence of $\mathbf{J}_{mv}$, the curl equation of **B** can be modified as

$\nabla\times\mathbf{B} = \mu_0(\mathbf{J}+\mathbf{J}_{mv}) = \mu_0(\mathbf{J}+\nabla\times\mathbf{M})$ or

$$\nabla\times\left(\frac{\mathbf{B}}{\mu_0}-\mathbf{M}\right)=\mathbf{J}.$$

Then, the magnetic field intensity **H** can be defined as

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}-\mathbf{M}\ \text{(A/m)}.$$

The curl equation can be rewritten as $\nabla\times\mathbf{H}=\mathbf{J}$ (A/m$^2$), and the Ampere's circuital law becomes

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I\ \text{(A)}.$$

In the same manner as defining the electric susceptibility, the magnetic susceptibility $\chi_m$ can be defined as

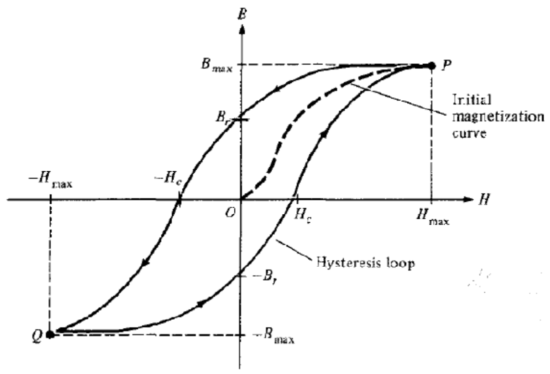
$\mathbf{M}=\chi_m\mathbf{H}$. Then

$$\mathbf{B}=\mu_0(1+\chi_m)\mathbf{H}=\mu_0\mu_r\mathbf{H}=\mu\mathbf{H};\mu=\mu_0\mu_r;\mu_r=\mu/\mu_0=1+\chi_m$$

where $\mu_r$ is the relative permeability.

Magnetic Material Magnetic materials are categorized as follows:

- *Diamagnetism* $\chi_m < 0$, $\mu_r \approx 1$ : Copper, lead
- *Paramagnetism* $\chi_m > 0$, $\mu_r \approx 1$ : Tungsten
- *Ferromagnetism* $\chi_m >> 0$, $\mu_r >> 1$ : Iron
- *Ferrimagnetism* $\chi_m >> 0$, $\mu_r >> 1$ : Ferrite




Hysteresis The relationship between **B** and **H** depends on the previous magnetization of the material —"magnetic history". Instead of having a simply linear relationship, it is only possible to represent it by a *magnetization curve* or **B**-**H** curve. The figure in the right shows a typical **B**-**H** curve. Assume that the material is initially *unmagnetized*, as $H$ increases from point O to point P, $\boldsymbol{B}$ increases from 0 to reach the saturation; this process is represented by the *initial magnetization curve*. After that, if $H$ is decreased, $B$ does not follow the initial curve but "lags behind" $H$, which is the meaning of the Greek word "hysteresis". If $H$ is reduced to zero, $B$ becomes $B_r$, which is called *the permanent flux density* or *the remnant flux density*. The value of $B_r$ depends on $H_{max}$ and its existence is the cause of having a permanent magnet. $B$ becomes zero when $H$ is reduced to $H_c$, which is called the *coercive field intensity*. The materials whose $H_c$ is small is categorized as "magnetically hard". Further decrease in $H$ to reach point Q and increasing $H$ again to reach point P creates a *hysteresis* *loop*, which varies from one material to another. The area of this loop represents the *hysteresis loss*.

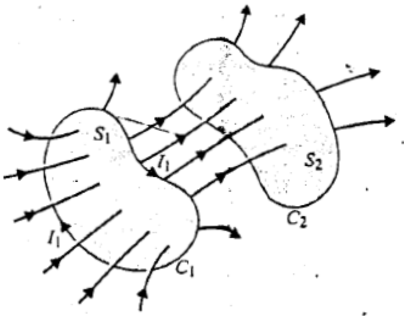
**5-9 Boundary Conditions for Magnetostatic Fields**

Using the same approaches as before, the boundary conditions for magnetic fields can be found to be

$B_{1n} = B_{2n}$ ; $H_{1t} - H_{2t} = J_{sn}$ or $\hat{\mathbf{a}}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$

In most materials (except conductors), $H_{1t}$=$H_{2t}$.

**5-10 Inductances and Inductors**




Consider two circuits $C_1$, $C_2$ with $C_1$ carries a current $I_1$. Some of magnetic flux due to $I_1$ will pass through $C_2$. This is called the mutual flux and is denoted by $\Phi_{12}$, which is given by

$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2$ (Wb).

From Biot-Savart law, **B** is proportional to *I*, and $\Phi_{12}$ is thus proportional to $I_1$. Therefore, it can be rewritten as $\Phi_{12} = L_{12} I_1$ (Wb), where $L_{12}$ is called the *mutual inductance* between $C_1$ and $C_2$. In case $C_2$ has $N_2$ turns, the flux linkage $\Lambda_{12}$ due to $\Phi_{12}$ is $\Lambda_{12} = N_2 \Phi_{12}$, thus

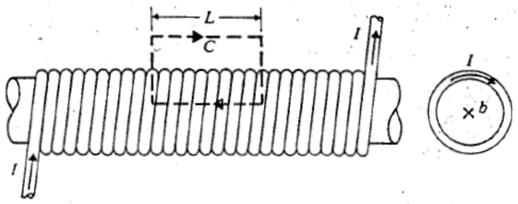
$\Lambda_{12} = L_{12} I_1$, and the mutual inductance can be generalized as

$$\boxed{L_{12} = \frac{\Lambda_{12}}{I_1}} \text{ (H).}$$

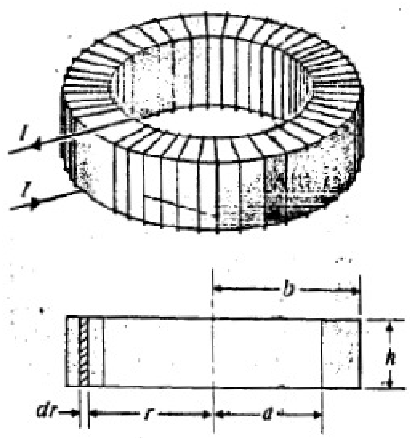
Likewise, the *self inductance* of loop $C_1$ is defined as *the magnetic flux linkage per unit current in the loop itself*, i.e.,

$$\boxed{L_{11} = \frac{\Lambda_{11}}{I_1}} \text{ (H).}$$

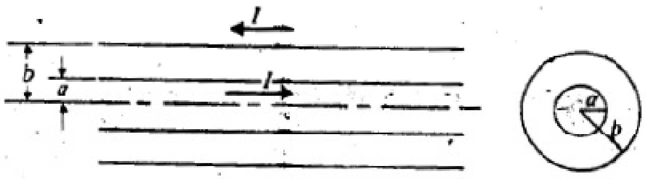
<u>EX 5-9</u> Find the inductance per unit length of a very long solenoid with air core having *n* turns per unit length.




<u>EX 5-8</u> Assume *N* turns of wire are tightly wound on a toroidal frame of a rectangular cross section with dimensions as shown below. Then assuming the permeability of the medium to be $\mu_0$, find the self-inductance of the toroidal coil.




EX 5-10 An air coaxial transmission line has a solid inner conductor of radius *a* and a very thin outer conductor of inner radius *b*. Determine the inductance per unit length of the line.




EX 5-11 Determine the inductance per unit length of two parallel conducting wires of radius *a*, separated by *d* with $d >> a$.

EX 5-12 Determine the mutual inductance between a conducting rectangular loop of size $w \times h$ and a very long straight wire, separated by $d$.

**5-11 Magnetic Energy**

Consider a single closed loop with a self-inductance $L_1$ in which the current $i_1$ increases from zero to $I_1$. An electromotive force (emf) will be induced in the loop that opposes the current change, and the work must be done to overcome this induced emf. Let $v_1 = L_1 di_1/dt$ be the voltage across the inductance, then the work required is

$$W_1 = \int v_1 i_1 dt = L_1 \int \frac{di_1}{dt} i_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2 ,$$

which is stored as *magnetic energy*. Next, if the second loop is introduced near the first loop and its current $i_2$ is increased from zero to $I_2$, then the work involved is

$$W_{21} = \int v_{21} I_1 dt = \pm L_{21} I_1 \int \frac{di_2}{dt} dt = \pm L_{21} I_1 \int_0^{I_2} di_2 = \pm L_{21} I_1 I_2 ,$$

where $L_{21}$ denotes the mutual inductance between the two loops. The ± sign depends on the direction of $\mathbf{B}_1$, $\mathbf{B}_2$ ($I_1$, $I_2$). Likewise the work related to the self-inductance of the second loop is $W_2 = L_2 I_2^2 / 2$. Hence, the total work involved in changing ($i_1$, $i_2$) from (0,0) to ($I_1$, $I_2$) is

$$W_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm L_{21} I_1 I_2 .$$

For a single loop carrying a current $I$ and has inductance $L$, the stored magnetic energy simply becomes

$$W_m = \frac{1}{2} L I^2 \text{ (J).}$$

The above result can be generalized for $N$ loops carrying currents $I_1, \ldots, I_N$, respectively, to be

$$W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k \text{ (J)}$$

Note that in linear media, the flux linkage $\Phi_k$ is given by

$$\Phi_k = \sum_{j=1}^{N} L_{jk} I_j ,$$

thus the total magnetic energy can be written as

$$W_m = \sum_{k=1}^{N} \Phi_k I_k \text{ (J).}$$

For continuous current elements, the flux linkage $\Phi_k$ is given by

$$\Phi_k = \int_{S_k'} \mathbf{B} \, \hat{\mathbf{a}}_n ds_k' = \oint_{C_k} \mathbf{A} \cdot d\mathbf{l}'_k \text{, thus}$$

$$W_m = \frac{1}{2} \sum_{k=1}^{N} \Delta I_k \oint_{C_k} \mathbf{A} \cdot d\mathbf{l}'_k .$$

Since $\Delta I_k d\mathbf{l}'_k = J\Delta a_k d\mathbf{l}'_k = \mathbf{J}\Delta v'_k$, then taking the limit as $N\to\infty$ yields

$$W_m = \frac{1}{2}\int_{V'}\mathbf{A}\cdot\mathbf{J}dv' \text{ (J).}$$

Since $\mathbf{B} = \nabla\times\mathbf{A}$ and the vector identity $\nabla\cdot(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{H}$,

$\mathbf{A}\cdot\mathbf{J} = \mathbf{A}\cdot\nabla\times\mathbf{H} = \mathbf{H}\cdot\nabla\times\mathbf{A} - \nabla\cdot(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot\mathbf{B} - \nabla\cdot(\mathbf{A}\times\mathbf{H})$. Thus,

$$W_m = \frac{1}{2}\int_{V'}\mathbf{A}\cdot\mathbf{J}dv' = \frac{1}{2}\int_{V'}\mathbf{H}\cdot\mathbf{B}dv' - \frac{1}{2}\int_{V'}\nabla\cdot(\mathbf{A}\times\mathbf{H})dv' = \frac{1}{2}\int_{V'}\mathbf{H}\cdot\mathbf{B}dv' - \frac{1}{2}\oint_{S'}\mathbf{A}\times\mathbf{H}\cdot d\mathbf{s}'.$$

At very far points, the second integral vanishes because |A|,|**H**| fall off as $1/r, 1/r^2$, respectively and the integrand goes to 0 as $r\to\infty$. Hence,

$$W_m = \frac{1}{2}\int_{V'}\mathbf{H}\cdot\mathbf{B}dv' = \int_{V'} w_m dv' \text{ (J); } w_m = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} \text{ (J/m}^3\text{)}.$$

In linear, isotropic media, $w_m = \mu H^2/2$.

EX 5-13 By using stored magnetic energy, determine the inductance per unit length of an air coaxial transmission line that has a solid inner conductor of radius *a* and a very thin outer conductor of inner radius *b*.

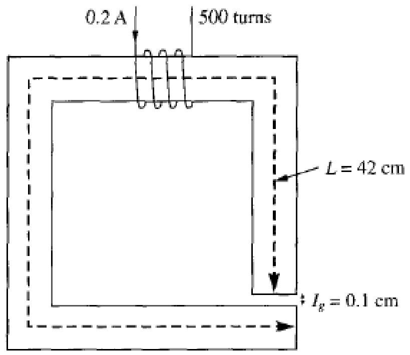
**Introduction to Magnetic Circuits**

Magnetic circuits can be defined analogous to electric circuits as follows:

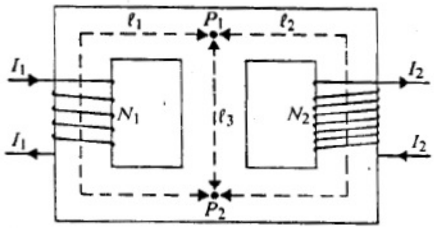
| | Electric Circuit | Magnetic Circuit |
|---|---|---|
| Potential | Electromotive force (emf)<br>$\mathbf{E} = -\nabla V$;<br>$V_{21} = -\int_{P_1}^{P_2}\mathbf{E}\cdot d\mathbf{l}$ (V) | magnetomotive force (mmf)<br>$\mathbf{H} = -\nabla V_m$;<br>$V_{m,21} = -\int_{P_1}^{P_2}\mathbf{H}\cdot d\mathbf{l}$ (A·t) |
| Ohm's law | $\mathbf{J} = \sigma\mathbf{E}$ (A/m$^2$) | $\mathbf{B} = \mu\mathbf{H}$ (Wb/m$^2$) |
| Total Current | $I = \int_S \mathbf{J}\cdot d\mathbf{s}$ (A) | $\Phi = \int_S \mathbf{B}\cdot d\mathbf{s}$ (Wb) |
| (Circuit)Ohm's law | $V = IR$ | $V_m = \Phi\mathcal{R}$ |
| Resistance | $R = \frac{\ell}{\sigma S}$ ($\Omega$)<br>Conductance $G = 1/R$ | $\mathcal{R} = \frac{\ell}{\mu S}$ (Reluctance, A·t/Wb)<br>Permeance $\mathcal{P} = 1/\mathcal{R}$ |
| Governing Equations | Kirchhoff's voltage law:<br>$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0$; $\sum_j V_j = \sum_k R_k I_k$<br>Kirchhoff's current law:<br>$\sum_k I_k = 0$ ($\nabla\cdot\mathbf{J} = 0$) | Ampere's law:<br>$\oint_C \mathbf{H}\cdot d\mathbf{l} = NI$; $\sum_j N_j I_j = \sum_k \mathcal{R}_k\Phi_k$<br>Conservation of magnetic flux:<br>$\sum_k \Phi_k = 0$ ($\nabla\cdot\mathbf{B} = 0$) |

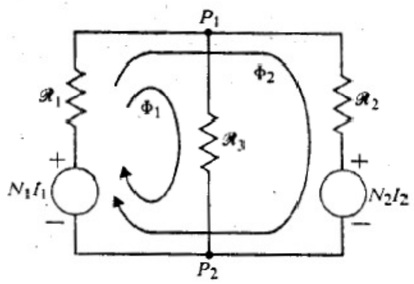
Ex Given an air-core toroid with 500 turns, a cross-sectional area of 6 cm$^2$, a mean radius of 15 cm, and a coil current of 4 A. Find $\Phi$, **B**, **H**.

Ex Consider the magnetic circuit shown below. Assuming that the core ($\mu = 1000\mu_0$) has a uniform cross section of 4 cm$^2$, determine the flux density in the air gap.




Ex Consider the magnetic circuit shown below. Steady currents $I_1$ and $I_2$ flow in windings of, respectively, $N_1$ and $N_2$ turns on the outside legs of the core. The core has a cross-sectional area $S_c$ and a permeability $\mu$. Determine the magnetic flux in the center leg.




**5-12 Magnetic Forces and Torques**

5-12.1 Forces and Torques on Current Carrying Conductors

Recall that $\mathbf{F}_m = q\mathbf{u}\times\mathbf{B}$ (N), now consider an element of conductor $dl$ with a cross-sectional area $S$. If there are $N$ charge carriers per unit volume moving with a velocity $\mathbf{u}$ in the directional of $dl$, then the magnetic force on the differential element is

$$d\mathbf{F}_m = Nq_1S\,|\,d\mathbf{l}\,|\,\mathbf{u}\times\mathbf{B} = Nq_1S\,|\,\mathbf{u}\,|\,d\mathbf{l}\times\mathbf{B} = Id\mathbf{l}\times\mathbf{B} \because I = Nq_1S\,|\,\mathbf{u}\,|.$$

Thus, the magnetic force on the complete circuit of contour $C$ carrying $I$ is given by

$\mathbf{F}_m = \oint_C d\mathbf{F}_m = I\oint_C d\mathbf{l}\times\mathbf{B}$ (N).

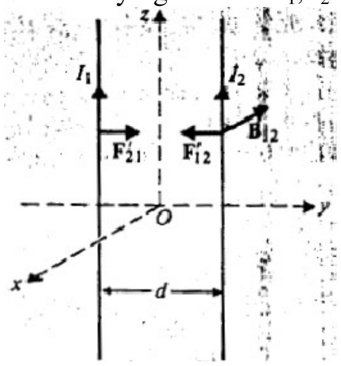
For two circuits with contours $C_1$,$C_2$ carrying currents $I_1$, $I_2$, let $\mathbf{B}_{12}$ be the magnetic field due to $I_1$ in $C_1$ at $C_2$, then the force $\mathbf{F}_{12}$ on circuit $C_2$ can be written as

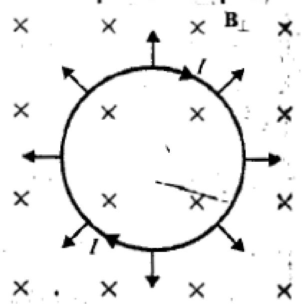
$\mathbf{F}_{12} = I_2\oint_C d\mathbf{l}_2\times\mathbf{B}_{12}$ . Since from Biot-Savart law, $\mathbf{B}_{12} = \frac{\mu_0 I_1}{4\pi}\oint_{C_1}\frac{d\mathbf{l}_1\times\hat{\mathbf{a}}_{R_{12}}}{R_{12}^2}$ ,

$$\mathbf{F}_{12} = I_2\oint_{C_2} d\mathbf{l}_2\times\frac{\mu_0 I_1}{4\pi}\oint_{C_1}\frac{d\mathbf{l}_1\times\hat{\mathbf{a}}_{R_{12}}}{R_{12}^2} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_{C_2}\oint_{C_1}\frac{d\mathbf{l}_2\times d\mathbf{l}_1\times\hat{\mathbf{a}}_{R_{12}}}{R_{12}^2} \text{ (N)}$$

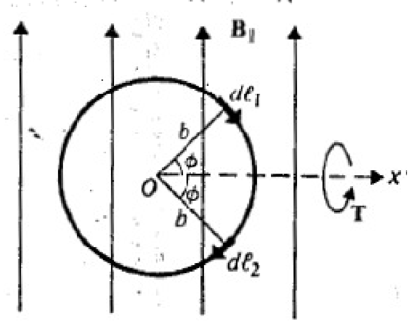
which is Ampere's law of force. From Newton's law of reaction, $\mathbf{F}_{12}$=-$\mathbf{F}_{21}$.

EX 5-14 Determine the force per unit length between two infinitely long, thin, parallel conducting wires carrying currents $I_1$, $I_2$ in the same direction. The wires are separated by a distance $d$.




Now, consider a small circular loop of radius $b$ and carrying a current $I$ in a uniform magnetic flux density $\mathbf{B}$. It is convenient to decompose $\mathbf{B}$ into $\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_\parallel$ as shown in figures below. The perpendicular component tends to expand the loop (or contract if $I$ is inversed). The parallel component produces an upward force $d\mathbf{F}_1$ (out of paper) on element $d\ell_1$ and a downward force $d\mathbf{F}_2$ = -$d\mathbf{F}_1$ on the symmetrically located element $d\ell_2$. Although the net force




is zero, a torque exists that tends to rotate the loop about the x axis in such a way as to align the magnetic field (due to $I$) with the external $\mathbf{B}$. The differential torque produced by $d\mathbf{F}_1$, $d\mathbf{F}_2$ is

$$d\mathbf{T} = \hat{\mathbf{x}}(dF)2b\sin\phi = \hat{\mathbf{x}}(Id\ell B_\parallel\sin\phi)2b\sin\phi$$
$$= \hat{\mathbf{x}}2Ib^2B_\parallel\sin^2\phi d\phi$$

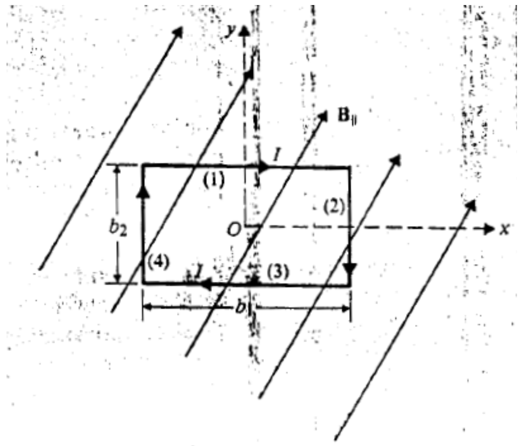
where $dF$=|$d\mathbf{F}_1$|=|$d\mathbf{F}_2$| and $d\ell$=|$d\ell_1$|=|$d\ell_2$|=$bd\phi$.

The total torque acting on the loop is then

$$\mathbf{T} = \int d\mathbf{T} = \hat{\mathbf{x}}2Ib^2B_\parallel\int_0^\pi\sin^2\phi d\phi = \hat{\mathbf{x}}I\pi b^2B_\parallel \text{ (N}\cdot\text{m)}.$$

Using $\mathbf{m} = \hat{\mathbf{a}}_n I\pi b^2 = \hat{\mathbf{a}}_n IS$, where $\hat{\mathbf{a}}_n$ is a unit vector in the direction normal to the plane of the loop, $\mathbf{T}$ can be rewritten as $\mathbf{T} = \mathbf{m}\times\mathbf{B}$ (N·m).

EX 5-15 A rectangular loop in the $xy$-plane with sides $b_1$,$b_2$ carrying a current $I$ lies in a uniform magnetic field $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$. Determine the force and torque on the loop.




5-12.3 Forces and Torques in terms of Stored Magnetic Energy

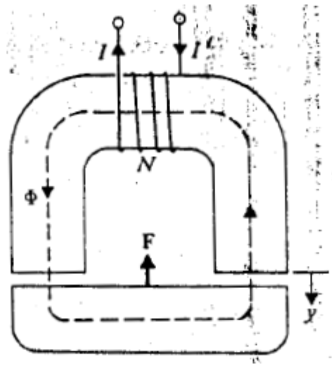
Using *the principle of virtual displacement*, the mechanical work $\mathbf{F}_\Phi \cdot d\boldsymbol{\ell}$ done by the system is at the expense of a decrease in the stored magnetic energy, $W_m$. ($\mathbf{F}_\Phi$ denotes the force under the constant-flux condition.) Thus,

$$\mathbf{F}_\Phi \cdot d\bar{\ell} = -dW_m = -\nabla W_m \cdot d\bar{\ell} \rightarrow \mathbf{F}_\Phi - \nabla W_m \text{ (N)}.$$

If the circuit is constrained to rotate about an axis, e.g., the z-axis, the mechanical work done by the system will be $(\mathbf{T}_\Phi)_z d\phi$, and

$$\left(\mathbf{T}_\Phi\right)_z = -\frac{\partial W_m}{\partial \phi} \text{ (N·m)}.$$

EX 5-16 Consider the electromagnet in which a current $I$ in an $N$-turn coil produces a flux Φ in the magnetic circuit. The cross-sectional area of the core is $S$. Determine the lifting force on the armature.




**Electrostatics-Magnetostatics Comparison**

| **Electrostatics** | **Magnetostatics** | **Electrostatics** | **Magnetostatics** |
|---|---|---|---|
| Static charge $q$ | Steady current $\mathbf{J}$ | ($\mathbf{E}$,$\mathbf{D}$,$\mathbf{F}_e$=$q\mathbf{E}$) | ($\mathbf{H}$,$\mathbf{B}$, $\mathbf{F}_m$=$q\mathbf{u}\times\mathbf{B}$) |
| $\nabla\times\mathbf{E} = \mathbf{0}; \nabla\cdot\mathbf{D} = \rho_v$ | $\nabla\times\mathbf{H} = \mathbf{J}; \nabla\cdot\mathbf{B} = 0$ | $\mathbf{E} = -\nabla V$ | $\mathbf{B} = \nabla\times\mathbf{A}$ |
| $\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{Q}{\varepsilon_0}$ | $\oint_C \mathbf{H}\cdot d\mathbf{l} = I$ | $V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho_v dv'}{R}$ | $\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}dv'}{R}$ |
| $\mathbf{p}=q\mathbf{d}$ | $\mathbf{m} = \hat{\mathbf{a}}_n IS$ | $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}$ | $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu\mathbf{H}$ |
| $\varepsilon_r = 1 + \chi_e = \varepsilon/\varepsilon_0$ | $\mu_r = 1 + \chi_m = \mu/\mu_0$ | $E_{1t} = E_{2t}$ | $B_{1n} = B_{2n}$ |
| $C=Q/V$ | $L=\Lambda/I$ | $D_{1n} - D_{2n} = \rho_s$ | $H_{1t} - H_{2t} = J_{sn}$ |
| $w_e = \mathbf{D}\cdot\mathbf{E}/2$ | $w_m = \mathbf{B}\cdot\mathbf{H}/2$ | $\mathbf{F}_Q = -\nabla W_e$ | $\mathbf{F}_\Phi = -\nabla W_m$ |