VECTOR ANALYSIS

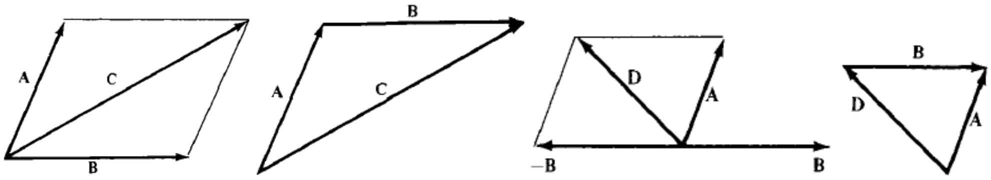
2-1 OVERVIEW

In electromagnetic model, some of the quantities are scalars (e.g., charge, current, power), while some others are vectors (e.g., electric and magnetic field intensities). Both scalars and vectors can be functions of both time and position. A scalar is a quantity that has only *magnitude*, while a vector is a quantity that has both *magnitude* and *direction*. In this chapter, the following topics will be discussed:

1. Vector algebra—addition, subtraction, and multiplication of vectors
2. Orthogonal coordinate systems—Cartesian, cylindrical, spherical coordinates
3. Vector calculus—differentiation and integration of vectors; gradient, divergence, and curl operations

2-2 VECTOR ADDITION AND SUBTRACTION

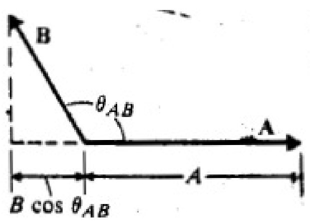
A vector **A** can be written as $\mathbf{A} = \hat{\mathbf{a}}_A A$; A = |**A**|; $\hat{\mathbf{a}}_A = \mathbf{A}/|\mathbf{A}| = \mathbf{A}/A$, where A denotes the magnitude of **A** and $\hat{\mathbf{a}}_A$ is the *unit* vector specifying the direction of **A**. The addition of two vectors, **C=A+B**, can be done by using the parallelogram rule or the head-to-tail rule as shown below. Likewise, the subtraction, **D**=A-**B**, can be done in the same manner.



Question If three vectors, **A**, **B**, and **C**, drawn in a head-to-tail fashion, form three sides of a triangle, what are **A+B+C** and **A+B-C**?

2-3 VECTOR MULTIPLICATION

2-3.1 Scalar or Dot Product



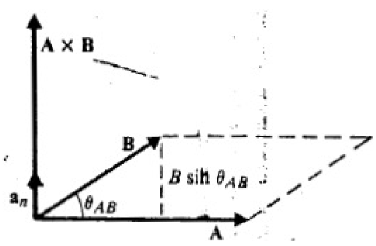
The scalar or dot product of two vectors, denoted by **A•B** ("**A** dot **B**"), is defined as the product of magnitudes of **A**,**B** and the cosine of the angle between them, i.e.,

$\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta_{AB}$. The following identities hold:

$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ (commutative) ; $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$ (distributive)

EX 2-1 Use vectors to prove the law of cosines for a triangle

2-3.2 Vector or Cross Product

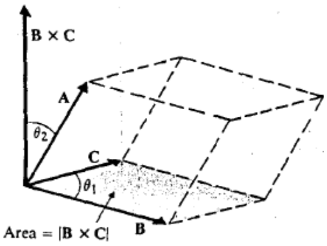


The vector or cross product of two vectors, denoted by **A×B** ("**A** cross **B**"), is defined as the vector whose magnitude is the area of the parallelogram formed by **A**,**B** and whose direction follows the thumb of the right hand when rotating from **A** to **B** (the right-hand rule), i.e.,

$\mathbf{A}\times\mathbf{B} \equiv \hat{\mathbf{a}}_n AB\sin\theta_{AB}$. The following identities hold:

$\mathbf{B}\times\mathbf{A} = -\mathbf{A}\times\mathbf{B}$, which is due to the right-hand rule; $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$.

EX 2-2 For given **A**, **B**, **C**, the following relationship holds regarding the scalar triple product:



$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$,

since the product represents the *volume* of the cubic formed by the three vectors as shown in the figure. Be careful about the order of the sequence {**A**, **B**, **C**}, {**B**, **C**, **A**}, {**C**, **A**, **B**}.

2-4 ORTHOGONAL COORDINATE SYSTEMS

Although the laws of electromagnetism are invariant with coordinate system, solution of practical problems requires that the relations derived from these laws be expressed in an appropriate coordinate system.

Since an electromagnetic (and other physics) problem generally exists in a three-dimensional space, where a point is specified by three coordinates, denoted by (u_1, u_2, u_3). Recall that an N-dimensional *vector space* requires N *base vectors* (*linearly independent*; typically all *unit* vectors) to expand the space, a three-dimensional coordinate system consists of three base vectors, $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3$. These base vectors, $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3$, are perpendicular to the coordinate surfaces, denoted by $u_1=c_1$, $u_2=c_2$, $u_3=c_3$, respectively, where c_1, c_2, c_3 are constants. When all base vectors are perpendicular to each other, the coordinate system is said to be *orthogonal*. Thus, for orthogonal coordinate systems, the following relationship holds:

$$\hat{\mathbf{u}}_i \cdot \hat{\mathbf{u}}_j = \begin{cases} 1 & i = j \\ 0 & else \end{cases}$$

Here, only *orthogonal* coordinate systems are discussed.

<u>2-4.1 Cartesian Coordinates</u>

In this most familiar system, (u_1, u_2, u_3) = (x, y, z) and the base vectors are $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. The right-hand rule gives $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}; \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}; \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$ and so on. A vector **A** can be represented by $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ where A_x, A_y, A_z are x, y, z components of **A**, respectively. It follows that

$$\mathbf{A} \cdot \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) = \hat{\mathbf{x}}A_xB_x + \hat{\mathbf{y}}A_yB_y + \hat{\mathbf{z}}A_zB_z,$$

$$\mathbf{A} \times \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \times (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z)$$

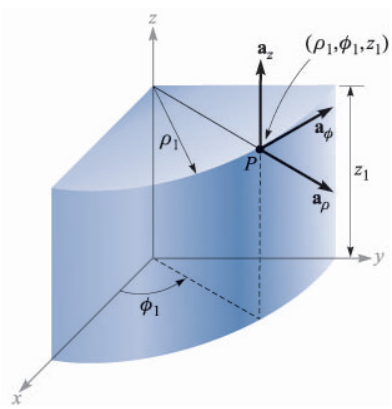
$$= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The differential length, $d\boldsymbol{\ell}$, and differential volume, dv, are given by

$$d\boldsymbol{\ell} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz; dv = dxdydz.$$

<u>EX 2-4</u> Given $\mathbf{A} = \hat{\mathbf{x}}5 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}, \mathbf{B} = -\hat{\mathbf{x}}3 + \hat{\mathbf{z}}4$, find **A•B**, **A×B** and θ_{AB}

<u>2-4.2 Cylindrical Coordinates</u>



In this system, (u_1, u_2, u_3) = (ρ, ϕ, z) and the base vectors are $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$. The right-hand rule gives $\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$; $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\boldsymbol{\rho}}$ $\hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}$ and vice versa. A vector **A** can be represented by $\mathbf{A} = \hat{\boldsymbol{\rho}}A_\rho + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$ where A_ρ, A_ϕ, A_z are ρ, ϕ, z components of **A**, respectively. The differential length, $d\boldsymbol{\ell}$, and differential volume, dv, are given by

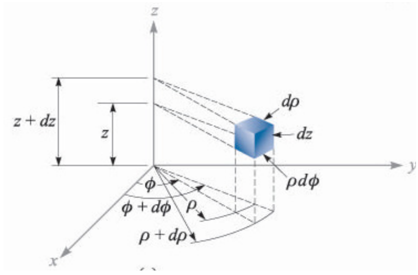
$$d\boldsymbol{\ell} = \hat{\boldsymbol{\rho}}d\rho + \hat{\boldsymbol{\phi}}\rho d\phi + \hat{\mathbf{z}}dz \text{ ; } dv = \rho d\rho d\phi dz.$$

The relationships between (ρ, ϕ) and (x, y) are given by $x = \rho\cos\phi$; $y = \rho\sin\phi$. Also, $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}$ are related to $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ by

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi; \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi.$$

Thus, $\hat{\boldsymbol{\rho}}\cdot\hat{\mathbf{x}}=\cos\phi;\hat{\boldsymbol{\rho}}\cdot\hat{\mathbf{y}}=\sin\phi;\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{x}}=-\sin\phi;\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{y}}=\cos\phi$ and it follows that the cylindrical to Cartesian coordinate transformation is given by

$$\begin{bmatrix}A_x\\A_y\\A_z\end{bmatrix}=\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}A_\rho\\A_\phi\\A_z\end{bmatrix}.$$

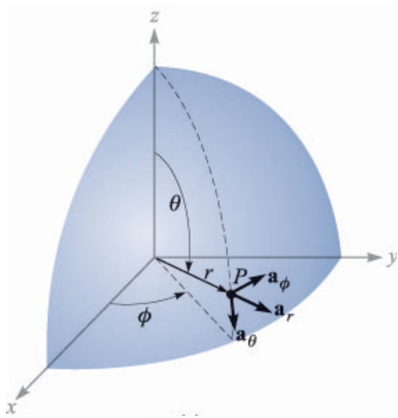


<u>EX 2-6</u> Given $\mathbf{A}=\hat{\boldsymbol{\rho}}3\cos\phi-\hat{\boldsymbol{\phi}}2\rho+\hat{\mathbf{z}}5$. (a) what is the field at P(4, 60°, 5)? (b) Express the field **A** at P, $\mathbf{A}_P$, in Cartesian coordinates (c) Express point P in Cartesian coordinates

<u>2-4.2 Spherical Coordinates</u>

In this system, $(u_1, u_2, u_3) = (r, \theta, \phi)$ and the base vectors are $\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}$. The right-hand rule gives $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\phi}}$; $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}}=\hat{\mathbf{r}}$; $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\theta}}$ and vice versa. A vector **A** can be represented by $\mathbf{A}=\hat{\mathbf{r}}A_r+\hat{\boldsymbol{\theta}}A_\theta+\hat{\boldsymbol{\phi}}A_\phi$ where A_r, A_θ, A_ϕ are r, θ, ϕ components of **A**, respectively. The differential length, $d\boldsymbol{\ell}$, and differential volume, dv, are given by

$$d\boldsymbol{\ell}=\hat{\mathbf{r}}dr+\hat{\boldsymbol{\theta}}rd\theta+\hat{\boldsymbol{\phi}}r\sin\theta d\phi;\ dv=r^2\sin\theta drd\theta d\phi.$$



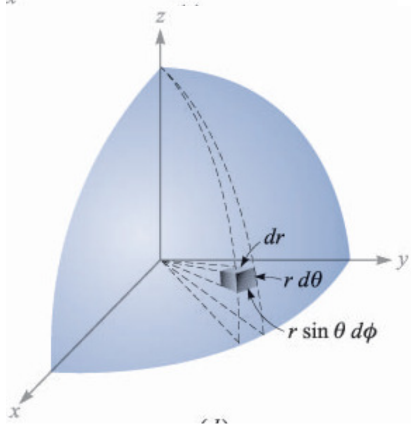
The relationships between (r, θ, ϕ) and (x, y, z) are given by $x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$. Also, $\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}$ are related to $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ by

$$\hat{\mathbf{r}}=\hat{\mathbf{x}}\sin\theta\cos\phi+\hat{\mathbf{y}}\sin\theta\sin\phi+\hat{\mathbf{z}}\cos\theta;$$

$$\hat{\boldsymbol{\theta}}=\hat{\mathbf{x}}\cos\theta\cos\phi+\hat{\mathbf{y}}\cos\theta\sin\phi-\hat{\mathbf{z}}\sin\theta;\quad \hat{\boldsymbol{\phi}}=-\hat{\mathbf{x}}\sin\phi+\hat{\mathbf{y}}\cos\phi$$

Thus, $\hat{\mathbf{r}}\cdot\hat{\mathbf{x}}=\sin\theta\cos\phi;\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}=\sin\theta\sin\phi;\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}=\cos\theta$ and vice versa. It follows that the spherical to Cartesian coordinate transformation is given by

$$\begin{bmatrix}A_x\\A_y\\A_z\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}A_r\\A_\theta\\A_\phi\end{bmatrix}.$$



<u>EX 2-8</u> Assuming that a cloud of electrons confined in a region between two spheres of radii 2 and 5 (cm) has a charge density of

$\dfrac{-3\times10^{-8}}{r^4}\cos^2\phi$ (C/m^3), find the total charge contained in the region.

<u>Summary</u>

Coordinates	Cartesian (x,y,z)	Cylindrical (ρ,ϕ,z)	Spherical (r,θ,ϕ)
Base vectors	$\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$	$\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\phi}},\hat{\mathbf{z}}$	$\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}$
Differential length $d\boldsymbol{\ell}$	$\hat{\mathbf{x}}dx+\hat{\mathbf{y}}dy+\hat{\mathbf{z}}dz$	$\hat{\boldsymbol{\rho}}d\rho+\hat{\boldsymbol{\phi}}\rho d\phi+\hat{\mathbf{z}}dz$	$\hat{\mathbf{r}}dr+\hat{\boldsymbol{\theta}}rd\theta+\hat{\boldsymbol{\phi}}r\sin\theta d\phi$
Differential volume dv	$dxdydz$	$\rho d\rho d\phi dz$	$r^2\sin\theta drd\theta d\phi$

2-5 GRADIENT OF A SCALAR FIELD

In general, a scalar field can be given by $V(t;\ u_1,\ u_2,\ u_3)$, where $(u_1,\ u_2,\ u_3)$ denotes the coordinates of the location. Note that for a static field, V is not a function of t.
The definition of Gradient is given by "*The vector that represents both the magnitude and the direction of the maximum space rate of a scalar*".

Consider Fig. 2-18,

$$\mathrm{grad}\,V = \nabla V \equiv \hat{a}_n \frac{dV}{dn}$$

where ∇ denotes the "del" operator.
Noticing that

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\theta$$

$$= \frac{dV}{dn}\hat{a}_n \cdot \hat{a}_l = \nabla V \cdot \hat{a}_l$$

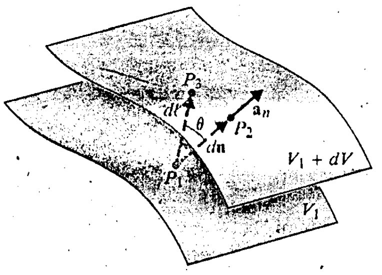


Fig. 2–18 Concerning gradient of a scalar.

and $\dfrac{dV}{dl} \le \dfrac{dV}{dn}$, since the direction of gradient is the direction of the maximum space rate of a scalar. Thus,

$$dV = \nabla V \cdot \hat{\mathbf{a}}_l d\ell = \nabla V \cdot d\boldsymbol{\ell} \qquad (1)$$

Now, considering the total differential dV from the point P_1 to the point P_3 along the direction of $d\boldsymbol{\ell}$ in the Cartesian coordinate system, i.e., $(u_1, u_2, u_3) = (x,y,z)$ and $(d\boldsymbol{\ell}_1, d\boldsymbol{\ell}_2, d\boldsymbol{\ell}_3) = (dx,dy,dz)$, one obtains

$$dV = \left(\hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}\right)\cdot\left(\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz\right) = \left(\hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}\right)\cdot d\boldsymbol{\ell} \qquad (2)$$

Comparing (1) and (2) yields

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)V$$

Operator ∇ is called "nabla". In the Cartesian coordinate system, it is given by

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z},$$

and one reads ∇V as "del V".

Example 2-9(a) The Electrostatic field intensity $\mathbf{E}$ is given by $\mathbf{E} = -\nabla V$. Determine $\mathbf{E}$ at the point (1,1,0) if $V = V_0 e^{-x}\sin\dfrac{\pi y}{4}$.

2-6 DIVERGENCE OF A VECTOR FIELD

It is convenient to represent field variations graphically by directed field lines, which are called *flux lines* or *streamlines*. The magnitude of the field at a point is depicted by the

density of the lines in the vicinity of the point, or in other words, it is measured by the number of flux lines per unit surface normal to the vector.
The definition of divergence is given by
div **A** ≡ the net outward flux of **A** per unit volume as the volume about the point tends to zero which can be given in terms of the following equation:

$$\text{div}\,\mathbf{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\cdot d\mathbf{s}}{\Delta v}$$

Consider a differential volume of sides Δx, Δy, and Δz centered about a point P in the field of a vector **A**. In Cartesian coordinates,

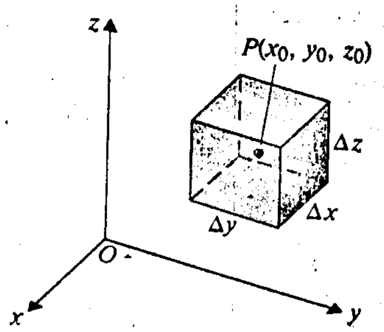


Fig. 2-19 A differential volume in Cartesian coordinates.

$$\oint_S \mathbf{A}\cdot d\mathbf{s} = \left[\int_{\substack{\text{front}\\ \text{face}}} + \int_{\substack{\text{back}\\ \text{face}}} + \int_{\substack{\text{right}\\ \text{face}}} + \int_{\substack{\text{left}\\ \text{face}}} + \int_{\substack{\text{top}\\ \text{face}}} + \int_{\substack{\text{bottom}\\ \text{face}}}\right]\mathbf{A}\cdot d\mathbf{s}$$

On the front face:

$$\int_{\substack{\text{front}\\ \text{face}}} \mathbf{A}\cdot d\mathbf{s} = \mathbf{A}_{\substack{front\\ face}}\cdot\Delta\mathbf{s}_{\substack{front\\ face}} = \mathbf{A}_{\substack{front\\ face}}\cdot\hat{\mathbf{x}}(\Delta y\Delta z) = A_x(x_0+\frac{\Delta x}{2}, y_0, z_0)\Delta y\Delta z$$

Using the Taylor series expansion to expand the term $A_x(x_0+\Delta x/2, y_0, z_0)$ yields:

$$A_x(x_0+\frac{\Delta x}{2}, y_0, z_0) = A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,y_0,z_0)} + \text{higher - order terms}$$

where higher-order terms (H.O.T.) contain the factor $(\Delta x/2)^2$, $(\Delta x/2)^3$ and so on.
Similarly, on the back face

$$\int_{\substack{\text{back}\\ \text{face}}} \mathbf{A}\cdot d\mathbf{s} = \mathbf{A}_{\substack{back\\ face}}\cdot\Delta\mathbf{s}_{\substack{back\\ face}} = \mathbf{A}_{\substack{back\\ face}}\cdot(-\hat{\mathbf{x}}\Delta y\Delta z) = -A_x(x_0-\frac{\Delta x}{2}, y_0, z_0)\Delta y\Delta z$$

$$= -A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T.}$$

Thus,

$$\left[\int_{\substack{front\\ face}} + \int_{\substack{back\\ face}}\right]\mathbf{A}\cdot d\mathbf{s} = \left(\frac{\partial A_x}{\partial x} + \text{H.O.T.}\right)\bigg|_{(x_0,y_0,z_0)} \Delta x\Delta y\Delta z$$

Following the same procedure, one obtains

$$\left[\int_{\substack{right\\ face}} + \int_{\substack{left\\ face}}\right]\mathbf{A}\cdot d\mathbf{s} = \left(\frac{\partial A_y}{\partial y} + \text{H.O.T.}\right)\bigg|_{(x_0,y_0,z_0)} \Delta x\Delta y\Delta z$$

$$\left[\int_{\substack{top\\ face}} + \int_{\substack{bottom\\ face}}\right]\mathbf{A}\cdot d\mathbf{s} = \left(\frac{\partial A_z}{\partial z} + \text{H.O.T.}\right)\bigg|_{(x_0,y_0,z_0)} \Delta x\Delta y\Delta z$$

Hence,

$$\oint_S \mathbf{A}\cdot d\mathbf{s} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\bigg|_{(x_0,y_0,z_0)} \Delta x\Delta y\Delta z + \text{H.O.T. in } \Delta x, \Delta y, \Delta z$$

Since $\Delta v = \Delta x \Delta y \Delta z$, taking the limit as Δv approaches 0 yields

$$\text{div } \mathbf{A} \equiv \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$\nabla \cdot \mathbf{A}$ read "del dot A".
If $\nabla \cdot \mathbf{A}=0$, **A** is called a solenoidal (divergenceless, divergence-free) field.

2-7 DIVERGENCE THEOREM

Since divergence of a vector field is defined as the net outward flux per unit volume, one can expect that *the volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume*, i.e.,

$$\boxed{\int_V (\nabla \cdot \mathbf{A}) dv = \oint_S \mathbf{A} \cdot d\mathbf{s}}$$: Divergence Theorem

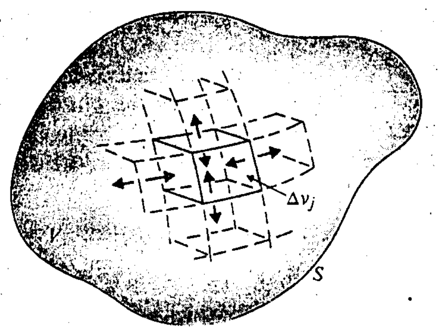


Fig. 2–20 Subdivided volume for proof of divergence theorem.

Consider a very small differential volume element Δv_j bounded by a surface s_j, the definition of divergence gives directly

$$(\nabla \cdot \mathbf{A})_j \Delta v_j = \oint_{S_j} \mathbf{A} \cdot d\mathbf{s}$$

Now, subdividing the entire volume V into N small differential volumes, and combining all differential volumes yields

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \Delta v_j \right] = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{S_j} \mathbf{A} \cdot d\mathbf{s} \right] \qquad (3)$$

The left hand side of (3) is, by definition, the volume integral of $\nabla \cdot \mathbf{A}$, i.e.,

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \Delta v_j \right] = \int_V (\nabla \cdot \mathbf{A}) dv .$$

While the right hand side of (3) is equal to

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{S_j} \mathbf{A} \cdot d\mathbf{s} \right] = \oint_S \mathbf{A} \cdot d\mathbf{s} ,$$

since the contributions from the internal surfaces of adjacent elements will cancel each other, resulting in the net contribution of the right side of (3) equal to the contributions of the external surface S bounding the volume V.

Example 2-12 Given $\mathbf{A} = \hat{\mathbf{x}}x^2 + \hat{\mathbf{y}}xy + \hat{\mathbf{z}}yz$, verify the divergence theorem over a cube one unit on each side.

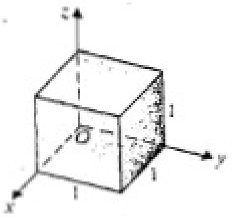


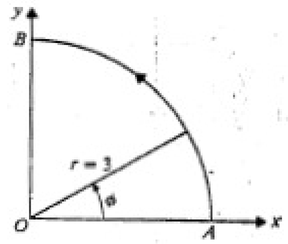
Fig. 2–21 A unit cube (Example 2–14).

2-8 CURL OF A VECTOR FIELD

A net outward flux of a vector **A** through a surface bounding a volume indicates the presence of a source; this source is called *a flow source*, and Div **A** is a measure of the strength of the flow source.
The other kind of source, called *vortex source*, causes a circulation of a vector field around it. The ***net circulation*** (or simply ***circulation***) of a vector field around a closed path is defined as

Circulation of **A** around contour C $\equiv \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}$

<u>Example 2-14</u> Given a vector field $\mathbf{F} = \hat{\mathbf{x}}xy - \hat{\mathbf{y}}2x$, find its circulation around the path OABO in the figure.



Curl of a vector field is the measure of the strength of a vortex source, which is given by

$$\text{curl}\,\mathbf{A} \equiv \nabla \times \mathbf{A} \equiv \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\hat{\mathbf{a}}_n \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}\right]_{\max}$$

or *a <u>vector</u> whose magnitude is the maximum net circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum.*

In the figure on the right

$$\left(\nabla \times \mathbf{A}\right)_u = \hat{\mathbf{a}}_{\mathbf{u}} \cdot \left(\nabla \times \mathbf{A}\right) = \lim_{\Delta s \to 0} \frac{1}{\Delta s_u}\left[\oint_{C_u} \mathbf{A} \cdot d\boldsymbol{\ell}\right]$$

Next, determine the three components of $\nabla\times\mathbf{A}$ in Cartesian coordinates. First, in order to find $(\nabla\times\mathbf{A})_x$ in the figure below, let $u \to x$, $\Delta S_u = \Delta y \Delta z$ and C denote the paths 1,2,3,4 indicated in the figure.

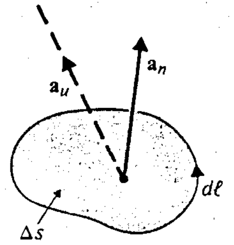


Fig. 2–23 Relation between $\mathbf{a}_n$ and $d\ell$ in defining curl.

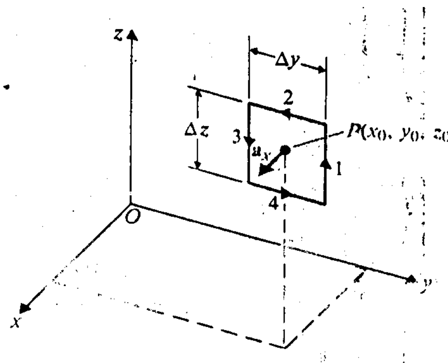


Fig. 2–24 Determining $(\nabla \times \mathbf{A})_x$.

$$(\nabla\times\mathbf{A})_x = \lim_{\Delta s\to 0}\frac{1}{\Delta y\Delta z}\left[\oint_{\substack{sides\\1,2,3,4}}\mathbf{A}\cdot d\boldsymbol{\ell}\right]$$

Since in Cartesian coordinates $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$,

Side 1: $d\boldsymbol{\ell} = \hat{\mathbf{z}}\Delta z$; $\mathbf{A}\cdot d\boldsymbol{\ell}=A_z(x_0, y_0+\Delta y/2, z_0)\,\Delta z$

$$A_z\left(x_0, y_0+\frac{\Delta y}{2}, z_0\right) = A_z(x_0,y_0,z_0)+\frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T.}$$

Thus,

$$\int_{\textbf{side 1}}\mathbf{A}\cdot d\boldsymbol{\ell} = \left\{A_z(x_0,y_0,z_0)+\frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T.}\right\}\Delta z$$

Side 3: $d\boldsymbol{\ell} = -\hat{\mathbf{z}}\Delta z$; $\mathbf{A}\cdot d\boldsymbol{\ell}=-A_z(x_0, y_0-\Delta y/2, z_0)\,\Delta z$

$$A_z\left(x_0, y_0-\frac{\Delta y}{2}, z_0\right) = A_z(x_0,y_0,z_0)-\frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T.}$$

Thus,

$$\int_{\textbf{side 3}}\mathbf{A}\cdot d\boldsymbol{\ell} = -\left\{A_z(x_0,y_0,z_0)-\frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T.}\right\}\Delta z$$

and

$$\int_{\substack{\textbf{side 1}+\\ side\,3}}\mathbf{A}\cdot d\boldsymbol{\ell} = \left\{\frac{\partial A_z}{\partial y}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T.}\right\}\Delta y\Delta z$$

Using the same approach yields

$$\int_{\substack{\textbf{side 2}+\\ side\,4}}\mathbf{A}\cdot d\boldsymbol{\ell} = \left\{-\frac{\partial A_y}{\partial z}\bigg|_{(x_0,y_0,z_0)} + \text{H.O.T.}\right\}\Delta y\Delta z$$

Hence, $(\nabla\times\mathbf{A})_x = \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}$.

$(\nabla\times\mathbf{A})_\mathbf{y}$, $(\nabla\times\mathbf{A})_\mathbf{z}$ can be found using the same procedure. It follows that curl in Cartesian coordinates can be given by,

$$\nabla\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

If $\nabla\times\mathbf{A} = \mathbf{0}$, **A** is called an *irrotational* (*conservative* or *curl-free*) *field.*

2-9 STOKES' THEOREM

By definition of curl, one obtains

$$(\nabla\times\mathbf{A})_j\cdot(\Delta\mathbf{s_j})=\oint_{c_j}\mathbf{A}\cdot d\boldsymbol{\ell}$$

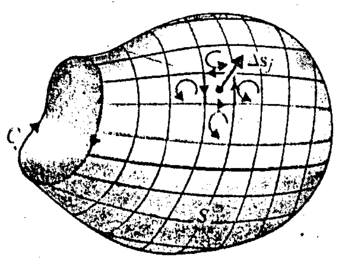


Fig. 2–25 Subdivided area for proof of Stokes's theorem.

Subdividing the entire surface into N small differential surfaces and summing all contributions yields

$$\lim_{\Delta s_j\to 0}\sum_{j=1}^{N}(\nabla\times\mathbf{A})_j\cdot(\Delta\mathbf{s_j})=\lim_{\Delta s_j\to 0}\sum_{j=1}^{N}\oint_{c_j}\mathbf{A}\cdot d\boldsymbol{\ell}$$

The left side is equals to

$$\lim_{\Delta s_j\to 0}\sum_{j=1}^{N}(\nabla\times\mathbf{A})_j\cdot(\Delta\mathbf{s_j})=\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{s}$$

while the right side equals to

$$\lim_{\Delta s_j\to 0}\sum_{j=1}^{N}\oint_{c_j}\mathbf{A}\cdot d\boldsymbol{\ell}=\oint_C\mathbf{A}\cdot d\boldsymbol{\ell}$$

Hence, Stokes' theorem can be given as follows:

$$\boxed{\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{s}=\oint_C\mathbf{A}\cdot d\boldsymbol{\ell}}$$

which means

the surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface

When the surface is *closed*, $\oint_S(\nabla\times\mathbf{A})\cdot d\mathbf{s}=0$.

Example 2-16 Verify Stokes' theorem for the vector field in Example 2-14 over a quarter-circular disk.

Green's Theorem Let C be a positively oriented, piecewise smooth, simple closed curve in the plane $\mathbb{R}^2$, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\boxed{\int_D\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)dxdy=\oint_C(Ldx+Mdy)}$$

Proof via Stokes' theorem Let $\mathbf{F}=\hat{\mathbf{x}}L(x,y)+\hat{\mathbf{y}}M(x,y)$, and D is a region on xy-plane, then from Stokes' theorem,

$$\int_D\nabla\times\mathbf{F}\cdot d\mathbf{s}=\int_D\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)dxdy=\oint_C\mathbf{F}\cdot d\boldsymbol{\ell}=\oint_C(Ldx+Mdy).$$

Note that Green's theorem is a special case of the Stokes' theorem, when applied to a region in the *xy*-plane.

2-10 TWO NULL IDENTITIES

1. Identity I: $\nabla\times(\nabla V)\equiv\mathbf{0}$

From Stokes' theorem

$$\int_S[\nabla\times(\nabla V)]\cdot d\mathbf{s}=\oint_C(\nabla V)\cdot d\boldsymbol{\ell}$$

However, since $dV = \nabla V \cdot d\boldsymbol{\ell}$, $\oint_C (\nabla V)\cdot d\boldsymbol{\ell} = \oint_C dV = 0$. Therefore, the surface integral of $\nabla\times(\nabla V)$ over any surface is zero. It follows that the integrand itself must therefore vanish, thus identity I is true.
A converse statement of Identity I can be made as follows: *If a vector field is curl-free, then it can be expressed as the gradient of a scalar field.* For example,
If $\nabla\times\mathbf{E} = \mathbf{0}$, **E** can be given by $\mathbf{E} = -\nabla V$, where *V* is a scalar field.
2. Identity II: $\nabla\cdot(\nabla\times\mathbf{A}) \equiv 0$
From divergence theorem,
$$\int_V \nabla\cdot(\nabla\times\mathbf{A})dv = \oint_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \int_{S_1}(\nabla\times\mathbf{A})\cdot\hat{\mathbf{a}}_{n1}ds + \int_{S_2}(\nabla\times\mathbf{A})\cdot\hat{\mathbf{a}}_{n2}ds = \oint_{C_1}\mathbf{A}\cdot d\mathbf{l} + \oint_{C_2}\mathbf{A}\cdot d\mathbf{l} = 0$$
Since this must hold for any volume, the identity II must be true.
It follows that since $\nabla\cdot(\nabla\times\mathbf{A}) = 0$, $\oint_S (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{s} = 0$ where *S* is a closed surface, as mentioned in 2-9.
Thus, *if a vector field is divergenceless, then it can be expressed as the curl of another vector field.* For example,
If $\nabla\cdot\mathbf{B} = 0$ then **B** can be expressed as $\mathbf{B} = \nabla\times\mathbf{A}$.

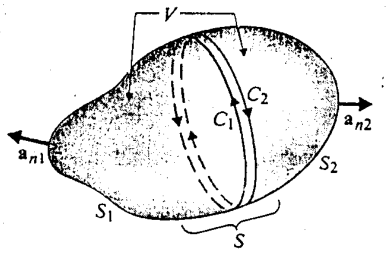


Fig. 2–26 An arbitraty volume *V* enclosed by surface *S*.

2-11 FIELD CLASSIFICATION AND HELMHOLTZ'S THEOREM

In general, vector fields can be classified as

1. Solenoidal and irrotational if $\nabla\cdot\mathbf{F} = 0$ and $\nabla\times\mathbf{F} = \mathbf{0}$
2. Solenoidal but not irrotational if $\nabla\cdot\mathbf{F} = 0$ and $\nabla\times\mathbf{F} \neq \mathbf{0}$
3. Irrotational but not solenoidal if $\nabla\cdot\mathbf{F} \neq 0$ and $\nabla\times\mathbf{F} = \mathbf{0}$
4. Neither irrotational nor solenoidal if $\nabla\cdot\mathbf{F} \neq 0$ and $\nabla\times\mathbf{F} \neq \mathbf{0}$

The most general vector field has both a nonzero divergence and a nonzero curl, and can be considered as the sum of a solenoidal field and an irrotational field.
Helmholtz's Theorem: *A vector field is determined if both its divergence and its curl are specified everywhere.*
Therefore, a vector field can generally be expressed as
$$\mathbf{F} = \mathbf{F}_i + \mathbf{F}_s$$
where $\mathbf{F}_i$ is a irrotational (conservative) field, and $\mathbf{F}_s$ is a solenoidal field. One can observe that
$\nabla\cdot\mathbf{F} = \nabla\cdot\mathbf{F}_i + \nabla\cdot\mathbf{F}_s = \nabla\cdot\mathbf{F}_i = g \Rightarrow$ conservative (irrotational) field component
$\nabla\times\mathbf{F} = \nabla\times\mathbf{F}_i + \nabla\times\mathbf{F}_s = \nabla\times\mathbf{F}_s = \mathbf{G} \Rightarrow$ solenoidal field component
Thus, both divergence and curl have to be specified. In other words, if both flow and vortex sources are specified, the vector field will be determined.
Since $\mathbf{F}_i$ is irrotational, it can be expressed as $\mathbf{F}_i = -\nabla V$. Likewise, since $\mathbf{F}_s$ is solenoidal, it can be expressed as $\mathbf{F}_s = \nabla\times\mathbf{A}$. Hence,
$$\mathbf{F} = \mathbf{F}_i + \mathbf{F}_s = -\nabla V + \nabla\times\mathbf{A}$$
In electromagnetic, *V*, **A** are called scalar potential, vector potential, respectively.
Example Given $\mathbf{F} = \hat{\mathbf{x}}(3y - c_1 z) + \hat{\mathbf{y}}(c_2 x - 2z) - \hat{\mathbf{z}}(c_3 y + z)$

(a) Determine the constants c_1, c_2, c_3 if **F** is irrotational.
(b) Determine the scalar potential *V* whose negative gradient equals **F**.