Time-Varying Fields & Maxwell's Equations

6-1 Overview

In this chapter, *time-varying* electric and magnetic fields, which form an *electromagnetic field*, will be discussed. Here, electric and magnetic fields are *coupled*. This first leads to electromagnetic induction described by Faraday's law and then electromagnetic waves governed by Maxwell's equations.

6-2 Electromagnetic Induction

Faraday discovered that a current was induced in a conducting loop when the magnetic flux linkage changed.

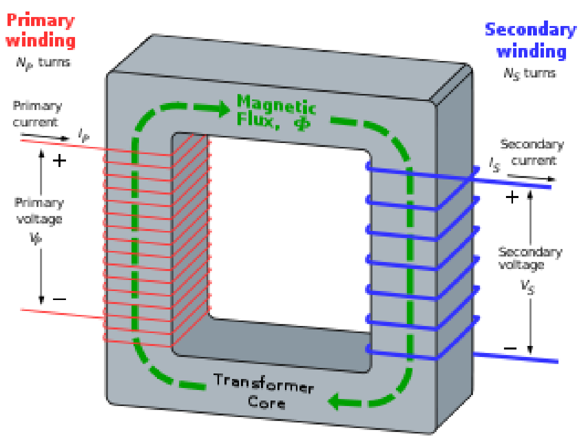
$\boxed{\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}}$ (**Faraday's Law**) Using Stokes' theorem: $\int_S \nabla\times\mathbf{E}\cdot d\mathbf{s} = \oint_C \mathbf{E}\cdot d\boldsymbol{\ell} = -\int_S \dfrac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{s}$

6-2.1 A Stationary Circuit in a Time-varying Magnetic Field

For a stationary circuit, $d\mathbf{s}$ is not a function of t and $\oint_C \mathbf{E}\cdot d\boldsymbol{\ell} = -\dfrac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$. If one define $\mathscr{V} = \oint_C \mathbf{E}\cdot d\boldsymbol{\ell}$: emf induced in circuit with contour C (V), then $\boxed{\mathscr{V} = -\dfrac{d\Phi}{dt}}$ (V) since $\Phi = \int_S \mathbf{B}\cdot d\mathbf{s}$: magnetic flux crossing surface S (Wb). This equation states that *the electromotive force induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.* This is a statement of Faraday's law of electromagnetic induction. The negative sign is an assertion that the induced emf will caused a current to flow in the closed loop in such a direction as to oppose the change in the linking magnetic flux. This assertion is known as Lenz's law. The emf induced in a stationary loop caused by a time-varying magnetic field is a *transformer emf.*

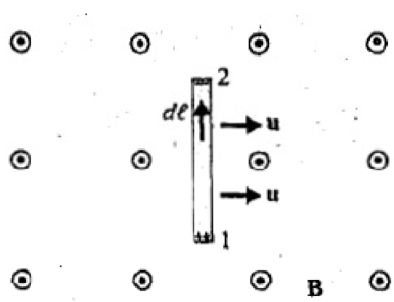
Ex 6-1 A circular loop of N turns of conducting wire lies in the x-y plane with its center at the origin of a magnetic field specified by $\mathbf{B} = \hat{\mathbf{z}}B_0\cos(\pi\rho/2b)\sin\omega t$, where b is the radius of the loop and ω is the angular frequency. Find the emf induced in the loop.

6-2.2 Transformers A transformer is an alternating-current (ac) device that transforms voltages, currents and impedances. It usually consists of two or more coils coupled *magnetically* through a common ferromagnetic core.



6-2.3 A Moving Conductor In a Static Magnetic Field

When a conductor moves with a velocity $\mathbf{u}$ in a static magnetic field $\mathbf{B}$, a force $\mathbf{F}_m$ will cause the freely movable electrons in the conductor to drift toward one end of the conductor and leave the other end positively charged. This separation of the positive and negative charges creates a Coulombian force of attraction. The charge-separation process continues until the electric and magnetic forces balance each other and a state of equilibrium is reached. At equilibrium, which is reached very rapidly, the net force on the free

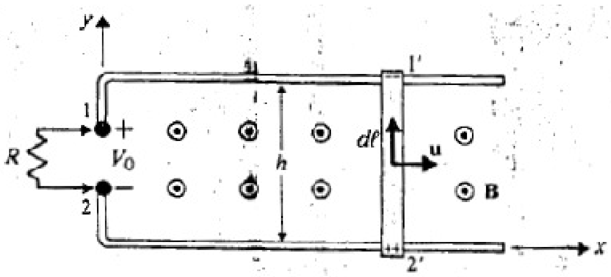


charges in the moving conductor is zero.

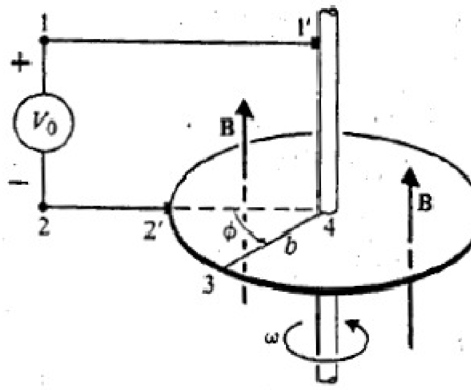
To an observer moving with q, there is no apparent motion, and the force per unit charge $\mathbf{F}_m / q = \mathbf{u} \times \mathbf{B}$ can be interpreted as the *induced* electric field acting along the conductor and producing a voltage: $V_{21} = \int_1^2 \mathbf{u} \times \mathbf{B} \cdot d\boldsymbol{\ell}$. If the moving conductor is a part of closed circuit C, the emf generated around the circuit is $\mathcal{V}' = \oint_C \mathbf{u} \times \mathbf{B} \cdot d\boldsymbol{\ell}$ (V).

This is referred to as *flux-cutting emf* or *motional emf.*

EX 6-2 A metal bar slides over a pair of conducting rails in a uniform magnetic field $\mathbf{B} = \hat{\mathbf{z}} B_0$ with a constant velocity $\mathbf{u}$ (a) Determine the open-circuit voltage V_0 across terminals 1 and 2. (b) Assuming that a resistance R is connected between the terminals, find the dissipated power in R. (c) Show that the electric power is equal to the mechanical power required to move the sliding bar.



EX 6-3 Faraday disk generator Assume the disk rotates with a constant angular velocity ω in a uniform and constant magnetic field $\mathbf{B} = \hat{\mathbf{z}} B_0$



6-2.4 A Moving Circuit in a Time-varying Magnetic Field

When a charge q moves with a velocity $\mathbf{u}$ in a region where both an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ exist, the electromagnetic force $\mathbf{F}$ on q, as measured by an observer, is given by Lorentz's force equation:

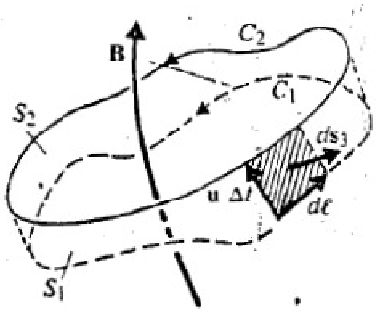
$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

To an observer moving with q, there is no apparent motion, and the force on q can be interpreted as caused by an electric field $\mathbf{E}'$ where $\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$ or $\mathbf{E} = \mathbf{E}' - \mathbf{u} \times \mathbf{B}$ and the induced emf is given by

$$\oint_C \mathbf{E}' \cdot d\boldsymbol{\ell} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} + \oint_C \mathbf{u} \times \mathbf{B} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C \mathbf{u} \times \mathbf{B} \cdot d\boldsymbol{\ell} \quad \text{(V)}$$

This is the general form of *Faraday's law* for a moving circuit in a time-varying magnetic field. The line integral on the left side is the emf induced in the moving frame of reference. The first term of the right-hand side is the *transformer emf* while the second term corresponds to the *motional emf.*

Proof of $\boxed{\dfrac{d\Phi}{dt}=\int_S \dfrac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}-\oint_C \mathbf{u}\times\mathbf{B}\cdot d\boldsymbol{\ell}}$



Consider a circuit with contour C that moves from C_1 at time t to C_2 at time $t+\Delta t$ in a time-varying **B**. The time-rate of change of Φ is

$\dfrac{d\Phi}{dt}=\dfrac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}=\lim_{\Delta t\to 0}\dfrac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}(t+\Delta t)\cdot d\mathbf{s}_2-\int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1\right]$. Using

$\mathbf{B}(t+\Delta t)=\mathbf{B}(t)+\dfrac{\partial \mathbf{B}(t)}{\partial t}\Delta t+\text{H.O.T.}$ yields

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}=\int_S \frac{\partial \mathbf{B}(t)}{\partial t}\cdot d\mathbf{s}+\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}(t)\cdot d\mathbf{s}_2-\int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1+\text{H.O.T.}\right]$$

Let S_3 denote the side surface, then $d\mathbf{s}_3=d\boldsymbol{\ell}\times\mathbf{u}\Delta t$. Applying the divergence theorem for $\mathbf{B}(t)$ yields

$$\int_V \nabla\cdot\mathbf{B}dv=\int_{S_2}\mathbf{B}(t)\cdot d\mathbf{s}_2-\int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1+\int_{S_3}\mathbf{B}(t)\cdot d\mathbf{s}_3=0\rightarrow\int_{S_2}\mathbf{B}(t)\cdot d\mathbf{s}_2-\int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1=-\Delta t\oint_C(\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell}.$$

Thus, one obtains the equation above.

Let $\mathscr{V}'=\oint_C \mathbf{E}'\cdot d\boldsymbol{\ell}=$ emf induced in circuit C measured in the moving frame, then

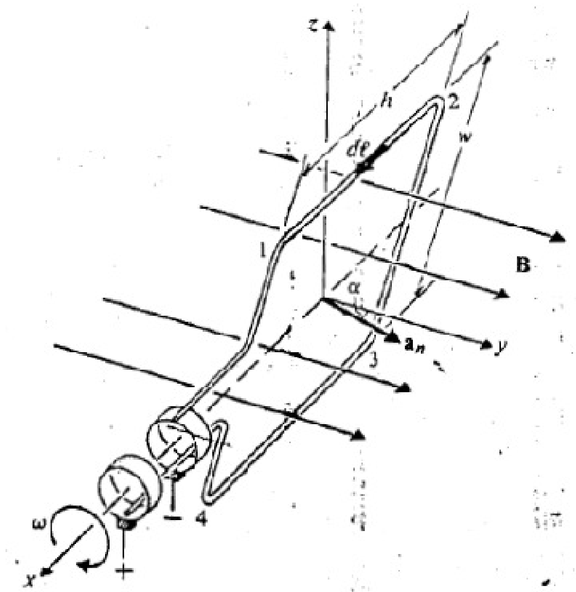
$\mathscr{V}'=-\dfrac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}=-\dfrac{d\Phi}{dt}$ (V)

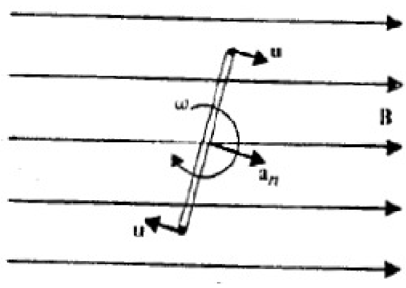
Clearly, $\mathscr{V}'$ reduces to $\mathscr{V}$ for non-moving circuits.

<u>EX 6-4</u> Open-circuit voltage of Faraday disk generator

<u>EX 6-2</u> Alternative method

<u>EX 6-5</u> An h by w rectangular conducting loop is situated in a changing magnetic field $\mathbf{B}=\hat{\mathbf{y}}B_0\sin\omega t$. The normal of the loop initially makes an angle α with the y-axis. Find the induced emf in the loop: (a) when the loop is at rest, and (b) when the loop rotates with an angular velocity ω about the x-axis.





6-3 MAXWELL'S EQUATIONS

Basic Equations

Differential form: $\nabla\times\mathcal{E}=-\dfrac{\partial\mathcal{B}}{\partial t}\,;\nabla\times\mathcal{H}=\dfrac{\partial\mathcal{D}}{\partial t}+\mathcal{J}\,;\nabla\cdot\mathcal{D}=\rho_v\,;\nabla\cdot\mathcal{B}=0$

Equation of continuity $\nabla\cdot\mathcal{J}=-\dfrac{\partial\rho_v}{\partial t}$ (Conservation of charge)

NOTE all equations are not independent.

Integral form: $\oint_C\mathcal{E}\cdot d\boldsymbol{\ell}=-\int_S\dfrac{\partial\mathcal{B}}{\partial t}\cdot d\mathbf{s}\,;\oint_C\mathcal{H}\cdot d\boldsymbol{\ell}=\int_S\dfrac{\partial\mathcal{D}}{\partial t}\cdot d\mathbf{s}+\int_S\mathcal{J}\cdot d\mathbf{s}\,;\oint_S\mathcal{D}\cdot d\mathbf{s}=\int_V\rho_v dv$

$\oint_S\mathcal{B}\cdot d\mathbf{s}=0\,;\ \oint_S\mathcal{J}\cdot d\mathbf{s}=-\int_V\dfrac{\partial\rho_v}{\partial t}dv$

Constitutive Relationships These are used to specify characteristics of the medium in which the field exists. These are required to formulate other independent equations. They are given by

$\mathcal{D}=\mathcal{D}(\mathcal{E},\mathcal{H})\,;\mathcal{B}=\mathcal{B}(\mathcal{E},\mathcal{H})\,;\mathcal{J}=\mathcal{J}(\mathcal{E},\mathcal{H})$.

In free space, $\mathcal{D}=\varepsilon_0\mathcal{E}\,;\mathcal{B}=\mu_0\mathcal{H}\,;\mathcal{J}=\mathbf{0}$, where μ_0=$4\pi\times10^{-7}$(H/m),ε_0=8.854×10^{-12}(F/m),c=$(\mu_0\varepsilon_0)^{-1/2}$(m/s).

For linear media, $\mathcal{D}=\varepsilon\mathcal{E}\,;\mathcal{B}=\mu\mathcal{H}\,;\mathcal{J}=\sigma\mathcal{E}$.

More general form: $\mathcal{D}=\varepsilon\mathcal{E}+\varepsilon_1\dfrac{\partial\mathcal{E}}{\partial t}+\varepsilon_2\dfrac{\partial^2\mathcal{E}}{\partial t^2}+\cdots$

Generalized Current Concept

$\mathcal{J}^t=\dfrac{\partial\mathcal{D}}{\partial t}+\mathcal{J}^c+\mathcal{J}^i\,;\mathcal{M}^t=\dfrac{\partial\mathcal{B}}{\partial t}+\mathcal{M}^i$

$\mathcal{J}^t$, $\mathcal{M}^t$: *Total* electric, magnetic currents ; $\mathcal{J}^i$, $\mathcal{M}^i$: *Impressed* electric, magnetic currents

$\mathcal{J}^c$: *conduction* electric current ; $\partial\mathcal{D}/\partial t$: *displacement* current

Revised Maxwell's equations $\nabla\times\mathcal{E}=-\mathcal{M}^t\,;\nabla\times\mathcal{H}=\mathcal{J}^t$

with $\nabla\cdot\mathcal{J}^t=0\,;\nabla\cdot\mathcal{M}^t=0$ (solenoidal or divergenceless -> continuity of current)

Example 6-6 An ac source of amplitude V_0 and angular frequency ω, v_C=$V_0\sin\omega t$, is connected across a parallel-plate capacitor C_1. Verify that the displacement current in the capacitor is the same as the conduction current in the wires.

ENERGY AND POWER

$\nabla\cdot(\mathcal{E}\times\mathcal{H})=\mathcal{H}\cdot\nabla\times\mathcal{E}-\mathcal{E}\nabla\times\mathcal{H}=-\mathcal{E}\cdot\mathcal{J}^t-\mathcal{H}\cdot\mathcal{M}^t$

Thus, $\nabla\cdot(\mathcal{E}\times\mathcal{H})+\mathcal{E}\cdot\mathcal{J}^t+\mathcal{H}\cdot\mathcal{M}^t=0$.

Integral form: $\oint_S(\mathcal{E}\times\mathcal{H})\cdot d\mathbf{s}+\int_V(\mathcal{E}\cdot\mathcal{J}^t+\mathcal{H}\cdot\mathcal{M}^t)dv=0$ -> conservation of power

Poynting vector is defined as

$\mathcal{P}=\mathcal{E}\times\mathcal{H}$ whose magnitude represents the power density

Outward "total" power: $\mathscr{P}_f=\oint_S\mathcal{P}\cdot d\mathbf{s}=\oint_S(\mathcal{E}\times\mathcal{H})\cdot d\mathbf{s}$

For linear media, $\mathcal{E}\cdot\mathcal{J}^t=\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon\mathscr{E}^2\right)+\sigma\mathscr{E}^2+\mathcal{E}\cdot\mathcal{J}^i$; $\mathcal{H}\cdot\mathcal{M}^t=\frac{\partial}{\partial t}\left(\frac{1}{2}\mu\mathscr{H}^2\right)+\mathcal{H}\cdot\mathcal{M}^i$.

Thus, the conservation of power can be rewritten as

$\mathscr{P}_s=\mathscr{P}_f+\mathscr{P}_d+\frac{d}{dt}(\mathscr{W}_e+\mathscr{W}_m)$ where

$\mathscr{P}_s=-\int_V\left(\mathcal{E}\cdot\mathcal{J}^i+\mathcal{H}\cdot\mathcal{M}^i\right)dv$: Source power

$\mathscr{P}_d=\int_V\sigma\mathscr{E}^2dv$: Dissipated (Ohmic) power

$\mathscr{W}_e=\int_V w_e dv=\frac{1}{2}\int_V\varepsilon\mathscr{E}^2dv$; $\mathscr{W}_m=\int_V w_m dv=\frac{1}{2}\int_V\mu\mathscr{H}^2dv$: Electric, Magnetic energy

<u>Boundary Conditions</u>

From $\oint_S\mathcal{D}\cdot d\mathbf{s}=\int_V\rho_v dv$, $\qquad\hat{\mathbf{n}}_1\cdot\left(\mathcal{D}_1-\mathcal{D}_2\right)=\rho_s$.

Likewise, from $\oint_S\mathcal{B}\cdot d\mathbf{s}=0$, $\qquad\hat{\mathbf{n}}_1\cdot\left(\mathcal{B}_1-\mathcal{B}_2\right)=0$.

$\oint_C\mathcal{E}\cdot d\boldsymbol{\ell}=-\int_S\frac{\partial\mathcal{B}}{\partial t}\cdot d\mathbf{s}$ -> $\qquad\hat{\mathbf{n}}_1\times\left(\mathcal{E}_1-\mathcal{E}_2\right)=\mathbf{0}$.

$\oint_C\mathcal{H}\cdot d\boldsymbol{\ell}=\int_S\frac{\partial\mathcal{D}}{\partial t}\cdot d\mathbf{s}+\int_S\mathcal{J}\cdot d\mathbf{s}$ -> $\qquad\hat{\mathbf{n}}_1\times\left(\mathcal{H}_1-\mathcal{H}_2\right)=\hat{\mathbf{t}}_1(\mathcal{J}_s\cdot\hat{\mathbf{t}}_1)+\hat{\mathbf{b}}_1(\mathcal{J}_s\cdot\hat{\mathbf{b}}_1)$,

where $\hat{\mathbf{n}}_1=\hat{\mathbf{b}}_1\times\hat{\mathbf{t}}_1$ (normal, binormal, tangential vectors)

<u>Exercise</u> Derive the boundary conditions given above.

Time-harmonic Electromagnetic Fields

Time-harmonic fields have the time variation given by $e^{j\omega t}$, then an *instantaneous* electric field intensity can be simplified to be

$\mathcal{E}(x,y,z;t)=\mathrm{Re}\{\mathbf{E}(x,y,z)e^{j\omega t}\}$ where $\omega=2\pi f=2\pi c/\lambda$ and $\mathbf{E}(x,y,z)$ is a vector phasor.

Then the differentiation and integration with respect to time are simply

$$\frac{\partial\mathcal{E}(x,y,z;t)}{\partial t}=j\omega\mathbf{E}(x,y,z)e^{j\omega t}=j\omega\mathcal{E}(x,y,z;t);\ \int\mathcal{E}(x,y,z;t)dt=\frac{\mathbf{E}(x,y,z)e^{j\omega t}}{j\omega}=\frac{\mathcal{E}(x,y,z;t)}{j\omega}.$$

Maxwell's equations for time-harmonic electromagnetic fields in *linear* media then become

$\nabla\times\mathbf{E}=-j\omega\mu\mathbf{H}$; $\nabla\times\mathbf{H}=j\omega\varepsilon\mathbf{E}+\mathbf{J}$; $\nabla\cdot\mathbf{E}=\rho_v/\varepsilon$; $\nabla\cdot\mathbf{B}=0$,

where **E,H,J** are all vector phasors independent of t.

The constitutive relationships then become

$$\mathbf{D}=\varepsilon\mathbf{E}+\varepsilon_1\frac{\partial\mathbf{E}}{\partial t}+\varepsilon_2\frac{\partial^2\mathbf{E}}{\partial t^2}+\cdots=\varepsilon\mathbf{E}+j\omega\varepsilon_1\mathbf{E}-\omega^2\varepsilon_2\mathbf{E}+\cdots=\varepsilon(\omega)\mathbf{E}.$$

Likewise, $\mathbf{J}=\sigma(\omega)\mathbf{E}$, $\mathbf{B}=\mu(\omega)\mathbf{H}$.