APPENDIX II

VECTOR ANALYSIS

II.1 VECTOR TRANSFORMATIONS

In this appendix we will indicate the vector transformations from rectangular-to-cylindrical (and vice versa), from cylindrical-to-spherical (and vice versa), and from rectangular-to-spherical (and vice versa). The three coordinate systems are shown in Figure II-1.

II.1.1 Rectangular-to-Cylindrical (and Vice Versa)

The coordinate transformation from rectangular (x, y, z) to cylindrical (ρ, ϕ, z) is given, referring to Figure II-1(b):

$$\begin{aligned} x &= \rho \cos\phi \\ y &= \rho \sin\phi \\ z &= z \end{aligned} \tag{II-1}$$

In the rectangular coordinate system we express a vector $\mathbf{A}$ as

$$\mathbf{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z \tag{II-2}$$

where $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ are the unit vectors and A_x, A_y, A_z are the components of the vector $\mathbf{A}$ in the rectangular coordinate system. We wish to write $\mathbf{A}$ as

$$\mathbf{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z \tag{II-3}$$

where $\hat{a}_\rho$, $\hat{a}_\phi$, $\hat{a}_z$ are the unit vectors and A_ρ, A_ϕ, A_z are the vector components in the cylindrical coordinate system. The z axis is common to both of them.

Referring to Figure II-2, we can write

$$\begin{aligned} \hat{a}_x &= \hat{a}_\rho \cos\phi - \hat{a}_\phi \sin\phi \\ \hat{a}_y &= \hat{a}_\rho \sin\phi + \hat{a}_\phi \cos\phi \\ \hat{a}_z &= \hat{a}_z \end{aligned} \tag{II-4}$$

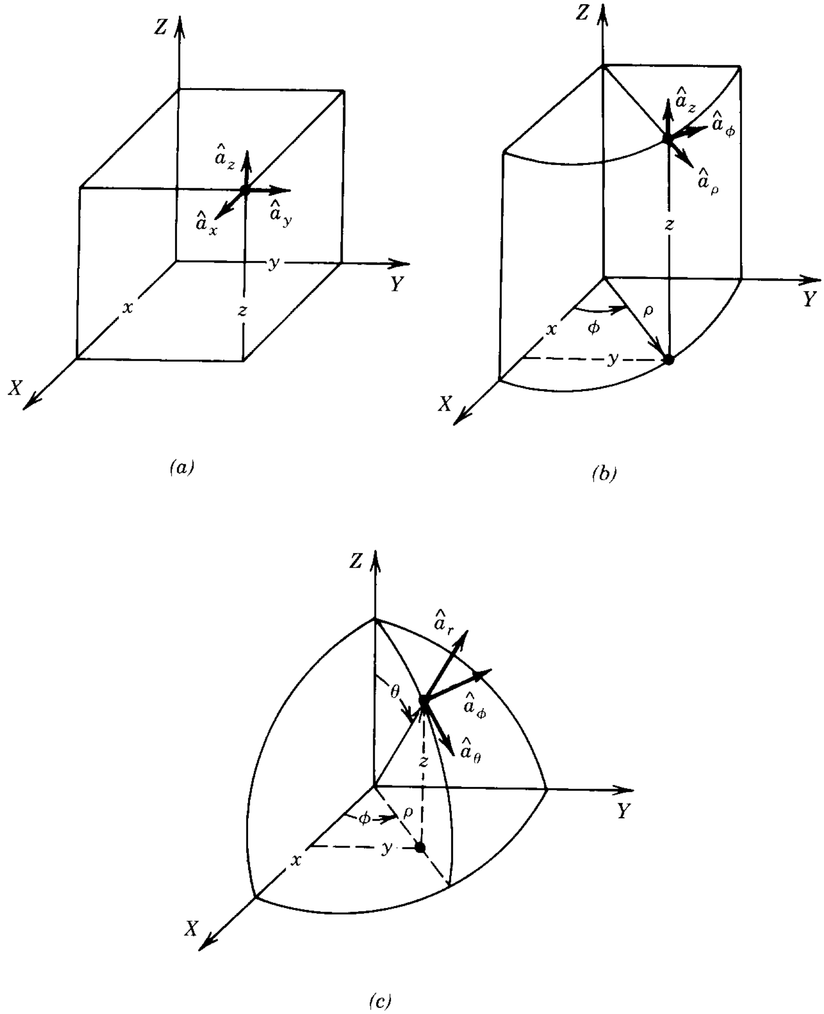

FIGURE II-1 (*a*) Rectangular, (*b*) cylindrical, and (*c*) spherical coordinate systems. (*Source:* C. A. Balanis, *Antenna Theory: Analysis and Design*; copyright © 1982, John Wiley & Sons, Inc.; reprinted by permission of John Wiley & Sons, Inc.)

Using (II-4) reduces (II-2) to

$$\mathbf{A} = (\hat{a}_\rho \cos\phi - \hat{a}_\phi \sin\phi)A_x + (\hat{a}_\rho \sin\phi + \hat{a}_\phi \cos\phi)A_y + \hat{a}_z A_z$$

$$\mathbf{A} = \hat{a}_\rho(A_x \cos\phi + A_y \sin\phi) + \hat{a}_\phi(-A_x \sin\phi + A_y \cos\phi) + \hat{a}_z A_z \qquad \text{(II-5)}$$

which when compared with (II-3) leads to

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$A_z = A_z \qquad \text{(II-6)}$$

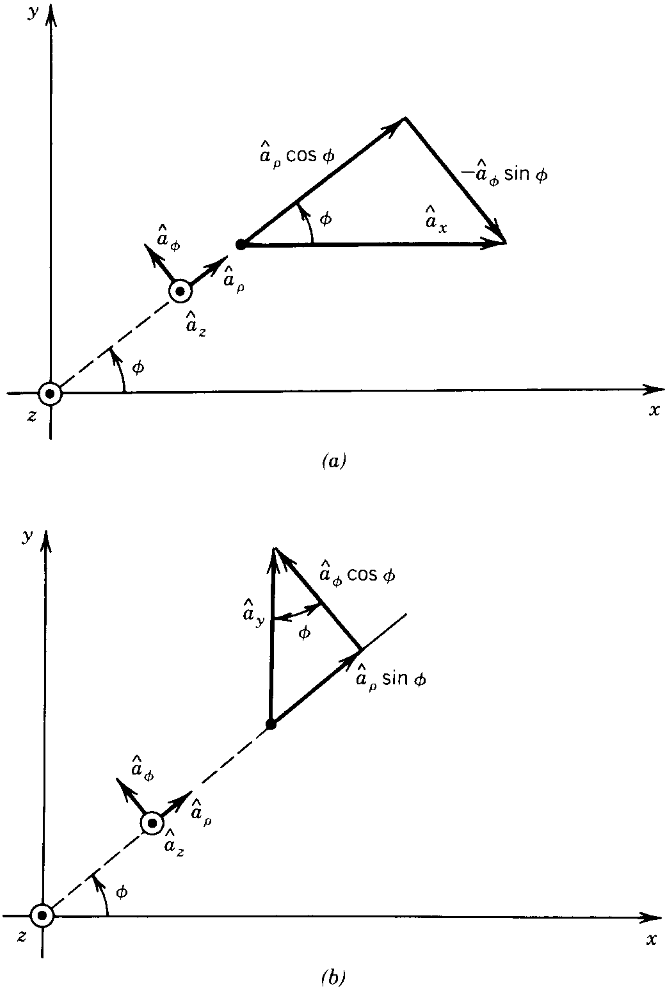

FIGURE II-2 Geometrical representation of transformation between unit vectors of rectangular and cylindrical coordinate systems. (*Source:* C. A. Balanis, *Antenna Theory: Analysis and Design*; copyright © 1982, John Wiley & Sons, Inc.; reprinted by permission of John Wiley & Sons, Inc.) (*a*) Geometry for unit vector $\hat{a}_x$. (*b*) Geometry for unit vector $\hat{a}_y$.

In matrix form, (II-6) can be written as

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{II-6a}$$

where

$$[A]_{rc} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{II-6b}$$

is the transformation matrix for rectangular-to-cylindrical components.

Since $[A]_{rc}$ is an orthonormal matrix (its inverse is equal to its transpose), we can write the transformation matrix for cylindrical-to-rectangular components as

$$[A]_{cr} = [A]_{rc}^{-1} = [A]_{rc}^{t} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{II-7}$$

or

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} \tag{II-7a}$$

or

$$\begin{aligned} A_x &= A_\rho \cos\phi - A_\phi \sin\phi \\ A_y &= A_\rho \sin\phi + A_\phi \cos\phi \\ A_z &= A_z \end{aligned} \tag{II-7b}$$

II.1.2 Cylindrical-to-Spherical (and Vice Versa)

Referring to Figure II-1(c), we can write that the cylindrical and spherical coordinates are related by

$$\begin{aligned} \rho &= r\sin\theta \\ z &= r\cos\theta \end{aligned} \tag{II-8}$$

In a geometrical approach similar to the one employed in the previous section, we can show that the cylindrical-to-spherical transformation of vector components is given by

$$\begin{aligned} A_r &= A_\rho \sin\theta + A_z \cos\theta \\ A_\theta &= A_\rho \cos\theta - A_z \sin\theta \\ A_\phi &= A_\phi \end{aligned} \tag{II-9}$$

or in matrix form by

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} \tag{II-9a}$$

Thus the cylindrical-to-spherical transformation matrix can be written as

$$[A]_{cs} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \tag{II-9b}$$

The $[A]_{cs}$ matrix is also orthonormal so that its inverse is given by

$$[A]_{sc} = [A]_{cs}^{-1} = [A]_{cs}^{t} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \tag{II-10}$$

and the spherical-to-cylindrical transformation is accomplished by

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} \tag{II-10a}$$

or

$$\begin{aligned} A_\rho &= A_r \sin\theta + A_\theta \cos\theta \\ A_\phi &= A_\phi \\ A_z &= A_r \cos\theta - A_\theta \sin\theta \end{aligned} \tag{II-10b}$$

This time the component A_ϕ and coordinate ϕ are the same in both systems.

II.1.3 Rectangular-to-Spherical (and Vice Versa)

Many times it may be required that a transformation be performed directly from rectangular-to-spherical components. By referring to Figure II-1, we can write that the rectangular and spherical coordinates are related by

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{II-11}$$

and the rectangular and spherical components by

$$\begin{aligned} A_r &= A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta \\ A_\theta &= A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta \\ A_\phi &= -A_x \sin\phi + A_y \cos\phi \end{aligned} \tag{II-12}$$

which can also be obtained by substituting (II-6) into (II-9). In matrix form, (II-12) can be written as

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{II-12a}$$

with the rectangular-to-spherical transformation matrix being

$$[A]_{rs} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \tag{II-12b}$$

The transformation matrix of (II-12b) is also orthonormal so that its inverse can be written as

$$[A]_{sr} = [A]_{rs}^{-1} = [A]_{rs}^{t} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \tag{II-13}$$

and the spherical-to-rectangular components related by

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} \tag{II-13a}$$

or

$$A_x = A_r \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$$
$$A_y = A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$$
$$A_z = A_r \cos\theta - A_\theta \sin\theta \tag{II-13b}$$

II.2 VECTOR DIFFERENTIAL OPERATORS

The differential operators of gradient of a scalar ($\nabla\psi$), divergence of a vector ($\nabla \cdot \mathbf{A}$), curl of a vector ($\nabla \times \mathbf{A}$), Laplacian of a scalar ($\nabla^2\psi$), and Laplacian of a vector ($\nabla^2\mathbf{A}$) frequently encountered in electromagnetic field analysis will be listed in the rectangular, cylindrical, and spherical coordinate systems.

II.2.1 Rectangular Coordinates

$$\nabla\psi = \hat{a}_x \frac{\partial\psi}{\partial x} + \hat{a}_y \frac{\partial\psi}{\partial y} + \hat{a}_z \frac{\partial\psi}{\partial z} \tag{II-14}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{II-15}$$

$$\nabla \times \mathbf{A} = \hat{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \tag{II-16}$$

$$\nabla \cdot \nabla\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{II-17}$$

$$\nabla^2\mathbf{A} = \hat{a}_x \nabla^2 A_x + \hat{a}_y \nabla^2 A_y + \hat{a}_z \nabla^2 A_z \tag{II-18}$$

II.2.2 Cylindrical Coordinates

$$\nabla\psi = \hat{a}_\rho \frac{\partial\psi}{\partial\rho} + \hat{a}_\phi \frac{1}{\rho} \frac{\partial\psi}{\partial\phi} + \hat{a}_z \frac{\partial\psi}{\partial z} \tag{II-19}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{II-20}$$

$$\nabla \times \mathbf{A} = \hat{a}_\rho \left(\frac{1}{\rho} \frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{a}_\phi \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho} \right)$$
$$+ \hat{a}_z \left(\frac{1}{\rho} \frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial\phi} \right) \tag{II-21}$$

$$\nabla^2\psi = \frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial\psi}{\partial\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{II-22}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \tag{II-23}$$

or in an expanded form

$$\nabla^2\mathbf{A} = \hat{a}_\rho\left(\frac{\partial^2 A_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\rho}{\partial\rho} - \frac{A_\rho}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial\phi^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial^2 A_\rho}{\partial z^2}\right)$$
$$+\hat{a}_\phi\left(\frac{\partial^2 A_\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\rho} - \frac{A_\phi}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi} + \frac{\partial^2 A_\phi}{\partial z^2}\right)$$
$$+\hat{a}_z\left(\frac{\partial^2 A_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\phi^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \quad \text{(II-23a)}$$

In the cylindrical coordinate system $\nabla^2\mathbf{A} \neq \hat{a}_\rho\nabla^2 A_\rho + \hat{a}_\phi\nabla^2 A_\phi + \hat{a}_z\nabla^2 A_z$ because the orientation of the unit vectors $\hat{a}_\rho$ and $\hat{a}_\phi$ varies with the ρ and ϕ coordinates.

II.2.3 Spherical Coordinates

$$\nabla\psi = \hat{a}_r\frac{\partial\psi}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \quad \text{(II-24)}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \quad \text{(II-25)}$$

$$\nabla\times\mathbf{A} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]$$
$$+\frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \quad \text{(II-26)}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \quad \text{(II-27)}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A} \quad \text{(II-28)}$$

or in an expanded form

$$\nabla^2\mathbf{A} = \hat{a}_r\left(\frac{\partial^2 A_r}{\partial r^2} + \frac{2}{r}\frac{\partial A_r}{\partial r} - \frac{2}{r^2}A_r + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2}\right.$$
$$\left. - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}A_\theta - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)$$
$$+\hat{a}_\theta\left(\frac{\partial^2 A_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\theta}{\partial\theta}\right.$$
$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)$$
$$+\hat{a}_\phi\left(\frac{\partial^2 A_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\phi}{\partial r} - \frac{1}{r^2\sin^2\theta}A_\phi + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial\theta^2}\right.$$
$$+ \frac{\cot\theta}{r^2}\frac{\partial A_\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi}$$
$$\left. + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right) \quad \text{(II-28a)}$$

Again note that $\nabla^2\mathbf{A} \neq \hat{a}_r\nabla^2 A_r + \hat{a}_\theta\nabla^2 A_\theta + \hat{a}_\phi\nabla^2 A_\phi$ since the orientation of the unit vectors $\hat{a}_r$, $\hat{a}_\theta$, and $\hat{a}_\phi$ varies with the r, θ, and ϕ coordinates.

II.3 VECTOR IDENTITIES

II.3.1 Addition and Multiplication

$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 \quad \text{(II-29)}$$

$$\mathbf{A}\cdot\mathbf{A}^* = |\mathbf{A}|^2 \quad \text{(II-30)}$$

$$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A} \quad \text{(II-31)}$$

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \quad \text{(II-32)}$$

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \quad \text{(II-33)}$$

$$(\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C}+\mathbf{B}\cdot\mathbf{C} \quad \text{(II-34)}$$

$$(\mathbf{A}+\mathbf{B})\times\mathbf{C} = \mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{C} \quad \text{(II-35)}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} \quad \text{(II-36)}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \quad \text{(II-37)}$$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= \mathbf{A}\cdot\mathbf{B}\times(\mathbf{C}\times\mathbf{D})\\ &= \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{D}\mathbf{C}-\mathbf{B}\cdot\mathbf{C}\mathbf{D})\\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \quad \text{(II-38)}\end{aligned}$$

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C}-(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D} \quad \text{(II-39)}$$

II.3.2 Differentiation

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \quad \text{(II-40)}$$

$$\nabla\times\nabla\psi = 0 \quad \text{(II-41)}$$

$$\nabla(\phi+\psi) = \nabla\phi+\nabla\psi \quad \text{(II-42)}$$

$$\nabla(\phi\psi) = \phi\nabla\psi+\psi\nabla\phi \quad \text{(II-43)}$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B} \quad \text{(II-44)}$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A}+\nabla\times\mathbf{B} \quad \text{(II-45)}$$

$$\nabla\cdot(\psi\mathbf{A}) = \mathbf{A}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{A} \quad \text{(II-46)}$$

$$\nabla\times(\psi\mathbf{A}) = \nabla\psi\times\mathbf{A}+\psi\nabla\times\mathbf{A} \quad \text{(II-47)}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}) \quad \text{(II-48)}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B} \quad \text{(II-49)}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\nabla\cdot\mathbf{B}-\mathbf{B}\nabla\cdot\mathbf{A}+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B} \quad \text{(II-50)}$$

$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A} \quad \text{(II-51)}$$

II.3.3 Integration

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} \quad \text{Stokes' theorem} \tag{II-52}$$

$$\oiint_S \mathbf{A} \cdot d\mathbf{s} = \iiint_V (\nabla \cdot \mathbf{A})\, dv \quad \text{divergence theorem} \tag{II-53}$$

$$\oiint_S (\hat{n} \times \mathbf{A})\, ds = \iiint_V (\nabla \times \mathbf{A})\, dv \tag{II-54}$$

$$\oiint_S \psi\, d\mathbf{s} = \iiint_V \nabla\psi\, dv \tag{II-55}$$

$$\oint_C \psi\, d\mathbf{l} = \iint_S \hat{n} \times \nabla\psi\, ds \tag{II-56}$$