WAVE EQUATION

Assume *time-harmonic* electromagnetic fields. In source-free region, i.e., $\mathbf{J} = \mathbf{0}; \rho_v = 0$, where the medium is linear and isotropic (ε,μ),

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} = -j\omega\mu\mathbf{H} \quad (1); \nabla \times \mathbf{H} = j\omega\mathbf{D} = j\omega\varepsilon\mathbf{E} \quad (2)$$

Taking curl of (1) yields

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu\nabla \times \mathbf{H} = -j\omega\mu(j\omega\varepsilon\mathbf{E}) = \omega^2\mu\varepsilon\mathbf{E}$$

Using $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ yields

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E} = \omega^2\mu\varepsilon\mathbf{E}$$

Therefore,

$\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = \mathbf{0}$: vector Helmholtz's equation (vector wave equation).

Likewise, taking curl of (2) yields

$$\nabla \times (\nabla \times \mathbf{H}) = j\omega\varepsilon\nabla \times \mathbf{E} = j\omega\varepsilon(-j\omega\mu\mathbf{H}) = \omega^2\mu\varepsilon\mathbf{H}$$

Thus, one obtains $\nabla^2\mathbf{H} + \omega^2\mu\varepsilon\mathbf{H} = \mathbf{0}$.

J. C. Maxwell made an assumption based on the above vector Helmholtz's equation that there existed propagating electromagnetic waves (1873), which was later verified by H. R. Hertz (1886).

SOLUTION OF WAVE EQUATION

In general, let the wave number k be

$$k^2 = \omega^2\mu\varepsilon \to k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v} = \frac{2\pi f}{v} = \frac{2\pi}{\lambda}; v = \frac{1}{\sqrt{\mu\varepsilon}}$$

In Cartesian coordinates, the wave equation can be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\mathbf{E} = \mathbf{0}.$$

Clearly, ∇^2 denotes the Laplacian operator mentioned before. The equation above can be decomposed as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)E_x = 0; \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)E_y = 0; \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)E_z = 0$$

which are called scalar wave equations.

Suppose the separation of variables method can be applied here, one can write E_x in terms of

$$E_x(x, y, z) = f(x)g(y)h(z)$$

Substituting it into the scalar wave equation for E_x yields

$$gh\frac{\partial^2 f}{\partial x^2} + fh\frac{\partial^2 g}{\partial y^2} + fg\frac{\partial^2 h}{\partial z^2} + k^2 fgh = 0 \to \frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{g}\frac{\partial^2 g}{\partial y^2} + \frac{1}{h}\frac{\partial^2 h}{\partial z^2} + k^2 = 0.$$

In order for solutions to be valid everywhere, i.e., $\forall x, \forall y, \forall z$, each term must be a constant. Hence,

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = -k_x^2; \frac{1}{g}\frac{\partial^2 g}{\partial y^2} = -k_y^2; \frac{1}{h}\frac{\partial^2 h}{\partial z^2} = -k_z^2; k_x^2 + k_y^2 + k_z^2 = k^2$$

Solving the above equations yields

$$f_1(x) = A_1 e^{-jk_x x} + A_2 e^{jk_x x}\text{(traveling wave)}; f_2(x) = B_1\cos k_x x + B_2 \sin k_x x\text{(standing wave)}$$

Obtaining $g(y)$, $h(z)$ using the same approach, then a general solution of E_x is determined. Finally, applying the "appropriate" boundary conditions to determine all constants, the complete solution can be obtained.

7-2 UNIFORM PLANE WAVE

One solution of the wave equation mentioned above can be written as

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}; \mathbf{k} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z; \mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$$

where $\mathbf{k} = k\hat{\mathbf{k}}$ denotes the wave number vector and $\hat{\mathbf{k}}$ represents the propagation direction. Also, $\mathbf{r}$ is the position vector, and $\mathbf{E}_0$ is a constant vector. This electric field has the same magnitude and direction, only its phase changes. *Wavefronts* (surfaces of constant phase) are "planes" of infinite extent and are parallel to each other as shown below. Thus, they are called a *Uniform plane wave*.

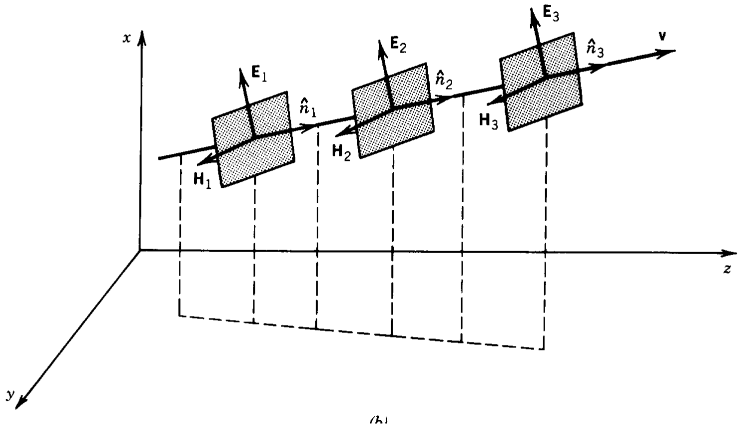


Figure 1: Phase front ของ Uniform plane wave

Consider $\mathbf{E} = \hat{\mathbf{x}}E_0 e^{-jkz}; \mathbf{k} = \hat{\mathbf{z}}k; \mathbf{r} = \hat{\mathbf{z}}z$, the instantaneous electric field can be written as

$\mathscr{E} = \text{Re}[\mathbf{E}e^{j\omega t}] = \text{Re}[\hat{\mathbf{x}}E_0 e^{-jkz} e^{j\omega t}] = \hat{\mathbf{x}}E_0 \cos(\omega t - kz)$.

Here, one can find a constant phase plane from $\omega t - kz = \text{constant}$. Thus, the velocity of the constant phase plane is given by

$$u_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}},$$

which is called the phase velocity. In free space (or air), $u_p = 1/\sqrt{\mu_0\varepsilon_0} = c \approx 3 \times 10^8 \text{(m/s)}$.

One important property of uniform plane waves $\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$ is that they are classified as *transverse electromagnetic waves* (TEM waves), i.e., there is no **E** and **H** in the propagation direction, as shown in Fig. 2. Other properties include

1. $\mathbf{k} \cdot \mathbf{E} = 0$
2. $\mathbf{H} = \dfrac{\hat{\mathbf{k}} \times \mathbf{E}}{\eta}$ where $\eta = \sqrt{\dfrac{\mu}{\varepsilon}}$ denotes the intrinsic impedance (or sometimes called wave impedance) of the medium. In free Space, it is approximately 377 (Ω). The definitition of an inntrinsic impedance is given by

$$\eta = \left|\frac{\mathbf{E}}{\mathbf{H}}\right|.$$

3. $\mathbf{E} \cdot \mathbf{H} = 0$, i.e., **E** is perpendicular to **H**.

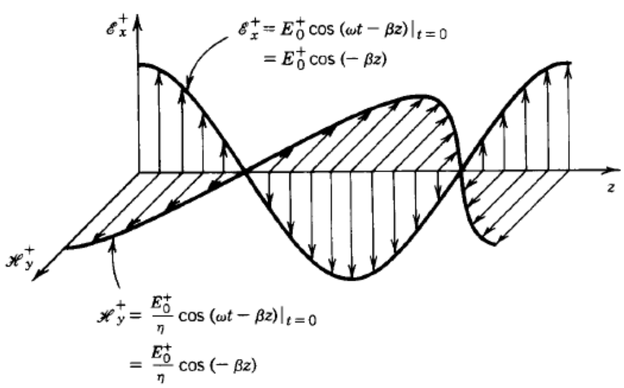


Figure 2 Instantaneous field of a uniform plane wave

Exercise Prove the above properties.

EX 7-1 A uniform plane wave with $\mathbf{E} = \hat{\mathbf{x}}E_x e^{-jkz}$ propagating in a lossless medium ($\varepsilon_r = 4$, $\mu_r = 1$, $\sigma = 0$). Assume that it is a sinusoidal wave with frequency 100 (MHz) and has its peak of 10^{-4} (V/m) at t=0 and z=1/8 (m).
a) Write the instantaneous expression for **E** for any t and z.
b) Write the instantaneous expression for **H**.
c) Determine the locations where E_x is a positive maximum when $t=10^{-8}$ (s).

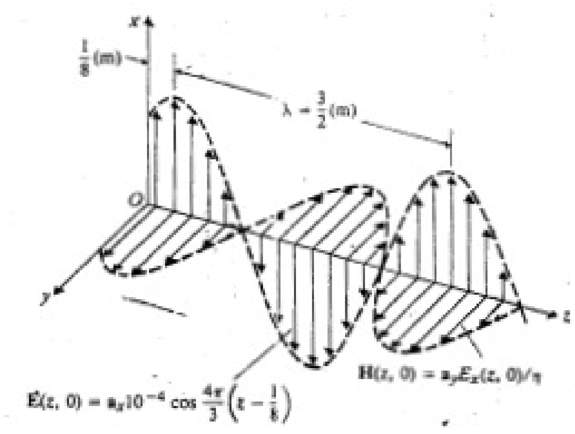
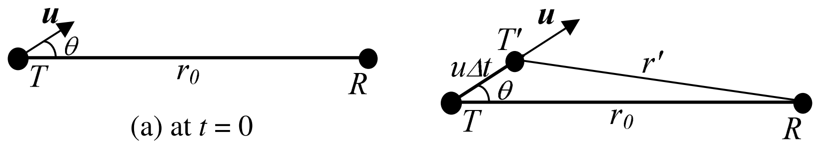


Figure 3: Example 7-1

7-2.1 DOPPLER EFFECT

When there is relative motion between a time-harmonic source and a receiver, the frequency of the wave detected by the receiver tends to be different from that emitted by the source. This phenomenon is known as **Doppler effect**. Assume that the source (transmitter) T of a time-harmonic wave of a frequency f moves with a velocity $\mathbf{u}$ at an angle θ relative to the direct line to a stationary receiver R as shown in Fig. 4 below. The EM wave emitted by T in air at t=0 will reach R at

$$t_1 = \frac{r_0}{c}.$$



(a) at $t = 0$

(b) at $t = \Delta t$

At a later time t=Δt, T has moved to the new position T', and the wave emitted by T' at that time will reach R at

$$t_2 = \Delta t + \frac{r'}{c} = \Delta t + \frac{1}{c}\left[r_0^2 - 2r_0(u\Delta t)\cos\theta + (u\Delta t)^2\right]^{1/2} \xrightarrow{(u\Delta t)^2 \ll r_0^2} t_2 = \Delta t + \frac{r_0}{c}\left(1 - \frac{u\Delta t}{r_0}\cos\theta\right).$$

Thus, the time elapsed at R, $\Delta t'$, corresponding to Δt at T is

$$\Delta t' = t_2 - t_1 = \Delta t\left(1 - \frac{u}{c}\cos\theta\right),$$

which is not equal to Δt. If Δt represents a period of the time-harmonic source, i.e., $\Delta t = 1/f$, then the frequency of the received wave at R is

$$f' = \frac{1}{\Delta t'} = \frac{f}{1 - \frac{u}{c}\cos\theta} \cong f\left(1 + \frac{u}{c}\cos\theta\right); \left(\frac{u}{c}\right)^2 \ll 1.$$

Thus, the frequency perceived at R is higher when T moves toward R, and is conversely lower when T moves away from R. The so-called ***red shift*** of the light spectrum emitted by a receding distant star in astronomy is due to this effect. (move away -> lower frequency -> red end)

7-2.3 POLARIZATION OF PLANE WAVES

The **polarization (pattern)** of a uniform plane wave describes the *time-varying* behavior of the electric field intensity vector at a given point in space, which can be explained in terms of the phase difference between two perpendicular components of electric field intensity. Consider the example shown in Fig. 5, the instantaneous electric field intensity, which is a plane wave traveling in the –z direction, is given by $\mathscr{E}(z;t) = \hat{\mathbf{x}}\mathscr{E}_x(z;t) + \hat{\mathbf{y}}\mathscr{E}_y(z;t)$ where

$$\mathcal{E}_x(z;t) = \mathrm{Re}[E_x e^{j(\omega t+kz)}] = \mathrm{Re}[E_{x0}e^{j\phi_x}e^{j(\omega t+kz)}] = E_{x0}\cos(\omega t + kz + \phi_x); E_x = E_{x0}e^{j\phi_x}$$
$$\mathcal{E}_y(z;t) = \mathrm{Re}[E_y e^{j(\omega t+kz)}] = \mathrm{Re}[E_{y0}e^{j\phi_y}e^{j(\omega t+kz)}] = E_{y0}\cos(\omega t + kz + \phi_y); E_y = E_{y0}e^{j\phi_y}$$

Polarizations can be classified as linear, circular, or elliptic as follows:

A Linear Polarization : the trace of **E** at one point draws a line.

Condition
$$\Delta\phi = \phi_y - \phi_x = n\pi, n = 0,1,2,\ldots \quad \text{OR}$$
$$\text{ONLY one component } (\mathcal{E}_x \text{ or } \mathcal{E}_y)$$

Example

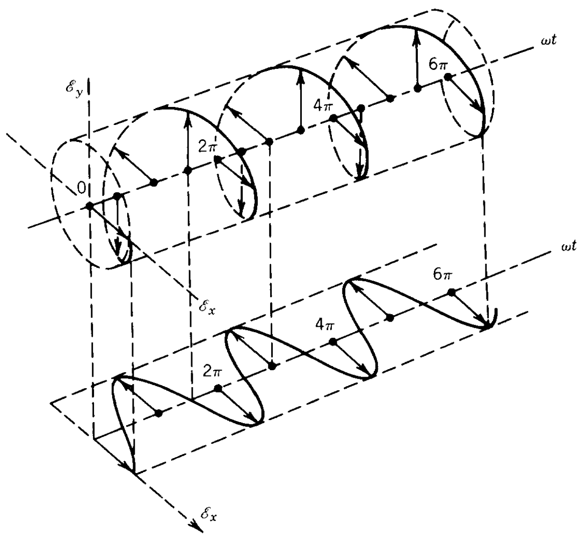


Figure 5: Rotation of a plane EM wave.

B Circular Polarization : the trace of **E** at one point draws a circle. Here, the *sense of rotation* is also needed to be specified, as clockwise (CW) or counterclockwise (CCW), *as observed along the direction of propagation*.

Condition
$$\Delta\phi = \phi_y - \phi_x = \begin{cases} (1/2+2n)\pi, n = 0,1,2,\ldots & \text{CW} \\ -(1/2+2n)\pi, n = 0,1,2,\ldots & \text{CCW} \end{cases} \quad \text{AND} \quad E_{x0} = E_{y0}(|\mathcal{E}_x| = |\mathcal{E}_y|)$$

Example

C Elliptic Polarization : the trace of **E** at one point draws an ellipse. Here, the *sense of rotation* is also needed to be specified as in the case of circular polarization.

Condition
$$1. \Delta\phi = \phi_y - \phi_x = \begin{cases} (1/2+2n)\pi, n = 0,1,2,\ldots & \text{CW} \\ -(1/2+2n)\pi, n = 0,1,2,\ldots & \text{CCW} \end{cases} \quad \text{AND} \quad E_{x0} \neq E_{y0}(|\mathcal{E}_x| \neq |\mathcal{E}_y|)$$
$$\text{OR} \quad 2. \Delta\phi = \phi_y - \phi_x \neq \frac{n}{2}\pi \begin{cases} >0 & \text{CW} \\ <0 & \text{CCW} \end{cases} \quad n = 0, \pm1, \pm2, \ldots$$

Example

7-3 PLANE WAVE IN LOSSY MEDIA

In medium which is conducting ($\sigma \neq 0$), a current $\mathbf{J} = \sigma\mathbf{E}$ will flow because of **E**. The time-harmonic equation regarding **H** should be changed to

$$\nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E} = j\omega\varepsilon_c\mathbf{E}; \varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} \text{ (F/m).}$$

Therefore, ε is replaced by the *complex permittivity* ε_c. In general, complex permittivity is given by

$\varepsilon_c = \varepsilon' - j\varepsilon''$ (F/m), where both $\varepsilon', \varepsilon''$ are generally functions of frequency. In conductors, since $\mathbf{J} = \sigma\mathbf{E}$,

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\varepsilon'\mathbf{E} = (\sigma + j\omega\varepsilon')\mathbf{E} = j\omega\varepsilon_c\mathbf{E}; \varepsilon_c = \varepsilon' + \frac{\sigma}{j\omega}; \varepsilon'' = \frac{\sigma}{\omega}.$$

The ratio $\varepsilon''/\varepsilon'$ is called a **loss tangent** because it is a measure of the power loss in the medium, i.e.,

$$\tan\delta_c = \frac{\varepsilon''}{\varepsilon'};\ \delta_c : \text{Loss Angle}.$$

Since the permittivity is complex, it follows that the wavenumber is also complex, i.e.,

$$k_c = \omega\sqrt{\mu\varepsilon_c}.$$

Hence, the wave equation is changed to $\nabla^2\mathbf{E} + \omega^2\mu\varepsilon_c\mathbf{E} = \nabla^2\mathbf{E} + k_c^2\mathbf{E} = \mathbf{0}$. Defining a new quantity *propagation constant* as follows

$$\gamma = jk_c = j\omega\sqrt{\mu\varepsilon_c} = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2} = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2}; \alpha \in \Re, \beta \in \Re,$$

then the wave equation can be rewritten as $\nabla^2\mathbf{E} - \gamma^2\mathbf{E} = \mathbf{0}$. Consider a uniform plane wave traveling in the +z direction which has only x component, the wave equation and solution are given by

$$\frac{d^2E_x}{dx^2} - \gamma^2E_x = 0 \rightarrow E_x = E_0e^{-\gamma z} = E_0e^{-\alpha z - j\beta z}.$$

Here, α,β are called *attenuation constant*, (unit: (Np/m)) and *phase constant*, (unit: (rad/m)), respectively, where 1 (Np (neper)/m) means a unit wave amplitude decreases to e^{-1} (0.368) after traveling 1 (m). In dB scale, 1 (Np/m) equals attenuation rate of $20\log_{10}e$ = -8.69 (dB/m). Note also that $k = \beta$ in lossless media.

7-3.1 LOW LOSS DIELECTRICS

In lossless media, $\sigma = 0$ OR $\varepsilon'' = 0$. In low loss dielectric media, $\varepsilon'' << \varepsilon'$ OR $\frac{\sigma}{\omega\varepsilon'} << 1$. Hence,

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2} \cong j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{2\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right).$$

Therefore,

$$\alpha = \mathrm{Re}[\gamma] = \omega\sqrt{\mu\varepsilon'}\frac{\varepsilon''}{2\varepsilon'} = \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} \quad \text{(Np/m)}$$

$$\beta = \mathrm{Im}[\gamma] = \omega\sqrt{\mu\varepsilon'}\left(1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right) \quad \text{(rad/m)}$$

The intrinsic impedance and the phase velocity then become

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \cong \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right) \quad (\Omega); \quad u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \quad \text{(m/s)}$$

7-3.2 GOOD CONDUCTORS

In good conductors, $\frac{\sigma}{\omega\varepsilon'} >> 1;\ \varepsilon_c = \varepsilon' + \frac{\sigma}{j\omega} \cong \frac{\sigma}{j\omega}$. Thus,

$$\gamma = \alpha + j\beta \cong j\omega\sqrt{\mu\varepsilon'}\sqrt{\frac{\sigma}{j\omega\varepsilon'}} = \sqrt{j\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\pi f\mu\sigma}$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \cong \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} \quad (\Omega); \quad u_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{(m/s)}.$$

For instance, copper has $\sigma = 5.80\times10^7$ (S/m) and $\mu = 4\pi\times10^{-7}$ (H/m) at the frequency 3 MHz, so u_p = 720 (m/s) and the wavelength in copper becomes

$$\lambda = \frac{2\pi}{\beta} = \frac{u_p}{f} = 2\sqrt{\frac{\pi}{f\mu\sigma}} \text{ (m)} .$$

Therefore, at 3 MHz, $\lambda = 0.24$ (m), which is quite short compared to the wavelength in air at this frequency ($\lambda_0 = c/f = 100$ (m)). At very high frequencies, the attenuation constant α for a good conductor tends to be very large. For example, the attenuation constant of copper at 3 (MHz) becomes

$$\alpha = \sqrt{\pi f \mu\sigma} = \sqrt{\pi(3\times10^6)(4\pi\times10^{-7})(5.80\times10^7)} = 2.62\times10^4 \text{ (Np/m)},$$

which means the wave will decay very rapidly inside copper. The reciprocal of α is called the *skin depth* or the *depth of penetration* of a conductor, i.e.,

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu\sigma}} = \frac{1}{\beta} = \frac{\lambda}{2\pi} \text{ (m)},$$

the distance through which the amplitude of a traveling wave decreases by a factor of $1/e$.

EX7-4 The electric field intensity of a linearly polarized uniform plane wave propagating in the +z direction in seawater ($\varepsilon_r = 72, \mu_r = 1, \sigma = 4$) is given by

$\mathcal{E}(z=0;t) = \hat{\mathbf{x}}100\cos(10^7\pi t)$ (V/m) at $z = 0$.

a) Determine α, β, η, u_p, λ, δ.

b) Find the distance at which the amplitude of **E** is 1% of its value at z=0.

c) Write the expressions for $\mathcal{E}(z;t)$ and $\mathcal{H}(z;t)$.

7-4 GROUP VELOCITY

The phase velocity u_p of a single-frequency plane wave is defined as the velocity of propagation of an equiphase wavefront, which is given by $u_p = \omega / \beta$ (m/s). In lossless media, $\beta = k = \omega\sqrt{\mu\varepsilon}$ is a linear function of ω, and thus $u_p = (\mu\varepsilon)^{-1/2}$ is a constant. However, in some cases (e.g., lossy media, transmission lines, waveguides), the phase constant is not a linear function of ω, therefore different frequencies will propagate at different velocities. Since information-baring signals consist of a band of frequencies, waves of the component frequencies will travel with different velocities, resulting in the *distortion* of the signal wave shape, or the signal "distorts". The phenomenon which the signal distortion is caused by the dependence of the phase velocity on frequency is called ***dispersion***, and such media are classified as ***dispersive*** media.

In general, an information-bearing signal normally has a small spread of frequencies (information) around a high carrier frequency, for example, an amplitude-modulation (AM) signal consists of sound signal (0-20 kHz) and carrier wave (e.g., 535 kHz - 1605 kHz, bandwidth/station 10 kHz). Such a signal comprises a "group" of frequencies and forms a wave packet. A *group velocity* is the velocity of propagation of the wave-packet envelope (of a group of frequencies). Typically, it is the velocity of information or power transmission.

Consider a wave packet that consists of two traveling waves having equal amplitude and slightly different angular frequencies $\omega_0 + \Delta\omega, \omega_0 - \Delta\omega; \Delta\omega << \omega_0$. The phase constants of two frequency components are given by $\beta_0 + \Delta\beta, \beta_0 - \Delta\beta$, thus

$$\begin{aligned} E(z;t) &= E_0 \cos[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z] + E_0 \cos[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z] \\ &= 2E_0 \cos(t\Delta\omega - z\Delta\beta)\cos(\omega_0 t - \beta_0 z) \end{aligned}.$$

Since $\Delta\omega << \omega_0$, the above expression represents a rapidly oscillating wave having an angular frequency ω_0 and an amplitude that varies slowly with an angular frequency $\Delta\omega$ as shown in Fig. 6.

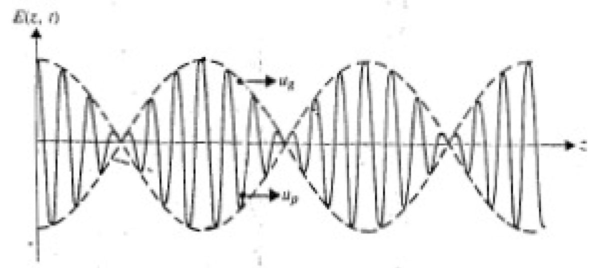


Figure 6: Sum of two time-harmonic traveling waves of equal amplitude and slightly different frequencies at a given *t*.

The phase velocity can be found from setting $\omega_0 t - \beta_0 z = \text{constant}$ as follows:

$$u_p = \frac{dz}{dt} = \frac{\omega_0}{\beta_0}$$

The velocity of the envelope (the *group velocity*) can be determined by setting the argument of the first cosine factor to a constant, i.e., $t\Delta\omega - z\Delta\beta = \text{constant}$:

$$u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\Delta\beta / \Delta\omega}$$

Taking the limit as $\Delta\omega \to 0$ yields

$$u_g = \frac{1}{d\beta / d\omega} \quad \text{(m/s)}.$$

When $u_g = u_p$, there is no dispersion, and there is no signal distortion.

7-5 FLOW OF ELECTROMAGNETIC POWER AND POYNTING VECTOR

Consider $\mathbf{E}(z)=\hat{\mathbf{x}}E_x(z)=\hat{\mathbf{x}}E_0e^{-(\alpha+j\beta)z}$. Thus, $\mathcal{E}(z;t)=\mathrm{Re}[\mathbf{E}(z)e^{j\omega t}]=\hat{\mathbf{x}}E_0e^{-\alpha z}\cos(\omega t-\beta z)$. Likewise, $\mathbf{H}(z)=\dfrac{\hat{\mathbf{z}}\times\mathbf{E}}{\eta_c}=\hat{\mathbf{y}}\dfrac{E_0}{|\eta_c|}e^{-\alpha z}e^{j\beta z}e^{-j\theta_\eta};\eta_c=|\eta_c|e^{j\theta_\eta}$, and

thus, $\mathcal{H}(z;t)=\mathrm{Re}[\mathbf{H}(z)e^{j\omega t}]=\hat{\mathbf{y}}\dfrac{E_0}{|\eta_c|}e^{-\alpha z}\cos(\omega t-\beta z-\theta_\eta)$.

The Poynting vector is then given by

$$\mathcal{P}=\mathcal{E}\times\mathcal{H}=\mathrm{Re}[\mathbf{E}(z)e^{j\omega t}]\times\mathrm{Re}[\mathbf{H}(z)e^{j\omega t}]=\hat{\mathbf{z}}\frac{E_0^2}{|\eta_c|}e^{-2\alpha z}\cos(\omega t-\beta z)\cos(\omega t-\beta z-\theta_\eta)$$

$$=\hat{\mathbf{z}}\frac{E_0^2}{2|\eta_c|}e^{-2\alpha z}[\cos\theta_\eta+\cos(2\omega t-2\beta z-\theta_\eta)]$$

The time-average Poynting vector then becomes

$$\mathbf{P}_{av}(z)=\frac{1}{T}\int_0^T\mathcal{P}(z;t)dt=\hat{\mathbf{z}}\frac{E_0^2}{2|\eta_c|}e^{-2\alpha z}\cos\theta_\eta\quad(\mathrm{W/m^2});\ T=\frac{2\pi}{\omega}=\frac{1}{f}.$$

In lossless media, since $\sigma=0,\eta_c\to\eta,\theta_\eta\to 0,\alpha\to 0$, $\mathbf{P}_{av}(z)=\hat{\mathbf{z}}\dfrac{E_0^2}{2\eta}\quad(\mathrm{W/m^2})$.

In general, the time-average Poynting vector can be found from

$$\mathbf{P}_{av}(z)=\frac{1}{T}\int_0^T\mathcal{P}(z;t)dt=\frac{1}{T}\int_0^T(\mathcal{E}\times\mathcal{H})dt=\frac{1}{T}\int_0^T(\mathrm{Re}[\mathbf{E}e^{j\omega t}]\times\mathrm{Re}[\mathbf{H}e^{j\omega t}])dt$$

$$=\frac{1}{T}\int_0^T\mathrm{Re}[\mathbf{E}\times\mathbf{H}^*]dt=\frac{1}{2}\mathrm{Re}[\mathbf{E}\times\mathbf{H}^*]$$

7-6 NORMAL INCIDENCE OF PLANE WAVES AT PLANE BOUNDARIES

Consider a uniform plane wave traveling from medium 1 to medium 2. Assume that the propagation direction is perpendicular to the interface between two media as shown in Fig. 7 and both media are lossless. Let the incident wave be given by

$$\mathbf{E}_i(z)=\hat{\mathbf{x}}E_{i0}e^{-j\beta_1 z};\mathbf{H}_i(z)=\frac{\hat{\mathbf{k}}_i\times\mathbf{E}_i}{\eta_1}=\frac{\hat{\mathbf{z}}\times\mathbf{E}_i}{\eta_1}=\hat{\mathbf{y}}\frac{E_{i0}e^{-j\beta_1 z}}{\eta_1};\eta_1=\sqrt{\frac{\mu_1}{\varepsilon_1}}$$

Since there is a discontinuity at z=0 plane, there are both reflected wave back to medium 1 and transmitted wave into medium 2, which can be given by

$$\mathbf{E}_r(z)=\hat{\mathbf{x}}E_{r0}e^{j\beta_1 z};\mathbf{H}_r(z)=\frac{\hat{\mathbf{k}}_r\times\mathbf{E}_i}{\eta_1}=\frac{-\hat{\mathbf{z}}\times\mathbf{E}_r}{\eta_1}=\hat{\mathbf{y}}\frac{E_{r0}e^{j\beta_1 z}}{\eta_1},$$

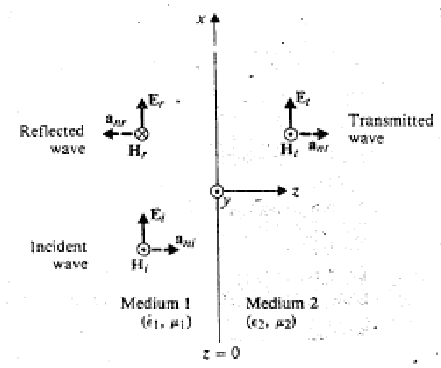


Figure 7: Normal Incidence

$$\mathbf{E}_t(z)=\hat{\mathbf{x}}E_{t0}e^{-j\beta_2 z};\mathbf{H}_t(z)=\frac{\hat{\mathbf{k}}_t\times\mathbf{E}_i}{\eta_2}=\frac{\hat{\mathbf{z}}\times\mathbf{E}_t}{\eta_2}=\hat{\mathbf{y}}\frac{E_{t0}e^{-j\beta_2 z}}{\eta_2};\eta_2=\sqrt{\frac{\mu_2}{\varepsilon_2}}$$

From the boundary condition $\hat{\mathbf{n}}\times\mathbf{E}_1=\hat{\mathbf{n}}\times\mathbf{E}_2$, one obtains

$$E_{i0}+E_{r0}=E_{t0}\qquad(3)$$

From the boundary condition $\hat{\mathbf{n}}\times\mathbf{H}_1=\hat{\mathbf{n}}\times\mathbf{H}_2$, one obtains

$$\frac{1}{\eta_1}(E_{i0}-E_{r0})=\frac{1}{\eta_2}E_{t0}\qquad(4)$$

Solving equations (3), (4) yields

$$E_{r0}=\frac{\eta_2-\eta_1}{\eta_2+\eta_1}E_{i0};E_{t0}=\frac{2\eta_2}{\eta_2+\eta_1}E_{i0}.$$

The reflection coefficient and the transmission coefficient are given by
$\Gamma=\frac{E_{r0}}{E_{i0}}=\frac{\eta_2-\eta_1}{\eta_2+\eta_1};\tau=\frac{E_{t0}}{E_{i0}}=\frac{2\eta_2}{\eta_2+\eta_1}$, respectively.
Note that $1+\Gamma=\tau$. The standing wave ratio (SWR) becomes
$$S=\frac{|\boldsymbol{E}|_{\max}}{|\boldsymbol{E}|_{\min}}=\frac{1+|\Gamma|}{1-|\Gamma|}(\text{dimensionless});1\le S\le\infty,|\Gamma|=\frac{S-1}{S+1}\le 1$$
The total electric field intensity in medium 1 becomes
$$\mathbf{E}_1(z)=\mathbf{E}_i+\mathbf{E}_r=\hat{\mathbf{x}}(E_{i0}e^{-j\beta_1 z}+\Gamma E_{i0}e^{j\beta_1 z})=\hat{\mathbf{x}}E_{i0}e^{-j\beta_1 z}(1+\Gamma e^{j2\beta_1 z});\mathbf{H}_1(z)=\hat{\mathbf{y}}\frac{E_{i0}}{\eta_1}(e^{-j\beta_1 z}-\Gamma e^{j\beta_1 z})$$

$$\mathbf{E}_2(z)=\mathbf{E}_t(z)=\hat{\mathbf{x}}\tau E_{i0}e^{-j\beta_2 z};\mathbf{H}_2(z)=\mathbf{H}_t(z)=\hat{\mathbf{y}}\frac{\tau E_0 ie^{-j\beta_2 z}}{\eta_2}$$

<u>EX 7-7</u> A uniform plane wave in a lossless medium with intrinsic impedance η_1 is incident normally onto another lossless medium with intrinsic impedance η_2 through a plane boundary. Obtain the expressions for the time-average power densities in both media and find the standing wave ratio in medium 1 if $2\eta_1=\eta_2$.

7-6.1 NORMAL INCIDENCE ON A GOOD CONDUCTOR

In good conductors, $\dfrac{\sigma}{\omega\varepsilon'} >> 1$, and in perfect electric conductors (PEC), $\sigma \to \infty$. Therefore, if medium 2 is a PEC, then $\eta_2 \cong \sqrt{\dfrac{j\omega\mu}{\sigma}} \xrightarrow{\sigma\to\infty} 0$. Hence, $\Gamma = -1; \tau = 0$. Note that the boundary condition $\hat{\mathbf{n}} \times \mathbf{E}_1 = \mathbf{0}$ on PEC yields $E_{i0} + E_{r0} = 0$, which implies $\Gamma = -1$.

Here, the electric and magnetic field intensities in medium 1 are given by

$$\mathbf{E}_1(z) = \mathbf{E}_i + \mathbf{E}_r = \hat{\mathbf{x}}(E_{i0}e^{-j\beta_1 z} + \Gamma E_{i0}e^{j\beta_1 z}) = \hat{\mathbf{x}}E_{i0}(e^{-j\beta_1 z} - e^{j\beta_1 z}) = -\hat{\mathbf{x}}j2E_{i0}\sin\beta_1 z;$$

$$\mathbf{H}_1(z) = \hat{\mathbf{y}}\frac{E_{i0}}{\eta_1}(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}) = \hat{\mathbf{y}}\frac{E_{i0}}{\eta_1}(e^{-j\beta_1 z} + e^{j\beta_1 z}) = \hat{\mathbf{y}}\frac{2E_{i0}}{\eta_1}\cos\beta_1 z$$

Therefore, it can be noticed that $\mathbf{E}_1$ lags behind $\mathbf{H}_1$ by $\pi/2$ (-j factor). Instantaneous fields can be found to be

$$\mathcal{E}_1(z;t) = \mathrm{Re}[\mathbf{E}_1(z)e^{j\omega t}] = \hat{\mathbf{x}}\,\mathrm{Re}[j2E_{i0}\sin\beta_1 z e^{j\omega t}] = \hat{\mathbf{x}}2E_{i0}\sin\beta_1 z\,\mathrm{Re}[e^{-j\pi/2}e^{j\omega t}]$$

$$= \hat{\mathbf{x}}2E_{i0}\sin\beta_1 z\cos(\omega t - \frac{\pi}{2}) = \hat{\mathbf{x}}2E_{i0}\sin\beta_1 z\sin\omega t$$

$$\mathcal{H}_1(z;t) = \mathrm{Re}[\mathbf{H}_1(z)e^{j\omega t}] = \hat{\mathbf{y}}\,\mathrm{Re}[\frac{2E_{i0}}{\eta_1}\cos\beta_1 z e^{j\omega t}] = \hat{\mathbf{y}}\frac{2E_{i0}}{\eta_1}\cos\beta_1 z\cos\omega t$$

7-7 OBLIQUE INCIDENCE OF PLANE WAVES AT PLANE BOUNDARIES

Here, the more general case of a uniform plane wave that impinges on a plane boundary *obliquely* is considered. Refer to Fig. 8, the z=0 plane is the interface between medium 1 (ε_1,μ_1) and medium 2 (ε_2,μ_2). The plane containing the *normal vector to the boundary surface* and the *wavenumber vector* is called the ***plane of incidence***, which is the xz plane in this case. Three angles in the figure, θ_i, θ_r, θ_t are called the *angle of incidence*, the *angle of reflection*, and the *angle of transmission*, respectively.

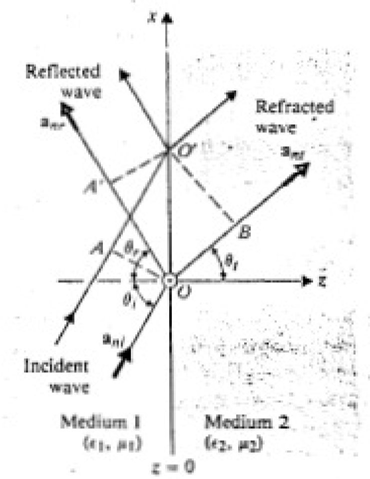


Figure 8: Oblique Incidence

In Fig. 8, lines AO, O'A' and O'B are the intersections of the wavefronts of the incident, reflected and transmitted waves respectively, with the plane of incidence. Since both incident and reflected waves are in medium 1, they must propagate with the same phase velocity, thus the distances $\overline{OA'}$ and $\overline{AO'}$ must be equal. Hence,

$\overline{OA'} = \overline{OO'}\sin\theta_r = \overline{AO'} = \overline{OO'}\sin\theta_i$ or

$\boxed{\theta_r = \theta_i}$: *Snell's law of reflection*

In medium 2, the time it takes for the transmitted wave to travel from O to B equals the time for the incident wave to travel from A to O'. Thus,

$$\frac{\overline{OB}}{u_{p2}} = \frac{\overline{AO'}}{u_{p1}} \to \frac{OB}{AO} = \frac{\overline{OO'}\sin\theta_t}{\overline{OO'}\sin\theta_i} = \frac{u_{p2}}{u_{p1}}$$, from which one obtains

$\boxed{\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{u_{p2}}{u_{p1}} = \dfrac{\beta_1}{\beta_2} = \dfrac{n_1}{n_2}}$: *Snell's law of refraction*

n in the above equation is called the *index of refraction*, which is the ratio of the speed of light in free space to that in the medium, i.e., $n = c/u_p$. If $\mu_1 = \mu_2$, then

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i}=\frac{n_1}{n_2}=\frac{\eta_2}{\eta_1}=\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}=\sqrt{\frac{\varepsilon_{r1}}{\varepsilon_{r2}}}\qquad(\mu_1=\mu_2)}$$

Alternative derivation of Snell's laws using Fermat's principle

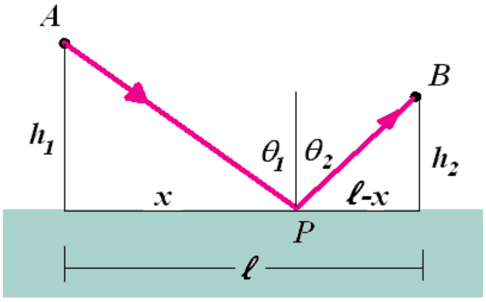
In optics, Fermat's principle or the principle of least time is the principle that the path taken between two points by a ray of light is the path that can be traversed in the least time. This can be used to derive Snell's laws as follows:

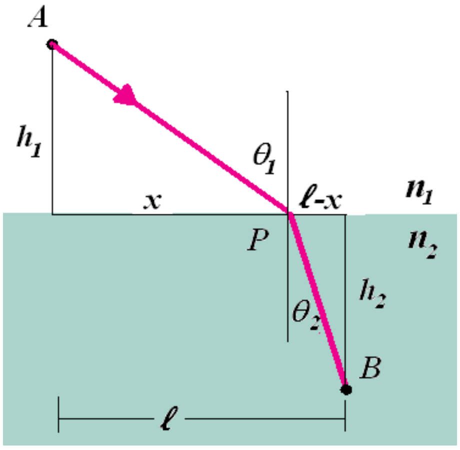
Law of reflection Consider the figure on the right, then the time required for the light to travel between points A and B is given by

$$t=\frac{\sqrt{x^2+h_1^2}}{c}+\frac{\sqrt{(\ell-x)^2+h_2^2}}{c}.$$

Taking the derivative with respect to x and setting it to 0 yields

$$\frac{x}{\sqrt{x^2+h_1^2}}-\frac{\ell-x}{\sqrt{(\ell-x)^2+h_2^2}}=0\text{, i.e., }\sin\theta_1=\sin\theta_2.$$





Law of refraction Likewise, from the figure on the right, the time required for the light to travel between points A and B is given by

$$t=\frac{\sqrt{x^2+h_1^2}}{c/n_1}+\frac{\sqrt{(\ell-x)^2+h_2^2}}{c/n_2}.$$

Taking the derivative with respect to x and setting it to 0 yields

$$\frac{n_1x}{\sqrt{x^2+h_1^2}}-\frac{n_2(\ell-x)}{\sqrt{(\ell-x)^2+h_2^2}}=0\text{, i.e., }n_1\sin\theta_1=n_2\sin\theta_2.$$

7-7.1 TOTAL REFLECTION

When $\varepsilon_1 > \varepsilon_2$ (wave in medium 1 is incident on a less dense medium 2) Snell's law of refraction dictates that $\theta_t > \theta_i$. Thus, it is possible that $\theta_t = \pi/2$, at which the refracted wave will glaze along the interface. The angle of incidence θ_c at which the total reflection occurs is called the *critical angle*. It can be found by setting $\theta_t = \pi/2$, thus,

$$\frac{\sin\theta_t}{\sin\theta_c}=\frac{1}{\sin\theta_c}=\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\ \text{หรือ}\ \boxed{\theta_c=\sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}=\sin^{-1}\left(\frac{n_2}{n_1}\right)\quad(\mu_1=\mu_2)}$$

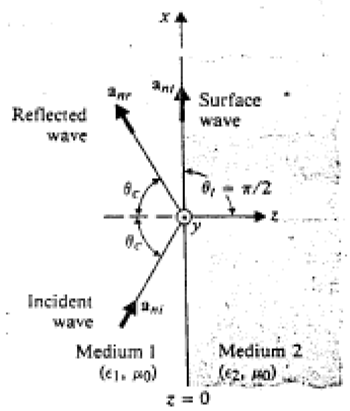


Figure 9: Plane wave incident at critical angle, $\varepsilon_1 > \varepsilon_2$.

Fig. 9 shows this situation. When $\theta_i > \theta_c$,

$$\cos\theta_t=\sqrt{1-\sin^2\theta_t}=\pm j\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i-1}$$ which becomes *imaginary*.

Here, the wavenumber vector in medium 2, $\mathbf{k}_t=\beta_2(\hat{\mathbf{x}}\sin\theta_t+\hat{\mathbf{z}}\cos\theta_t)$

Therefore, $e^{-jk_t\cdot r}$ becomes

$$e^{-j\mathbf{k}_t\cdot\mathbf{r}}=e^{-j\beta_2(x\sin\theta_t+z\cos\theta_t)}=e^{-\alpha_2z}e^{-j\beta_{2x}x};\alpha_2=\beta_2\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i-1};\beta_{2x}=\beta_2\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i,$$

which means the wave will decay very rapidly away from the interface. In other words, the wave is tightly bound to the interface and is thus called the *surface wave*.

EX 7-9 The permittivity of water at optical frequencies is $1.75\varepsilon_0$. It is found that an isotropic light source at a distance *d* under water yields an illuminated circular area of a radius 5 (m). Determine *d*.

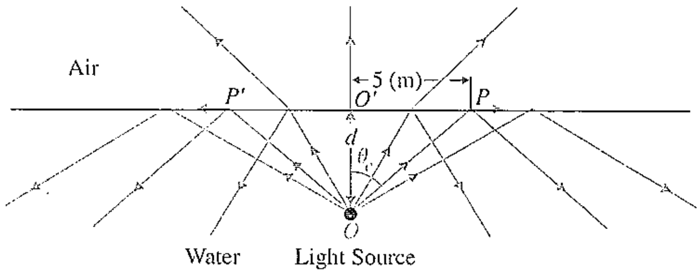


Figure10: An underwater light source in example 7-9.

Ex Optical fibers use the total reflection to keep the light inside the core as shown in Fig. 11. The numerical aperture (NA) of an optical fiber is the number indicating the size of the cone which can be used to bring in the light (Fig. 12). Let the index of refraction of core and cladding be n_1, n_2, respectively, find NA.

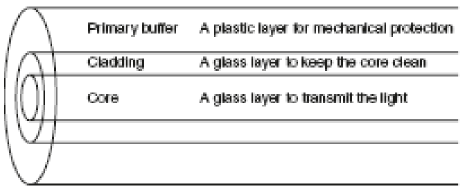


Figure 11: An optical fiber.

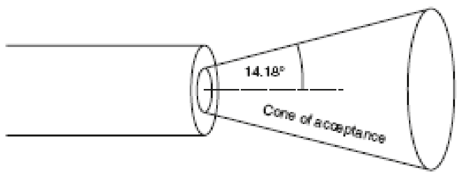


Figure 12: Acceptance angle = $\sin^{-1}$NA

7-7.3 PERPENDICULAR POLARIZATION

Refer to Fig. 13, the xz-plane is the plane of incidence here and the propagation direction is given by $\hat{\mathbf{k}}_i = \hat{\mathbf{x}}\sin\theta_i + \hat{\mathbf{z}}\cos\theta_i$. The electric and magnetic field intensities become

$$\mathbf{E}_i(x,z) = \hat{\mathbf{y}}E_{i0}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)};$$

$$\mathbf{H}_i(x,z) = \frac{E_{i0}}{\eta_1}(-\hat{\mathbf{x}}\cos\theta_i + \hat{\mathbf{z}}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

In lossless media, $\beta_1 = k_1$. The reflected and transmitted waves can be written as

$$\hat{\mathbf{k}}_r = \hat{\mathbf{x}}\sin\theta_r - \hat{\mathbf{z}}\cos\theta_r; \mathbf{E}_r(x,z) = \hat{\mathbf{y}}E_{r0}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)};$$

$$\mathbf{H}_r(x,z) = \frac{E_{r0}}{\eta_1}(\hat{\mathbf{x}}\cos\theta_r + \hat{\mathbf{z}}\sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

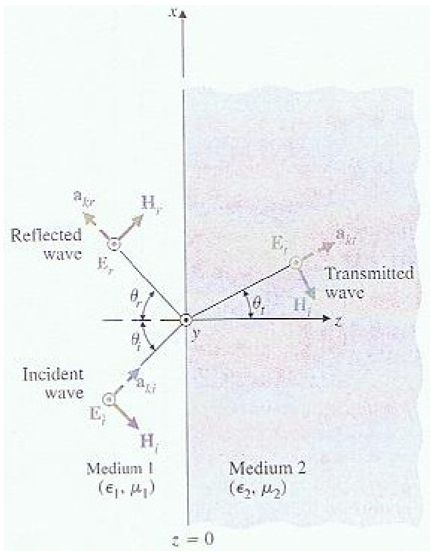


Figure 13 Perpendicular polarization

$\hat{\mathbf{k}}_t = \hat{\mathbf{x}}\sin\theta_t + \hat{\mathbf{z}}\cos\theta_t; \mathbf{E}_t(x,z) = \hat{\mathbf{y}}E_{t0}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)};$

$\mathbf{H}_t(x,z) = \dfrac{E_{t0}}{\eta_2}(-\hat{\mathbf{x}}\cos\theta_t + \hat{\mathbf{z}}\sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$

Applying the boundary conditions $\hat{\mathbf{n}}\times\mathbf{E}_1 = \hat{\mathbf{n}}\times\mathbf{E}_2$ and $\hat{\mathbf{n}}\times\mathbf{H}_1 = \hat{\mathbf{n}}\times\mathbf{H}_2$ yields,

$$E_{i0}e^{-j\beta_1 x\sin\theta_i} + E_{r0}e^{-j\beta_1 x\sin\theta_r} = E_{t0}e^{-j\beta_2 x\sin\theta_t}; \frac{1}{\eta_1}(E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} - E_{r0}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}) = \frac{E_{t0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}\,\forall x$$

In order for the above conditions are satisfied everywhere, $e^{-j\beta_1 x\sin\theta_i} = e^{-j\beta_1 x\sin\theta_r} = e^{-j\beta_2 x\sin\theta_t}$ or

$$\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t,$$

which is called the *phase-matching* conditions. Hence, $\theta_i = \theta_r; \sin\theta_t / \sin\theta_i = \beta_1/\beta_2 = n_1/n_2$ (the Snell's laws given above). Thus, the boundary conditions become $E_{i0} + E_{r0} = E_{t0}; \dfrac{\cos\theta_i}{\eta_1}(E_{i0} - E_{r0}) = \dfrac{E_{t0}}{\eta_2}\cos\theta_t$.

Solving for E_{r0} and E_{t0}, the reflection and transmission coefficients can be found to be

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}; \tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}.$$

Note that the case where $\theta_i = 0 \to \theta_r = \theta_t = 0$ reduces to the normal incidence in 7-6. Furthermore, the relationship between the reflection and transmission coefficients is the same as that of the normal incidence, i.e., $1 + \Gamma_\perp = \tau_\perp$. When the medium 2 is PEC, $\eta_2 = 0$, thus $\Gamma_\perp = -1(E_{r0} = -E_{i0}); \tau_\perp = 0(E_{t0} = 0)$.

<u>EX 7-12</u> The instantaneous expression for the electric field of a uniform plane wave in air is $\mathcal{E}_i(x,z;t) = \hat{\mathbf{y}}10\cos(\omega t + 3x - 4z)$ (V/m). This wave is incident on a PEC boundary at z = 0.

a) Find β_1, ω, θ_i.

b) Find $\mathbf{E}_r(x,z)$

c) Find $\mathcal{E}_1(x,z;t)$

7-7.4 PARALLEL POLARIZATION

Let the xz-plane be the plane of incidence, and the propagation direction $\hat{\mathbf{k}}_i = \hat{\mathbf{x}}\sin\theta_i + \hat{\mathbf{z}}\cos\theta_i$ as shown in Fig. 14. The electric and magnetic field intensities become

$$\mathbf{E}_i(x,z) = E_{i0}(\hat{\mathbf{x}}\cos\theta_i - \hat{\mathbf{z}}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)};$$

$$\mathbf{H}_i(x,z) = \hat{\mathbf{y}}\frac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

The reflected and transmitted waves then become

$$\hat{\mathbf{k}}_r = \hat{\mathbf{x}}\sin\theta_r - \hat{\mathbf{z}}\cos\theta_r;$$

$$\mathbf{E}_r(x,z) = E_{r0}(\hat{\mathbf{x}}\cos\theta_r + \hat{\mathbf{z}}\sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)};$$

$$\mathbf{H}_r(x,z) = -\hat{\mathbf{y}}\frac{E_{r0}}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

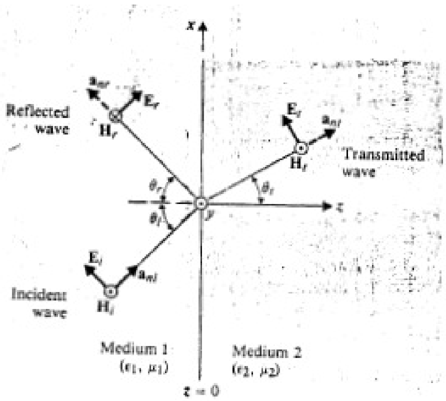


Figure 14: Parallel polarization

$$\hat{\mathbf{k}}_t = \hat{\mathbf{x}}\sin\theta_t + \hat{\mathbf{z}}\cos\theta_t; \mathbf{E}_t(x,z) = E_{t0}(\hat{\mathbf{x}}\cos\theta_t - \hat{\mathbf{z}}\sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}; \mathbf{H}_t(x,z) = \hat{\mathbf{y}}\frac{E_{t0}}{\eta_2}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

Applying the boundary conditions $\hat{\mathbf{n}}\times\mathbf{E}_1 = \hat{\mathbf{n}}\times\mathbf{E}_2$, $\hat{\mathbf{n}}\times\mathbf{H}_1 = \hat{\mathbf{n}}\times\mathbf{H}_2$ and both Snell's laws (i.e., Phase-matching conditions) yield

$$(E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t; \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}.$$

Solving for E_{r0} and E_{t0}, then the reflection and transmission coefficients can be found to be

$$\Gamma_{\parallel} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}; \tau_{\parallel} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}.$$

Note that if $\theta_i = 0 \to \theta_r = \theta_t = 0$, the results given above reduce to those found for the normal Incidence in 7-6. Furthermore, the reflection coefficient is related to the transmission coefficient as

$$1 + \Gamma_{\parallel} = \tau_{\parallel}\frac{\cos\theta_t}{\cos\theta_i},$$

which is different from the perpendicular polarization case except when $\theta_i = 0$ (normal incidence). If medium 2 is a PEC, then $\eta_2 = 0$ and $\Gamma_{\parallel} = -1 (E_{r0} = -E_{i0}); \tau_{\parallel} = 0 (E_{t0} = 0)$.

Fig. 15 shows a comparison between reflection coefficients for both polarizations. As can be seen, those for perpendicular polarization are higher except the case of normal incidence.

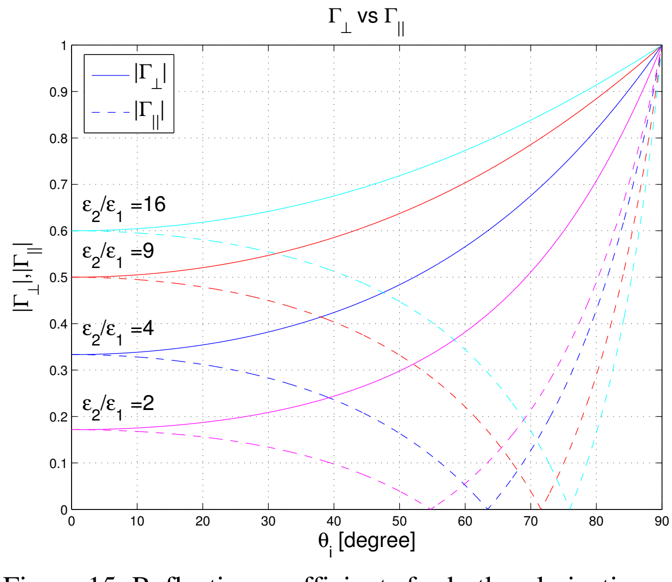


Figure 15: Reflection coefficients for both polarizations.

7-7.5 BREWSTER ANGLE OF NO REFLECTION (TOTAL TRANSMISSION)

Brewster angle is the *incident* angle at which the total transmission (Γ=0) occurs. In the case of perpendicular polarization,

$$\Gamma_{\perp} = 0 = \frac{\eta_2\cos\theta_{B,\perp} - \eta_1\cos\theta_t}{\eta_2\cos\theta_{B,\perp} + \eta_1\cos\theta_t} \to \eta_2\cos\theta_{B,\perp} = \eta_1\cos\theta_t$$

Since $\sin\theta_t / \sin\theta_{B,\perp} = \beta_1 / \beta_2$,

$$\sin^2\theta_{B,\perp} = \left(\frac{\beta_2}{\beta_1}\right)^2 \sin^2\theta_t = \frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}(1-\cos^2\theta_t) = \frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}\left(1-\left(\frac{\eta_2}{\eta_1}\right)^2\cos^2\theta_{B,\perp}\right) = \frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}\left(1-\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}(1-\sin^2\theta_{B,\perp})\right)$$

$$\therefore \sin^2\theta_{B,\perp} = \frac{\dfrac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}\left(1-\dfrac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)}{1-\left(\dfrac{\mu_2}{\mu_1}\right)^2} = \frac{\dfrac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}-\left(\dfrac{\mu_2}{\mu_1}\right)^2}{1-\left(\dfrac{\mu_2}{\mu_1}\right)^2} = \frac{1-\dfrac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}}{1-\left(\dfrac{\mu_1}{\mu_2}\right)^2}$$

Therefore, <u>if $\mu_1 = \mu_2$ total transmission will not occur</u>.

In the case of parallel polarization,

$$\Gamma_{\parallel} = 0 = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_{B,\parallel}}{\eta_2\cos\theta_t + \eta_1\cos\theta_{B,\parallel}} \rightarrow \eta_2\cos\theta_t = \eta_1\cos\theta_{B,\parallel}$$

$$\sin^2\theta_{B,\parallel} = \left(\frac{\beta_2}{\beta_1}\right)^2 \sin^2\theta_t = \frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}(1-\cos^2\theta_t) = \frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}\left(1-\left(\frac{\eta_1}{\eta_2}\right)^2\cos^2\theta_{B,\parallel}\right) = \frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}\left(1-\frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}(1-\sin^2\theta_{B,\parallel})\right)$$

$$\therefore \sin^2\theta_{B,\parallel} = \frac{\dfrac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}\left(1-\dfrac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}\right)}{1-\left(\dfrac{\varepsilon_2}{\varepsilon_1}\right)^2} = \frac{\dfrac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}-\left(\dfrac{\varepsilon_2}{\varepsilon_1}\right)^2}{1-\left(\dfrac{\varepsilon_2}{\varepsilon_1}\right)^2} = \frac{1-\dfrac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}{1-\left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2}$$

$$\therefore \sin\theta_{B,\parallel} = (1-\varepsilon_1/\varepsilon_2)^{-1/2} \quad (\mu_1 = \mu_2)$$

$$\therefore \tan\theta_{B,\parallel} = (\varepsilon_1/\varepsilon_2)^{-1/2} = \sqrt{\varepsilon_2/\varepsilon_1} \text{ OR } \theta_{B,\parallel} = \tan^{-1}\sqrt{\varepsilon_2/\varepsilon_1} = \tan^{-1}(n_2/n_1) \qquad (\mu_1 = \mu_2)$$

<u>EX 7-13</u> The dielectric constant of pure water is 80. (a) Determine the Brewster angle for parallel polarization, $\theta_{B\parallel}$, and the corresponding angle of transmission. (b) A plane wave with perpendicular polarization is incident from air on water surface at $\theta_i = \theta_{B\parallel}$. Find the reflection and transmission coefficients.

$$\theta_{B,\parallel} = \sin^{-1}(1-\varepsilon_1/\varepsilon_2)^{-1/2} = \sin^{-1}(1-1/\varepsilon_r)^{-1/2} = 81.0°$$

$$\sin\theta_t = \beta_1\sin\theta_{B,\parallel}/\beta_2 = (1/\sqrt{\varepsilon_r})(1/\sqrt{1+1/\varepsilon_r}) = 1/\sqrt{1+\varepsilon_r} = 1/\sqrt{81} \rightarrow \theta_t = 6.38°$$

$$\Gamma_{\perp} = \frac{\eta_2\cos\theta_{B,\parallel} - \eta_1\cos\theta_t}{\eta_2\cos\theta_{B,\parallel} + \eta_1\cos\theta_t} = \frac{40.1\cos 81° - 377\cos 6.38°}{40.1\cos 81° + 377\cos 6.38°} = -0.967; \tau_{\perp} = 1+\Gamma_{\perp} = 0.033$$