Steady Electric Currents

4-1 Overview

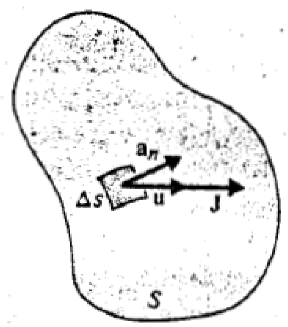
There are two types of electric current caused by the motion of free charges:

(a) *Convection currents* are due to the motion of positively or negatively charged particles in a vacuum or rarefied gas. This type of currents is not governed by Ohm's law.

(b) *Conduction currents* are due to the organized motion of the conduction electrons (in conductors) resulting from the applied external electric field. The relation between conduction current density and electric field intensity is represented in terms of the point-form Ohm's law.

4-2 Current Density and Ohm's law

<u>A. Convection Current</u>



Consider the steady motion of one kind of charge carriers, each of charge q, across an element of surface Δs with a velocity $\mathbf{u}$. If N is the number of charge carriers per unit volume, then in time Δt each charge carrier moves a distance $\mathbf{u}\Delta t$, and the amount of charge passing through the surface Δs is

$$\Delta Q = Nq\mathbf{u}\cdot\hat{\mathbf{a}}_n\Delta s\Delta t \ \text{(C)}$$

Since the current is the change rate of charge with respect to time,

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u}\cdot\hat{\mathbf{a}}_n\Delta s = Nq\mathbf{u}\cdot\Delta\mathbf{s}$$

The (volume) current density (unit: A/m^2) is defined as

$$\mathbf{J} = Nq\mathbf{u} \ \text{(A/m}^2\text{)}.$$

Thus, $\Delta I = \mathbf{J}\cdot\Delta\mathbf{s}$. The total current I flowing through a surface S is then

$$I = \int_S \mathbf{J}\cdot d\mathbf{s} \ \text{(A)}.$$

The product Nq denotes the free charge per unit volume, i.e., volume charge density, thus

$$\mathbf{J} = \rho_v\mathbf{u} \ \text{(A/m}^2\text{)},$$

which is the relation between the *convection current density* and the velocity of the charge carrier.

<u>B. Conduction Current</u>

In the case of conduction currents there may be more than one kind of charge carriers (electrons, holes and ions) drifting with different velocities. Hence,

$$\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \quad \text{(A/m}^2\text{)}.$$

For most conducting materials, the average drift velocity is directly proportional to the electric field intensity. For metallic conductors,

$$\mathbf{u}_e = -\mu_e\mathbf{E} \quad \text{(m/s)},$$

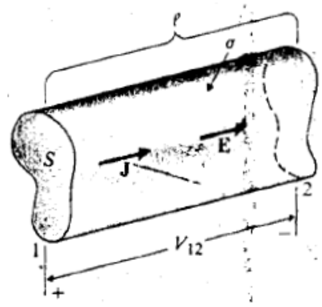
where μ_e is the electron *mobility* measured in (m^2/V·s). For copper, $\mu_e = 3.2\times10^{-3}$, while $\mu_e = 5.2\times10^{-3}$ and $\mu_e = 1.4\times10^{-4}$ for silver and aluminum, respectively. The average drift velocity of electrons is very low because they collide with the atoms in the course of the motion, dissipating part of their kinetic energy as heat. This phenomenon presents a resistance to current flow. It follows that

$$\mathbf{J} = -\rho_e\mu_e\mathbf{E},$$

where $\rho_e = -Ne$ is the charge density of the drifting electrons and is a negative quantity. Defining the *conductivity* σ as $\sigma = -\rho_e\mu_e$ (A/V·m), the point form of *Ohm's law* can be written as

$$\mathbf{J} = \sigma\mathbf{E}.$$

The conductivity of copper is 5.80×10^7, while that of rubber, which is a good insulator) is only 10^{-15}.



Now, consider the figure in the right. According to the Ohm's law, $V_{12} = RI$, where V_{12} is the voltage across a resistance R, and I is the current flows from point 1 and point 2. Assume that the material is *homogeneous* and has uniform cross section. Within the conducting material, $\mathbf{J} = \sigma\mathbf{E}$, where **J**, **E** are in the direction of the current flow. The potential difference (or voltage) between terminals 1 and 2 is
$V_{12} = E\ell$, or $E = V_{12}/\ell$. The total current is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS \text{ , or } J = I/S.$$

Thus, $J = I/S = \sigma E = \sigma V_{12}/\ell$, or

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI \; ; R = \frac{\ell}{\sigma S} .$$

This resistance formula is for a straight piece of homogeneous material of a uniform cross section.

4-3 Equation of Continuity and Kirchhoff's Current Law

The *principle of conservation of charge* is one of the fundamental postulates of physics, like the conservation of power or energy. Consider arbitrary volume V bounded by surface S. A net charge Q exists within this region. If a net current I flows across the surface *out of* (*into*) this region, the charge in V must *decrease* (*increase*) at a rate that equals the current. The current leaving this region is the total outward flux of the current density vector through the surface S, thus

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho_v dv .$$

Assuming that V is stationary, applying the divergence theorem yields

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = \int_V \nabla \cdot \mathbf{J} dv = -\int_V \frac{\partial \rho_v}{\partial t} dv \text{ , thus}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} ,$$

which is called the *equation of continuity*. For steady current, charge density does not vary with time, thus $\nabla \cdot \mathbf{J} = 0$, which implies that steady currents are divergenceless or solenoidal. This relationship also holds at points where $\rho_v = 0$ (no flow source). Over any enclosed surface,

$$\int_V \nabla \cdot \mathbf{J} dv = \oint_S \mathbf{J} \cdot d\mathbf{s} = 0 \text{, which can be written as } \sum_j I_j = 0 .$$

This is essentially an expression of *Kirchhoff's current law*. It states that *the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero*.

Recall that inside *perfect conductors*, $\rho_v = 0$ and $\mathbf{E} = \mathbf{0}$ under equilibrium conditions. For non-perfect conductors, applying the Ohm's law in the equation of continuity yields

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_v}{\partial t} \text{ . In a simple medium, } \nabla \cdot \mathbf{E} = \rho_v/\varepsilon \text{, thus}$$

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0 .$$

The solution to the above differential equation is

$$\rho_v = \rho_0 e^{-(\sigma/\varepsilon)t} \text{ (C/m}^3\text{)},$$

where ρ_0 denotes the initial charge density at $t = 0$.The charge density at a given location will decay to $1/e$ or 36.8% of its value in a time equal to

$$\tau = \frac{\varepsilon}{\sigma} \text{ (s)},$$

which is called the *relaxation time*. For a good conductor such as copper, $\sigma = 5.80\times10^{7}$ (S/m), $\varepsilon \approx \varepsilon_0 = 8.854\times10^{-12}$ (F/m), τ equals 1.53×10^{-19} (s).

4-4 Power Dissipation and Joule's Law

Under the influence of an electric field, conduction electrons in a conductor undergo a drift motion *macroscopically. Microscopically*, these electrons collide with atoms on lattice sites. Energy is thus transmitted from the electric field to the atoms in thermal vibration. The work Δw done by an electric field $\mathbf{E}$ in moving a charge q a distance $\Delta\ell$ is $q\mathbf{E}\cdot\Delta\ell$, which corresponds to a dissipated power

$$p = \lim_{\Delta t\to 0}\frac{\Delta w}{\Delta t} = q\mathbf{E}\cdot\mathbf{u},$$

where $\mathbf{u}$ is the drift velocity. The total power delivered to all the charge carriers in a volume dv is

$dP = \sum_i p_i = \mathbf{E}\cdot\left(\sum_i N_i q_i \mathbf{u}_i\right)dv = \mathbf{E}\cdot\mathbf{J}dv$ or $\dfrac{dP}{dv} = \mathbf{E}\cdot\mathbf{J}$ (W/m^3). Thus, $\mathbf{E}\cdot\mathbf{J}$ denotes a power density under steady current conditions. For a given volume V the total *dissipated* power is

$P = \int_V \mathbf{E}\cdot\mathbf{J}dv$ (W), which is known as *Joule's law*.

In a conductor of a uniform cross section, $dv = ds d\ell$, where $d\ell$ is measured in the direction $\mathbf{J}$,

$$P = \int_V \mathbf{E}\cdot\mathbf{J} ds d\ell = \int_L E d\ell \int_S J ds = VI = I^2 R \text{ (W)},$$

which is a familiar formula for Ohmic power or Ohmic loss.

4-5 Governing Equations for Steady Current Density

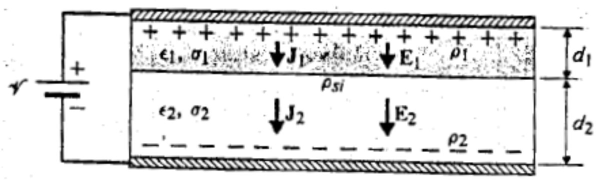
The governing equations for steady current density are obtained by the steady current condition (divergence equation) and curl equation of $\mathbf{E}$ together with Ohm's law, and are given below:

$\nabla\cdot\mathbf{J} = 0$; $\nabla\times(\mathbf{J}/\sigma) = \mathbf{0}$ (differential form)

$\oint_S \mathbf{J}\cdot d\mathbf{s} = 0$; $\oint_C \frac{1}{\sigma}\mathbf{J}\cdot d\ell = 0$ (integral form)

These equations lead to the following boundary conditions between two media with different conductivities, σ_1 and σ_2, given as:

$J_{1n} = J_{2n}$; $J_{1t}/\sigma_1 = J_{2t}/\sigma_2$.



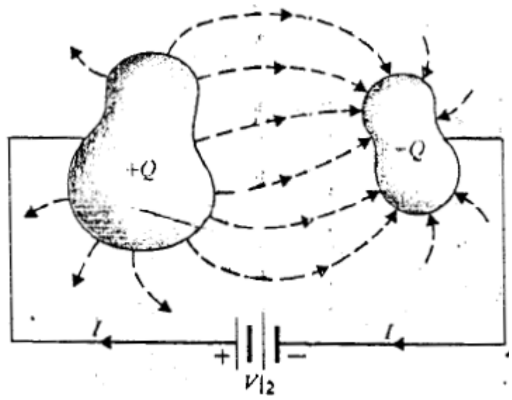
Example An emf V is applied across a parallel-plate capacitor of area S. The space between the conductive plates is filled with two different lossy dielectrics of thicknesses d_1 and d_2, permittivities ε_1 and ε_2, and conductivities σ_1 and σ_2 respectively. Determine

(a) the current density between the plates, (b) the electric filed intensities in both dielectrics

(c) the surface charge densities on the plates and at the interface.

4-6 Resistance Calculations

Refer to the figure in the right, the total charge is related to the voltage by $Q = CV$, thus the capacitance can be found from



$$C=\frac{Q}{V}=\frac{\oint_S \mathbf{D}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot d\mathbf{l}}=\frac{\oint_S \varepsilon\mathbf{E}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot d\mathbf{l}}.$$

When the medium is lossy, a current will flow from the positive to the negative conductor, and a current-density field will be created in the medium. Assume that the medium is isotropic (**J** and **E** are in the same direction), the resistance between two conductors is

$$R=\frac{V}{I}=\frac{-\int_L \mathbf{E}\cdot d\mathbf{l}}{\oint_S \mathbf{J}\cdot d\mathbf{s}}=\frac{-\int_L \mathbf{E}\cdot d\mathbf{l}}{\oint_S \sigma\mathbf{E}\cdot d\mathbf{s}}.$$

Comparison of both equations above shows the following relationship between C and R:

$$RC=\frac{C}{G}=\frac{\varepsilon}{\sigma}.$$

The procedure to find R is similar to that for finding C. One can assume V, then find I and obtain R from the ratio V/I. Alternatively, one can first assume I, find V and then R as well. In transmission line analysis, this approach is used to determine the *shunt* conductance of the line, which is due to the loss in dielectrics.

Example 4-3 Find the per unit length (leakage) conductance (the reciprocal of resistance)

(a) between the inner and outer conductors of a coaxial cable that has an inner conductor of radius a, an outer conductor of inner radius b, and a medium with conductivity σ.

(b) of a parallel-wire transmission line consisting of wires of radius a separated by a distance D in a medium with conductivity σ.