Lecture 5-6

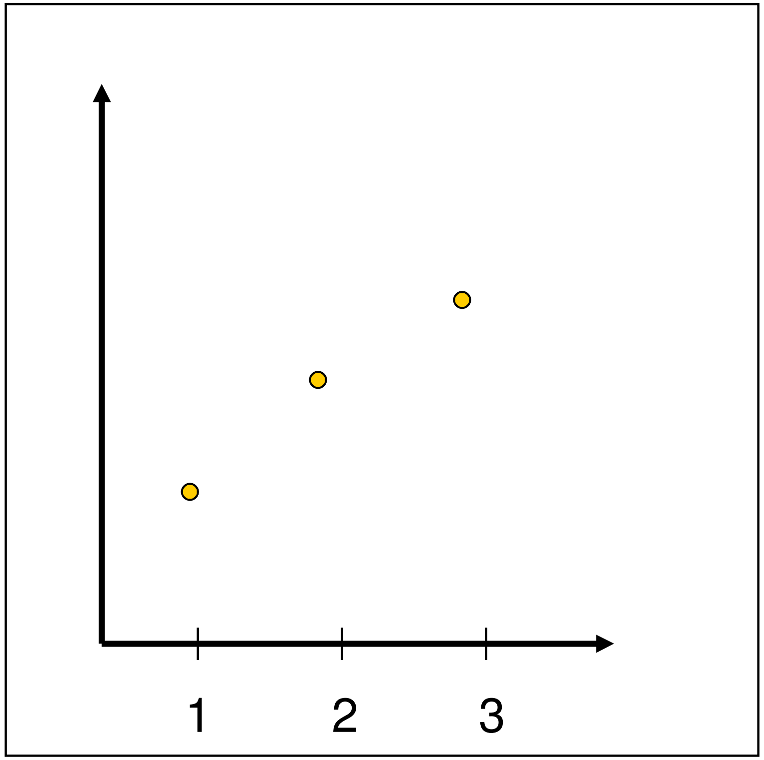
Curve Fitting

- Least-square Regression
- Linear Regression & Normal Equations
- Linear and Quadratic Interpolation
- Newton's Divided Difference Method
- Lagrange Interpolation
- Spline Interpolation

Motivation

Given a set of experimental data:

x	1	2	3
y	5.1	5.9	6.3



- The relationship between ***x*** and ***y*** may not be clear.
- Find a function ***f(x)*** that best fit the data

Motivation

- In engineering, two types of applications are encountered:
  - ***<u>Trend analysis:</u>*** Predicting values of dependent variable, may include extrapolation beyond data points or interpolation between data points.
  - ***<u>Hypothesis testing:</u>*** Comparing existing mathematical model with measured data.

1. What is the best mathematical function ***f*** that represents the dataset?
2. What is the best criterion to assess the fitting of the function ***f*** to the data?

Curve Fitting

Given a set of tabulated data, find a curve or a function that *best represents the data*.

Given:

1. The tabulated **data**
2. The **form** of the function
3. The curve fitting **criteria**

Find the **unknown** coefficients

Least Squares Regression

Linear Regression

- Fitting a straight line to a set of paired observations:

  $(x_1, y_1), (x_2, y_2),\ldots,(x_n, y_n)$.

  $y=a_0+a_1x+e$

  a_1-slope.

  a_0-intercept.

  e-error, or residual, between the model and the observations.

Selection of the Functions

Linear $f(x) = a + bx$

Quadratic $f(x) = a + bx + cx^2$

Polynomial $f(x) = \sum_{k=0}^{n} a_k \; x^k$

General $f(x) = \sum_{k=0}^{m} a_k \; g_k(x)$

$g_k(x)$ are known.

Decide on the Criterion

1. Least Squares Regression :

$$\text{minimize} \quad \sum_{i=1}^{n} (y_i - f(x_i))^2$$

Minimize Squared Error

2. Exact Matching (Interpolation) :

$$y_i = f(x_i)$$

Predict the “in-between” values

Least Squares Regression

Given:

x_i	x_1	x_2	....	x_n
y_i	y_1	y_2	....	y_n

The form of the function is assumed to be known but the coefficients are unknown.

$$e_i^{\,2} = (y_i - f(x_i))^2 = (f(x_i) - y_i)^2$$

The difference is assumed to be the result of experimental error.

Determine the Unknowns

We want to find a and b to minimize:

$$\Phi(a,b) = \sum_{i=1}^{n} (a + bx_i - y_i)^2$$

How do we obtain a and b to minimize: $\Phi(a,b)$?

Necessary condition for the minimum:

$$\frac{\partial\ \Phi(a,b)}{\partial\ a} = 0 \text{ AND } \frac{\partial\ \Phi(a,b)}{\partial\ b} = 0$$

Normal Equations

$$\frac{\partial\ \Phi(a,b)}{\partial\ a} = \sum_{i=1}^{n} 2(a + bx_i - y_i) = 0$$

$$\rightarrow n\, a + \left(\sum_{i=1}^{n} x_i\right) b = \left(\sum_{i=1}^{n} y_i\right)$$

$$\frac{\partial\ \Phi(a,b)}{\partial\ b} = \sum_{i=1}^{n} 2(a + bx_i - y_i) x_i = 0$$

$$\rightarrow \left(\sum_{i=1}^{n} x_i\right) a + \left(\sum_{i=1}^{n} x_i^2\right) b = \left(\sum_{i=1}^{n} x_i\, y_i\right)$$

Normal Equations

From Linear Algebra

- Let

$$\mathbf{A} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_k \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}; \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_k \\ \vdots \\ y_n \end{bmatrix}; \mathbf{c} = \begin{bmatrix} a \\ b \end{bmatrix}$$

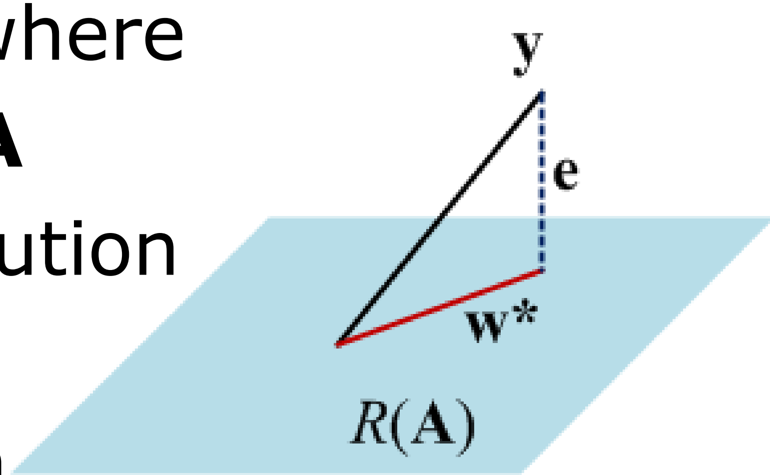
- Then the matrix equation becomes

$$\mathbf{Ac} = \mathbf{y}$$

n equations 2 unknowns -> no solution unless only 2 linearly independent equations

Normal equation in matrix form

- Let $\mathbf{e} = \mathbf{y} - \mathbf{w} = \mathbf{y} - \mathbf{Ac}$ where $\mathbf{w} \in R(\mathbf{A})$, column space of $\mathbf{A}$
- Then the least square solution occurs when
$\mathbf{e} \perp R(\mathbf{A})$ -> $\mathbf{e} \perp \boldsymbol{a}_i$, i=1,…,n
($\boldsymbol{a}_i$: column vector of $\mathbf{A}$)
- Thus, one obtains



$$\mathbf{A}^T\mathbf{e} = \mathbf{0} \rightarrow \boxed{\mathbf{A}^T\mathbf{A}\mathbf{c} = \mathbf{A}^T\mathbf{y}}$$

Normal equation

Normal Equations in matrix form

- In matrix form:

$$\mathbf{A} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_k \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}; \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_k \\ \vdots \\ y_n \end{bmatrix}; \mathbf{c} = \begin{bmatrix} a \\ b \end{bmatrix}; \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{c} = \mathbf{A}^{\mathrm{T}}\mathbf{y}$$

- Thus

$$b = \frac{n\left(\sum_{i=1}^{n} x_i\, y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}; a = \frac{1}{n}\left(\left(\sum_{i=1}^{n} y_i\right) - b\left(\sum_{i=1}^{n} x_i\right)\right)$$

Example 1: Linear Regression

Assume :

$$f(x) = a + bx$$

x	1	2	3
y	5.1	5.9	6.3

Equations :

$$n\,a + \left(\sum_{i=1}^{n} x_i\right) b = \left(\sum_{i=1}^{n} y_i\right)$$

$$\left(\sum_{i=1}^{n} x_i\right) a + \left(\sum_{i=1}^{n} x_i^2\right) b = \left(\sum_{i=1}^{n} x_i\; y_i\right)$$

Example 1: Linear Regression

i	1	2	3	sum
x_i	1	2	3	6
y_i	5.1	5.9	6.3	17.3
x_i^2	1	4	9	14
$x_i\ y_i$	5.1	11.8	18.9	35.8

Equations:

$$3a + 6b = 17.3$$

$$6a + 14b = 35.8$$

$$Solving: \quad a = 4.5667 \qquad b = 0.60$$

Multiple Linear Regression

Example:

Given the following data:

t	0	1	2	3
x	0.1	0.4	0.2	0.2
y	3	2	1	2

Determine a function of two variables:

$$f(x,t) = a + b\,x + c\,t$$

That best fits the data with the least sum of the square of errors.

Solution of Multiple Linear Regression

Construct Φ , the sum of the square of the error and derive the necessary conditions by equating the partial derivatives with respect to the unknown parameters to zero, then solve the equations.

t	0	1	2	3
x	0.1	0.4	0.2	0.2
y	3	2	1	2

Solution of Multiple Linear Regression

$$f(x,t) = a + bx + ct, \quad \Phi(a,b,c) = \sum_{i=1}^{n} (a + bx_i + ct_i - y_i)^2$$

Necessary conditions:

$$\frac{\partial \Phi(a,b,c)}{\partial a} = 2\sum_{i=1}^{n} (a + bx_i + ct_i - y_i) \quad = 0$$

$$\frac{\partial \Phi(a,b,c)}{\partial b} = 2\sum_{i=1}^{n} (a + bx_i + ct_i - y_i) x_i = 0$$

$$\frac{\partial \Phi(a,b,c)}{\partial c} = 2\sum_{i=1}^{n} (a + bx_i + ct_i - y_i) t_i = 0$$

System of Equations (Normal Equations)

$$a\,n + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} y_i$$

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} (x_i)^2 + c\sum_{i=1}^{n} (x_i\ t_i) = \sum_{i=1}^{n} (x_i\ y_i)$$

$$a\sum_{i=1}^{n} t_i + b\sum_{i=1}^{n} (x_i\ t_i) + c\sum_{i=1}^{n} (t_i)^2 = \sum_{i=1}^{n} (t_i\ y_i)$$

Normal Equations in Matrix Form

$$\mathbf{A} = \begin{bmatrix} 1 & x_1 & t_1 \\ \vdots & \vdots & \vdots \\ 1 & x_k & t_k \\ \vdots & \vdots & \vdots \\ 1 & x_n & t_n \end{bmatrix}; \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_k \\ \vdots \\ y_n \end{bmatrix}; \mathbf{c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix};$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{c} = \mathbf{A}^{\mathrm{T}}\mathbf{y}$$

Example 2: Multiple Linear Regression

i	1	2	3	4	Sum
t_i	0	1	2	3	6
x_i	0.1	0.4	0.2	0.2	0.9
y_i	3	2	1	2	8
x_i^2	0.01	0.16	0.04	0.04	0.25
$x_i t_i$	0	0.4	0.4	0.6	1.4
$x_i y_i$	0.3	0.8	0.2	0.4	1.7
t_i^2	0	1	4	9	14
$t_i y_i$	0	2	2	6	10

Example 2: System of Equations

$$4a + 0.9b + 6c = 8$$

$$0.9a + 0.25b + 1.4c = 1.7$$

$$6a + 1.4b + 14c = 10$$

$$Solving:$$

$$a = 2.9574, \ b = -1.7021, \quad c = -0.38298$$

$$f(x,t) = a + bx + ct = 2.9574 - 1.7021\,x - 0.38298\,t$$

Polynomial Regression

- The least squares method can be extended to fit the data to a higher-order polynomial

$$f(x) = a + bx + cx^2, \;\; e_i^2 = (f(x) - y_i)^2$$

$$\text{Minimize} \quad \Phi(a,b,c) = \sum_{i=1}^{n} \left(a + bx_i + cx_i^2 - y_i\right)^2$$

Necessary conditions :

$$\frac{\partial \Phi(a,b,c)}{\partial a} = 0, \quad \frac{\partial \Phi(a,b,c)}{\partial b} = 0, \quad \frac{\partial \Phi(a,b,c)}{\partial c} = 0$$

Equations for Quadratic Regression

Minimize $\Phi(a,b,c) = \sum_{i=1}^{n} \left(a + bx_i + cx_i^2 - y_i \right)^2$

$$\frac{\partial \Phi(a,b,c)}{\partial a} = 2\sum_{i=1}^{n} \left(a + bx_i + cx_i^2 - y_i \right) = 0$$

$$\frac{\partial \Phi(a,b,c)}{\partial b} = 2\sum_{i=1}^{n} \left(a + bx_i + cx_i^2 - y_i \right) x_i = 0$$

$$\frac{\partial \Phi(a,b,c)}{\partial c} = 2\sum_{i=1}^{n} \left(a + bx_i + cx_i^2 - y_i \right) x_i^2 = 0$$

Normal Equations

$$a\,n + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i$$

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{n} x_i\,y_i$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4 = \sum_{i=1}^{n} x_i^2\,y_i$$

Example 3: Polynomial Regression

Fit a second-order polynomial to the following data

x_i	0	1	2	3	4	5	$\sum=15$
y_i	2.1	7.7	13.6	27.2	40.9	61.1	$\sum=152.6$
x_i^2	0	1	4	9	16	25	$\sum=55$
x_i^3	0	1	8	27	64	125	225
x_i^4	0	1	16	81	256	625	$\sum=979$
$x_i\ y_i$	0	7.7	27.2	81.6	163.6	305.5	$\sum=585.6$
$x_i^2\ y_i$	0	7.7	54.4	244.8	654.4	1527.5	$\sum=2488.8$

Example 3: Equations and Solution

$$6a + 15b + 55c = 152.6$$

$$15a + 55b + 225c = 585.6$$

$$55a + 225b + 979c = 2488.8$$

Solving…

$$a = 2.4786,\ b = 2.3593,\ c = 1.8607$$

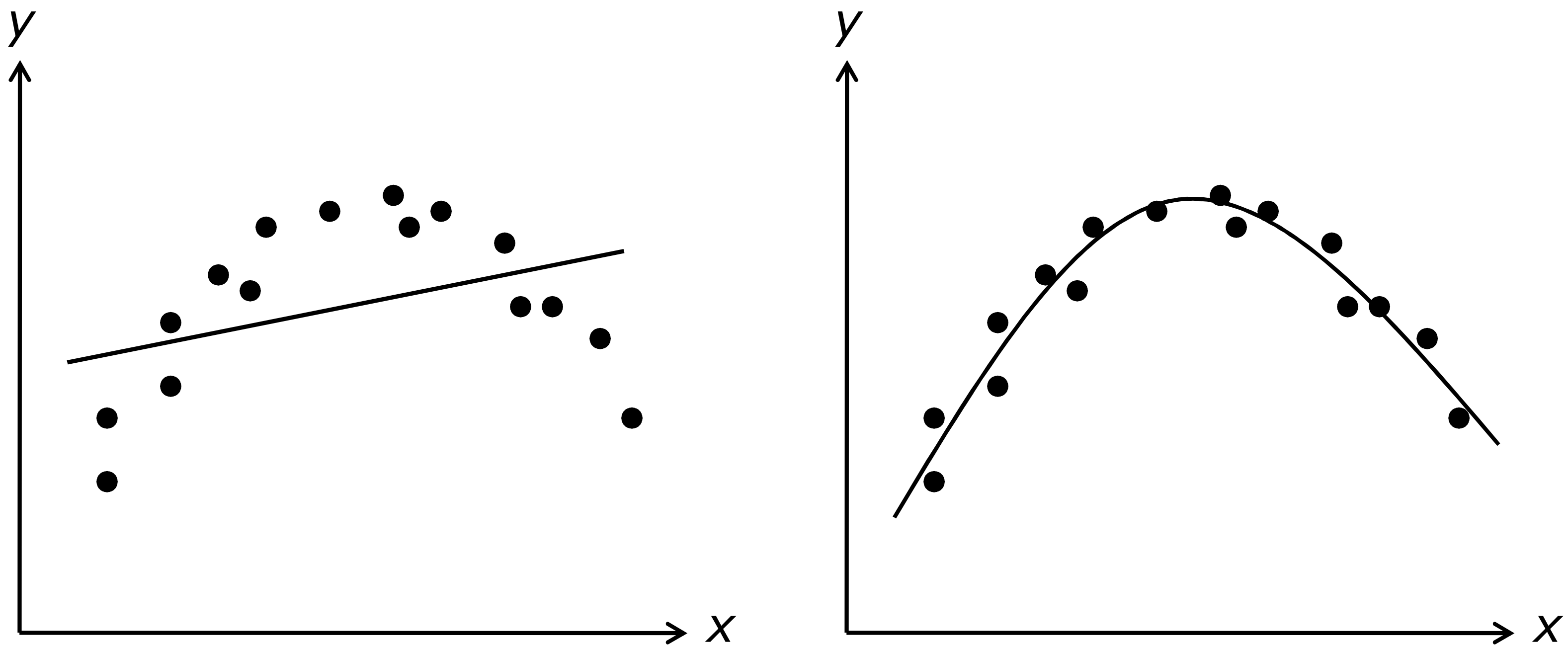
$$f(x) = 2.4786 + 2.3593x + 1.8607x^2$$

How Do You Judge Functions?

Given two or more functions to fit the data,
How do you select the best?

Answer :
Determine the parameters for each function, then compute Φ for each one. The function resulting in smaller Φ (least sum of the squares of the errors) is the best.

Example showing that Quadratic is preferable than Linear Regression



Linear Regression **Quadratic Regression**

Fitting with Nonlinear Functions

x_i	0.24	0.65	0.95	1.24	1.73	2.01	2.23	2.52
y_i	0.23	-0.23	-1.1	-0.45	0.27	0.1	-0.29	0.24

It is required to find a function of the form:

$$f(x) = a \ln(x) + b \cos(x) + c\, e^x$$

to fit the data.

$$\Phi(a,b,c) = \sum_{i=1}^{n} (f(x_i) - y_i)^2$$

Fitting with Nonlinear Functions

$$\Phi(a,b,c) = \sum_{i=1}^{n} (a \ln(x_i) + b \cos(x_i) + c\, e^{x_i} - y_i)^2$$

$$\text{Necessary condition for the minimum:}$$

$$\left.\begin{aligned} \frac{\partial\, \Phi(a,b,c)}{\partial\, a} &= 0 \\ \frac{\partial\, \Phi(a,b,c)}{\partial\, b} &= 0 \\ \frac{\partial\, \Phi(a,b,c)}{\partial\, c} &= 0 \end{aligned}\right\} \Rightarrow \textit{Normal Equations}$$

Normal Equations

$$a\sum_{i=1}^{n}(\ln x_i)^2 + b\sum_{i=1}^{n}(\ln x_i)(\cos x_i) + c\sum_{i=1}^{n}(\ln x_i)(e^{x_i}) = \sum_{i=1}^{n} y_i(\ln x_i)$$

$$a\sum_{i=1}^{n}(\ln x_i)(\cos x_i) + b\sum_{i=1}^{n}(\cos x_i)^2 + c\sum_{i=1}^{n}(\cos x_i)(e^{x_i}) = \sum_{i=1}^{n} y_i(\cos x_i)$$

$$a\sum_{i=1}^{n}(\ln x_i)(e^{x_i}) + b\sum_{i=1}^{n}(\cos x_i)(e^{x_i}) + c\sum_{i=1}^{n}(e^{x_i})^2 = \sum_{i=1}^{n} y_i(e^{x_i})$$

Evaluate the sums and solve the normal equations.

Example 4: Evaluating Sums

xi	0.24	0.65	0.95	1.24	1.73	2.01	2.23	2.52	∑=11.57
yi	0.23	-0.23	-1.1	-0.45	0.27	0.1	-0.29	0.24	∑=-1.23
$(\ln xi)^2$	2.036	0.1856	0.0026	0.0463	0.3004	0.4874	0.6432	0.8543	∑=4.556
ln(xi) cos(xi)	-1.386	-0.3429	-0.0298	0.0699	-0.0869	-0.2969	-0.4912	-0.7514	∑=-3.316
$\ln(xi) * e^{xi}$	-1.814	-0.8252	-0.1326	0.7433	3.0918	5.2104	7.4585	11.487	∑=25.219
yi * ln(xi)	-0.328	0.0991	0.0564	-0.0968	0.1480	0.0698	-0.2326	0.2218	∑=-0.0625
$\cos(xi)^2$	0.943	0.6337	0.3384	0.1055	0.0251	0.1808	0.3751	0.6609	∑=3.26307
$\cos(xi) * e^{xi}$	1.235	1.5249	1.5041	1.1224	-0.8942	-3.1735	-5.696	-10.104	∑=-14.481
yi*cos(xi)	0.223	-0.1831	-0.6399	-0.1462	-0.0428	-0.0425	0.1776	-0.1951	∑=-0.8485
$(e^{xi})^2$	1.616	3.6693	6.6859	11.941	31.817	55.701	86.488	154.47	∑=352.39
$yi * e^{xi}$	0.2924	-0.4406	-2.844	-1.555	1.523	0.7463	-2.697	2.9829	∑=-1.9923

Example 4: Equations & Solution

$$4.55643\,a - 3.31547\,b + 25.2192\,c = -0.062486$$

$$-3.31547\,a + 3.26307\,b - 14.4815\,c = -0.848514$$

$$25.2192\,a - 14.4815\,b + 352.388\,c = -1.992283$$

Solving the above equations :

$$a = -0.88815, \quad b = -1.1074, \quad c = 0.012398$$

Therefore,

$$f(x) = -0.88815\ \ln(x) - 1.1074\cos(x) + 0.012398\,e^{x}$$

Example 5

Given:

x_i	1	2	3
y_i	2.4	5	9

Find a function $f(x) = ae^{bx}$ that best fits the data.

$$\Phi(a,b) = \sum_{i=1}^{n} \left(ae^{bx_i} - y_i \right)^2$$

Normal Equations are obtained using:

$$\frac{\partial \Phi}{\partial a} = 2\sum_{i=1}^{n} \left(ae^{bx_i} - y_i \right) e^{bx_i} = 0$$

Difficult to Solve

$$\frac{\partial \Phi}{\partial b} = 2\sum_{i=1}^{n} \left(ae^{bx_i} - y_i \right) a\, x_i\, e^{bx_i} = 0$$

Linearization Method

Find a function $f(x) = ae^{bx}$ that best fits the data.

Define $g(x) = \ln(f(x)) = \ln(a) + b\,x$

Define $z_i = \ln(y_i) = \ln(a) + bx_i$

Let $\alpha = \ln(a)$ *and* $z_i = \ln(y_i)$

Instead of minimizing: $\Phi(a,b) = \sum_{i=1}^{n} \left(ae^{bx_i} - y_i\right)^2$

Minimize: $\Phi(\alpha,b) = \sum_{i=1}^{n} (\alpha + bx_i - z_i)^2$ (Easier to solve)

Example 5: Equations

$$\Phi(\alpha, b) = \sum_{i=1}^{n} (\alpha + b\, x_i - z_i)^2$$

Normal Equations are obtained using:

$$\frac{\partial \Phi}{\partial \alpha} = 2\sum_{i=1}^{n} (\alpha + b\, x_i - z_i) = 0$$

$$\frac{\partial \Phi}{\partial b} = 2\sum_{i=1}^{n} (\alpha + b\, x_i - z_i)\, x_i = 0$$

$$\alpha\, n + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} z_i \quad \text{and} \quad \alpha \sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} (x_i\, z_i)$$

Evaluating Sums and Solving

x_i	1	2	3	$\Sigma=6$
y_i	2.4	5	9	
$z_i=\ln(y_i)$	0.875469	1.609438	2.197225	$\Sigma=4.68213$
x_i^2	1	4	9	$\Sigma=14$
$x_i\ z_i$	0.875469	3.218876	6.591674	$\Sigma=10.6860$

Equations:

$$3\alpha + 6b = 4.68213$$

$$6\alpha + 14b = 10.686$$

Solving Equations:

$$\alpha = 0.23897,\quad b = 0.66087$$

$$\alpha = \ln(a),\quad a = e^{\alpha}$$

$$a = e^{0.23897} = 1.26994$$

$$f(x) = ae^{bx} = 1.26994\, e^{0.66087x}$$

More Linearization Method

- For a function of the form

$$y = f(x) = \frac{ax}{b+x}$$

- Can use the following linearization:

$$\frac{1}{y} = \frac{b+x}{ax} = \frac{b}{ax} + \frac{1}{a} \Rightarrow w = \alpha + \beta z$$

$$w = \frac{1}{y}; z = \frac{1}{x}; \alpha = \frac{1}{a}; \beta = \frac{b}{a}$$

Normal Equations in Matrix Form

$$\mathbf{A} = \begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_k \\ \vdots & \vdots \\ 1 & z_n \end{bmatrix} = \begin{bmatrix} 1 & 1/x_1 \\ \vdots & \vdots \\ 1 & 1/x_k \\ \vdots & \vdots \\ 1 & 1/x_n \end{bmatrix}; \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_k \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 1/y_1 \\ \vdots \\ 1/y_k \\ \vdots \\ 1/y_n \end{bmatrix}; \mathbf{c} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{c} = \mathbf{A}^{\mathrm{T}}\mathbf{w}$$

Then

$$a = \frac{1}{\alpha}; b = a\beta; y = \frac{ax}{b+x}$$

Other Transformations for Linearization

No.	Non-linear equation	Linear form	Relationship to $\hat{y} = a + b\hat{x}$	Values for least squares regression
1.	$y = cx^m$	$\ell n(y) = m\ell n(x) + \ell n(c)$	$\hat{y} = \ell n(y),\ \hat{x} = \ell n(x)$ $b = m,\ a = \ell n(c)$	$\ell n(x_i)$ and $\ell n(y_i)$
2.	$y = c\, e^{mx}$	$\ell n(y) = mx + \ell n(c)$	$\hat{y} = \ell n(y),\ \hat{x} = x$ $b = m,\ a = \ell n(c)$	x_i and $\ell n(y_i)$
3.	$y = c\, 10^{mx}$	$\log(y) = mx + \log c$	$\hat{y} = \log(y),\ \hat{x} = x$ $b = m,\ a = \log(c)$	x_i and $\ell n(y_i)$
4.	$y = \frac{1}{mx + c}$	$\frac{1}{y} = mx + c$	$\hat{y} = \frac{1}{y},\ \hat{x} = x$ $b = m,\ a = c$	x_i and $\frac{1}{y_i}$
5.	$y = \frac{mx}{c + x}$	$\frac{1}{y} = \frac{c}{mx} + \frac{1}{m}$	$\hat{y} = \frac{1}{y},\ \hat{x} = \frac{1}{x}$ $b = \frac{c}{m},\ a = \frac{1}{m}$	$\frac{1}{x_i}$ and $\frac{1}{y_i}$
6.	$xy^c = d$ Gas equation	$\log y = \frac{1}{c}\log d - \frac{1}{c}\log x$	$\hat{y} = \log y,\ \hat{x} = \log x$ $a = \frac{1}{c}\log d,\ b = -\frac{1}{c}$	$\log x_i$ and $\log y_i$
7.	$y = cd^x$	$\log y = \log c + x \log d$	$\hat{y} - \log y,\ \hat{x} = x$ $a = \log c,\ b = \log d$	x_i and $\log y_i$
8.	$y = c + d\sqrt{x}$	$y = c + d\hat{x}$ where $\hat{x} = \sqrt{x}$	$\hat{y} = y$ and $\hat{x} = \sqrt{x}$ $a = c$ and $b = d$	$\sqrt{x_i}$ and y_i

Interpolation : Introduction

Interpolation was used for long time to provide an estimate of a tabulated function at values that are not available in the table.

What is sin (0.15)?

x	sin(x)
0	0.0000
0.1	0.0998
0.2	0.1987
0.3	0.2955
0.4	0.3894

Using **Linear Interpolation** sin (0.15) ≈ **0.1493**

True value (4 decimal digits) sin (0.15) = **0.1494**

The Interpolation Problem

Given a set of n+1 points,

$$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots., \left(x_n, f(x_n)\right)$$

Find an n[th] order polynomial $f_n(x)$ that passes through all points, such that:

$$f_n(x_i) = f(x_i) \quad for\ i = 0, 1, 2, \ldots, n$$

Example

An experiment is used to determine the viscosity of water as a function of temperature. The following table is generated:

Temperature (degree)	Viscosity
0	1.792
5	1.519
10	1.308
15	1.140

Problem: Estimate the viscosity when the temperature is 8 degrees.

Interpolation Problem

Find a polynomial that fits the data points exactly.

$$V(T) = \sum_{k=0}^{n} a_k T^k$$

$$V_i = V(T_i)$$

V	: Viscosity
T	: Temperature
a_k	: Polynomial coefficients

Linear Interpolation: V(T)= 1.73 – 0.0422 T

(Assume straight line @[5,10]) V(8)= 1.3924

Linear Regression : V(T)= 1.765 – 0.433T

Existence and Uniqueness

Given a set of n+1 points:

$$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots., \left(x_n, f(x_n)\right)$$

Assumption: $x_0, x_1, \ldots, x_n$ are distinct

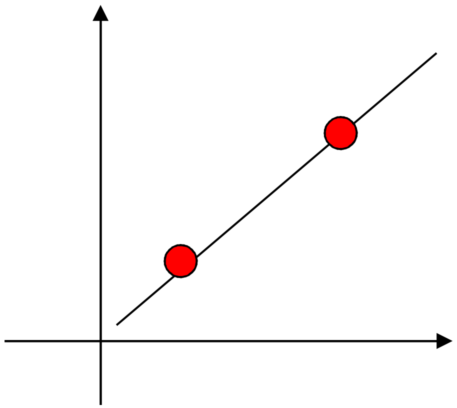
Theorem:

There is a unique polynomial ***f***$\boldsymbol{f_n(x)}$ of order ≤ n such that:

$$f_n(x_i) = f(x_i) \qquad for \quad i = 0,1,\ldots,n$$

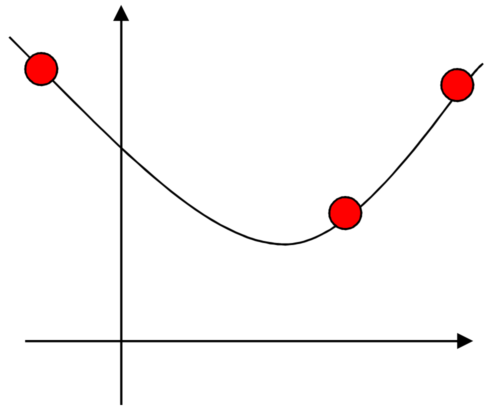
Examples of Polynomial Interpolation

Linear Interpolation



- Given any two points, there is one polynomial of order ≤ 1 that passes through the two points.

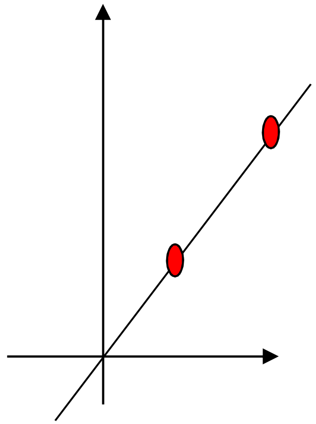
Quadratic Interpolation



Given any three points there is one polynomial of order ≤ 2 that passes through the three points.

Linear Interpolation

Given any two points, $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right)$



The line that interpolates the two points is:

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}\left(x - x_0\right)$$

Example :

Find a polynomial that interpolates (1,2) and (2,4).

$$f_1(x) = 2 + \frac{4-2}{2-1}\left(x-1\right) = 2x$$

Quadratic Interpolation

- Given any three points: $(x_0, f(x_0)),\ (x_1, f(x_1)),\ and\ (x_2, f(x_2))$
- The polynomial that interpolates the three points is:

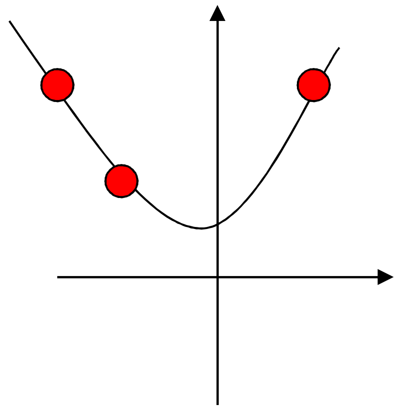
$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$where:$

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_0, x_1, x_2] = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



General nth Order Interpolation

Given any n+1 points: $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$

The polynomial that interpolates all points is:

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \ldots + b_n(x - x_0)\ldots(x - x_{n-1})$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1]$$

....

$$b_n = f[x_0, x_1, \ldots, x_n]$$

Newton's Divided Differences (DD)

$$f[x_k] = f(x_k)$$ Zeroth order DD

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$ First order DD

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$ Second order DD

....

$$f[x_0, x_1, ..., x_k] = \frac{f[x_1, x_2, ..., x_k] - f[x_0, x_1, ..., x_{k-1}]}{x_k - x_0}$$

Divided Difference Table

x	F[ ]	F[,]	F[, ,]	F[, , ,]
x_0	F[x0]	$F[x_0,x_1]$	$F[x_0,x_{1,}x_2]$	$F[x_0,x_1,x_2,x_3]$
x_1	$F[x_1]$	$F[x_1,x_2]$	$F[x_1,x_2,x_3]$	
x_2	$F[x_2]$	$F[x_2,x_3]$		
x_3	$F[x_3]$			

$$f_n(x) = \sum_{i=0}^{n} \left\{ F[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} \left(x - x_j \right) \right\}$$

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	$f(x_i)$
0	-5
1	-3
-1	-15

Entries of the divided difference table are obtained from the data table using simple operations.

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	$f(x_i)$
0	-5
1	-3
-1	-15

The first two column of the

table are the data columns.

Third column: First order differences.

Fourth column: Second order differences.

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{-3-(-5)}{1-0} = 2$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{-15-(-3)}{-1-1} = 6$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{6-(2)}{-1-(0)} = -4$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	y_i
0	-5
1	-3
-1	-15

$$f_2(x) = -5 + 2(x-0) - 4(x-0)(x-1)$$

$f_2(x)$ $= F[x_0] + F[x_0, x_1]\,(x-x_0) + F[x_0, x_1, x_2]\,(x-x_0)(x-x_1)$

Two Examples

Obtain the interpolating polynomials for the two examples:

x	y
1	0
2	3
3	8

x	y
2	3
1	0
3	8

What do you observe?

Two Examples

x	Y		
1	0	3	1
2	3	5	
3	8		

$$P_2(x) = 0 + 3(x-1) + 1(x-1)(x-2)$$
$$= x^2 - 1$$

x	Y		
2	3	3	1
1	0	4	
3	8		

$$P_2(x) = 3 + 3(x-2) + 1(x-2)(x-1)$$
$$= x^2 - 1$$

Property: Ordering the points should not affect the divided difference:

$$f[x_0, x_1, x_2] = f[x_1, x_2, x_0] = f[x_2, x_1, x_0]$$

Example

- Find a polynomial to interpolate the data.

x	f(x)
2	3
4	5
5	1
6	6
7	9

Example

x	f(x)	f[,]	f[, ,]	f[, , ,]	f[, , , ,]
2	3	1	-5/3	37/24	-27/40
4	5	-4	9/2	-11/6	
5	1	5	-1		
6	6	3			
7	9				

$$f_4 = 3 + 1(x-2) - 1.6667(x-2)(x-4) + 1.5417(x-2)(x-4)(x-5)$$
$$- 0.6750(x-2)(x-4)(x-5)(x-6)$$

Interpolation with Equally Spaced Data

If data is equally spaced and in ascending order, then the independent variable assumes values of

$$x_1 = x_0 + h$$
$$x_2 = x_0 + 2h$$
$$\vdots$$
$$x_n = x_0 + nh$$

where h is the interval, or step size, between the data. On this basis, the finite divided differences can be expressed in concise form. For example, the second forward divided difference is

$$f[x_0, x_1, x_2] = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

which can be expressed as

$$f[x_0, x_1, x_2] = \frac{f(x_2) - 2\, f(x_1) + f(x_0)}{2h^2} \qquad \text{(B18.2.1)}$$

because $x_1 - x_0 = x_2 - x_1 = (x_2 - x_0)/2 = h$. Now recall that the second forward difference is equal to [numerator of Eq. (4.24)]

$$\Delta^2 f(x_0) = f(x_2) - 2\, f(x_1) + f(x_0)$$

Therefore, Eq. (B18.2.1) can be represented as

$$f[x_0, x_1, x_2] = \frac{\Delta^2 f(x_0)}{2!h^2}$$

or, in general,

$$f[x_0, x_1, \ldots, x_n] = \frac{\Delta^n f(x_0)}{n!h^n} \qquad \text{(B18.2.2)}$$

Using Eq. (B18.2.2), we can express Newton's interpolating polynomial [Eq. (18.15)] for the case of equispaced data as

$$f_n(x) = f(x_0) + \frac{\Delta f(x_0)}{h}(x - x_0) + \frac{\Delta^2 f(x_0)}{2!h^2}(x - x_0)(x - x_0 - h) + \cdots + \frac{\Delta^n f(x_0)}{n!h^n}(x - x_0)(x - x_0 - h) \cdots [x - x_0 - (n-1)h] + R_n \quad \text{(B18.2.3)}$$

where the remainder is the same as Eq. (18.16). This equation is known as *Newton's formula,* or the *Newton-Gregory forward formula.* It can be simplified further by defining a new quantity, α:

$$\alpha = \frac{x - x_0}{h}$$

This definition can be used to develop the following simplified expressions for the terms in Eq. (B18.2.3):

$$x - x_0 = \alpha h$$
$$x - x_0 - h = \alpha h - h = h(\alpha - 1)$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$x - x_0 - (n-1)h = \alpha h - (n-1)h = h(\alpha - n + 1)$$

which can be substituted into Eq. (B18.2.3) to give

$$f_n(x) = f(x_0) + \Delta f(x_0)\alpha + \frac{\Delta^2 f(x_0)}{2!}\alpha(\alpha - 1) + \cdots + \frac{\Delta^n f(x_0)}{n!}\alpha(\alpha - 1)\cdots(\alpha - n + 1) + R_n \tag{B18.2.4}$$

where

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}\alpha(\alpha - 1)(\alpha - 2)\cdots(\alpha - n)$$

Lagrange Interpolation

Problem:

Given

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

Find the polynomial of least order $f_n(x)$ such that: $f_n(x_i) = f(x_i) \quad for \quad i = 0,1,\ldots,n$

Lagrange Interpolation Formula:

$$f_n(x) = \sum_{i=0}^{n} f(x_i)\ \ell_i(x)$$

$$\ell_i(x) = \prod_{j=0, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

Lagrange Interpolation

$\ell_i(x)$ are called the cardinals.

The cardinals are n^{th} order polynomials :

$$\ell_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Lagrange Interpolation Example

x	1/3	1/4	1
y	2	-1	7

$$P_2(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x)$$

$$\ell_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}\frac{(x - x_2)}{(x_0 - x_2)} = \frac{(x - 1/4)}{(1/3 - 1/4)}\frac{(x - 1)}{(1/3 - 1)}$$

$$\ell_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}\frac{(x - x_2)}{(x_1 - x_2)} = \frac{(x - 1/3)}{(1/4 - 1/3)}\frac{(x - 1)}{(1/4 - 1)}$$

$$\ell_2(x) = \frac{(x - x_0)}{(x_2 - x_0)}\frac{(x - x_1)}{(x_2 - x_1)} = \frac{(x - 1/3)}{(1 - 1/3)}\frac{(x - 1/4)}{(1 - 1/4)}$$

$$P_2(x) = 2\{-18(x - 1/4)(x - 1)\} - 1\{16(x - 1/3)(x - 1)\}$$
$$+ 7\{2(x - 1/3)(x - 1/4)\}$$

Example

Find a polynomial to interpolate:

x	y
0	1
1	3
2	2
3	5
4	4

Both Newton's interpolation method and Lagrange interpolation method must give the same answer.

Newton's Interpolation Method

0	1	2	-3/2	7/6	-5/8
1	3	-1	2	-4/3	
2	2	3	-2		
3	5	-1			
4	4				

Interpolating Polynomial

$$f_4(x) = 1 + 2(x) - \frac{3}{2}x(x-1) + \frac{7}{6}x(x-1)(x-2)$$
$$-\frac{5}{8}x(x-1)(x-2)(x-3)$$

$$f_4(x) = 1 + \frac{115}{12}x - \frac{95}{8}x^2 + \frac{59}{12}x^3 - \frac{5}{8}x^4$$

Interpolating Polynomial Using Lagrange Interpolation Method

$$f_4(x) = \sum_{i=0}^{4} f(x_i)\, \ell_i = \ell_0 + 3\ell_1 + 2\ell_2 + 5\ell_3 + 4\ell_4$$

$$\ell_0 = \frac{(x-1)}{(0-1)}\,\frac{(x-2)}{(0-2)}\,\frac{(x-3)}{(0-3)}\,\frac{(x-4)}{(0-4)} = \frac{(x-1)(x-2)(x-3)(x-4)}{24}$$

$$\ell_1 = \frac{(x-0)}{(1-0)}\,\frac{(x-2)}{(1-2)}\,\frac{(x-3)}{(1-3)}\,\frac{(x-4)}{(1-4)} = \frac{x(x-2)(x-3)(x-4)}{-6}$$

$$\ell_2 = \frac{(x-0)}{(2-0)}\,\frac{(x-1)}{(2-1)}\,\frac{(x-3)}{(2-3)}\,\frac{(x-4)}{(2-4)} = \frac{x(x-1)(x-3)(x-4)}{4}$$

$$\ell_3 = \frac{(x-0)}{(3-0)}\,\frac{(x-1)}{(3-1)}\,\frac{(x-2)}{(3-2)}\,\frac{(x-4)}{(3-4)} = \frac{x(x-1)(x-2)(x-4)}{-6}$$

$$\ell_4 = \frac{(x-0)}{(4-0)}\,\frac{(x-1)}{(4-1)}\,\frac{(x-2)}{(4-2)}\,\frac{(x-3)}{(4-3)} = \frac{x(x-1)(x-2)(x-3)}{24}$$

Inverse Interpolation

Problem : Given a table of values

Find x such that : $f(x) = y_k$, where y_k is given

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

One approach:

Use polynomial interpolation to obtain $f_n(x)$ to interpolate the data then use Newton's method to find a solution to x

$$f_n(x) = y_k$$

Inverse Interpolation

Inverse interpolation:

1. Exchange the roles of x and y.

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

y_i	y_0	y_1	....	y_n
x_i	x_0	x_1	....	x_n

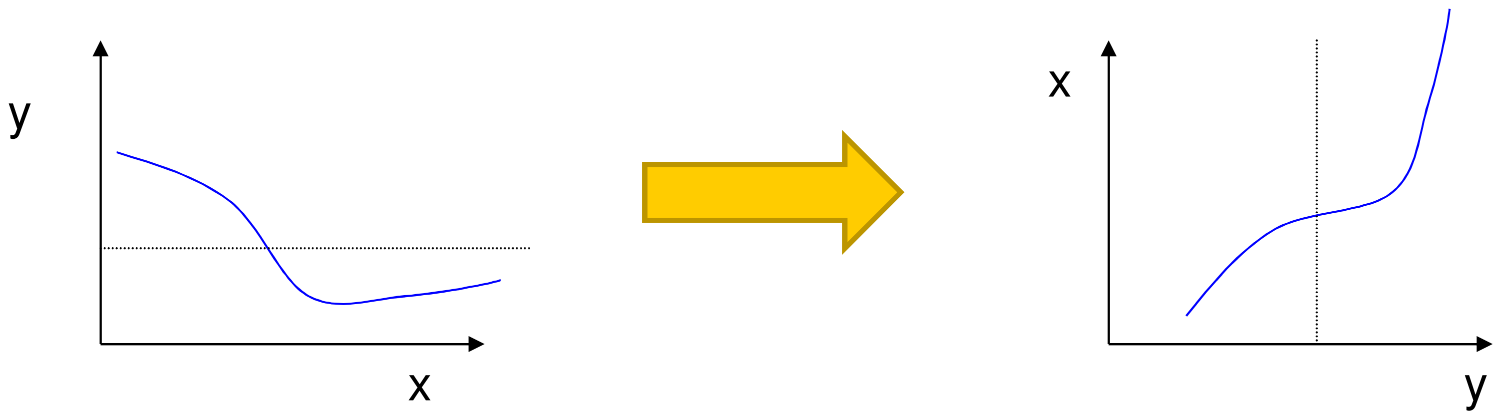
2. Perform polynomial

Interpolation on the new table.

3. Evaluate

$$x = f_n(y_k)$$

Inverse Interpolation



Inverse Interpolation

Question:

What is the limitation of inverse interpolation?

- The original function has an inverse.
- $y_1, y_2, \ldots, y_n$ must be distinct.

Inverse Interpolation

Example

x	1	2	3
y	3.2	2.0	1.6

Problem :

Given the table. Find x such that $f(x) = 2.5$

3.2	1	-.8333	1.0417
2.0	2	-2.5	
1.6	3		

$$x = f_2(y) = 1 - 0.8333(y - 3.2) + 1.0417(y - 3.2)(y - 2)$$

$$x = f_2(2.5) = 1 - 0.8333(-0.7) + 1.0417(-0.7)(0.5) = 1.2187$$

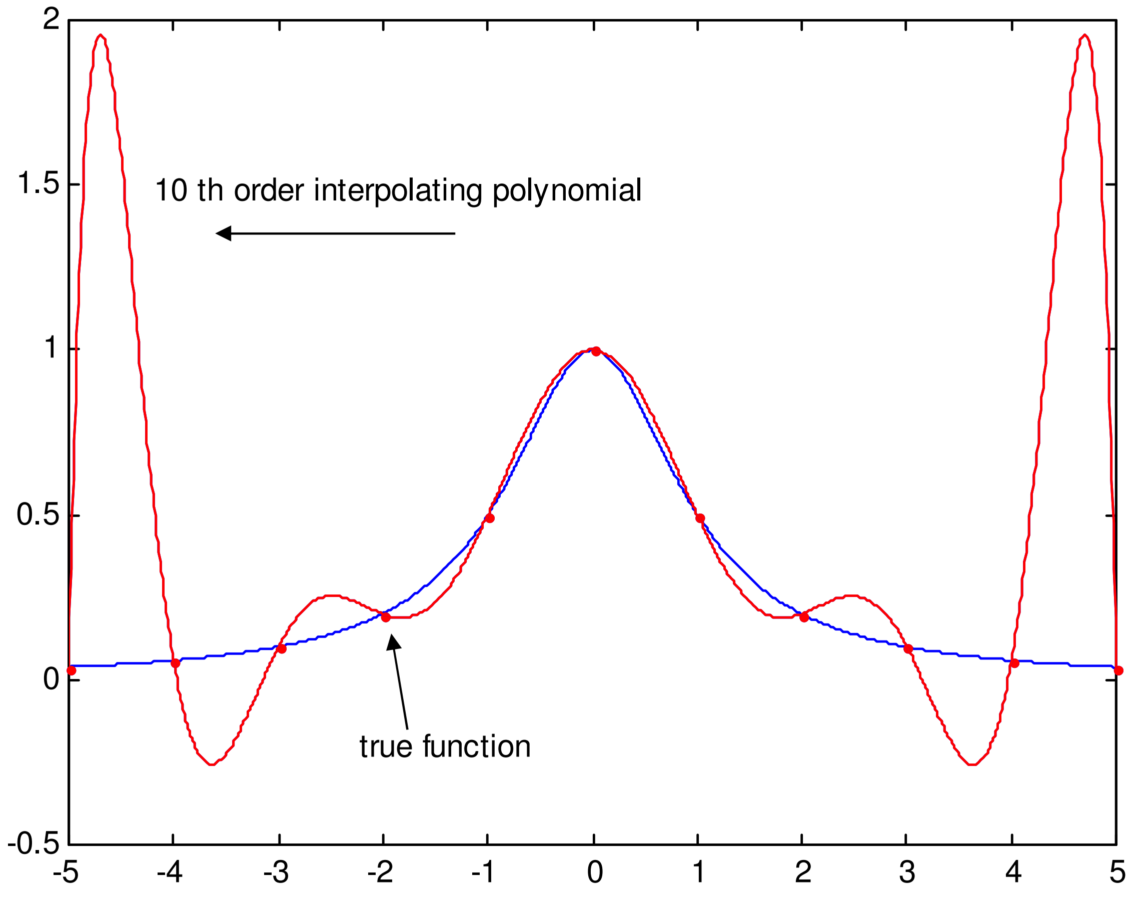
Errors in polynomial Interpolation

- Polynomial interpolation may lead to large errors (especially for high order polynomials).

BE CAREFUL

- When an n^{th} order interpolating polynomial is used, the error is related to the $(n+1)^{th}$ order derivative.

10^{th} Order Polynomial Interpolation



Errors in polynomial Interpolation

Theorem

Let $f(x)$ be a function such that :

$f^{(n+1)}(x)$ is continuous on [a, b], and $\left|f^{(n+1)}(x)\right| \leq M$.

Let $P(x)$ be any polynomial of degree $\leq$ n

that interpolates f at $n+1$ equally spaced points

in [a, b] (including the end points). Then :

$$\left|f(x)\text{-}P(x)\right| \leq \frac{M}{4(n+1)}\left(\frac{b-a}{n}\right)^{n+1}$$

Example

$f(x) = \sin(x)$

We want to use 9^{th} order polynomial to interpolate f(x)

(using 10 equally spaced points) in the interval [0,1.6875].

$$\left| f^{(n+1)} \right| \leq 1 \quad for\ n > 0$$

$$M = 1,\ n = 9$$

$$\left| f(x)\text{-}P(x) \right| \leq \frac{M}{4(n+1)} \left(\frac{b-a}{n} \right)^{n+1}$$

$$\left| f(x)\text{-}P(x) \right| \leq \frac{1}{4(10)} \left(\frac{1.6875}{9} \right)^{10} = 1.34 \times 10^{-9}$$

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

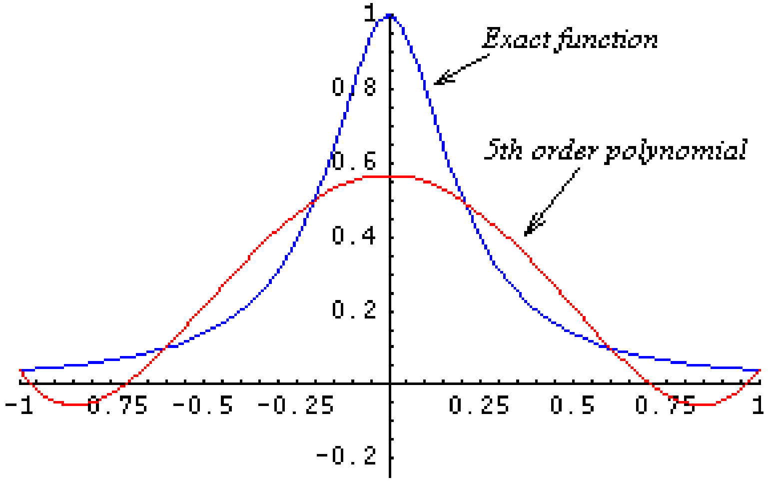


Figure : 5th order polynomial vs. exact function

Why Splines ?

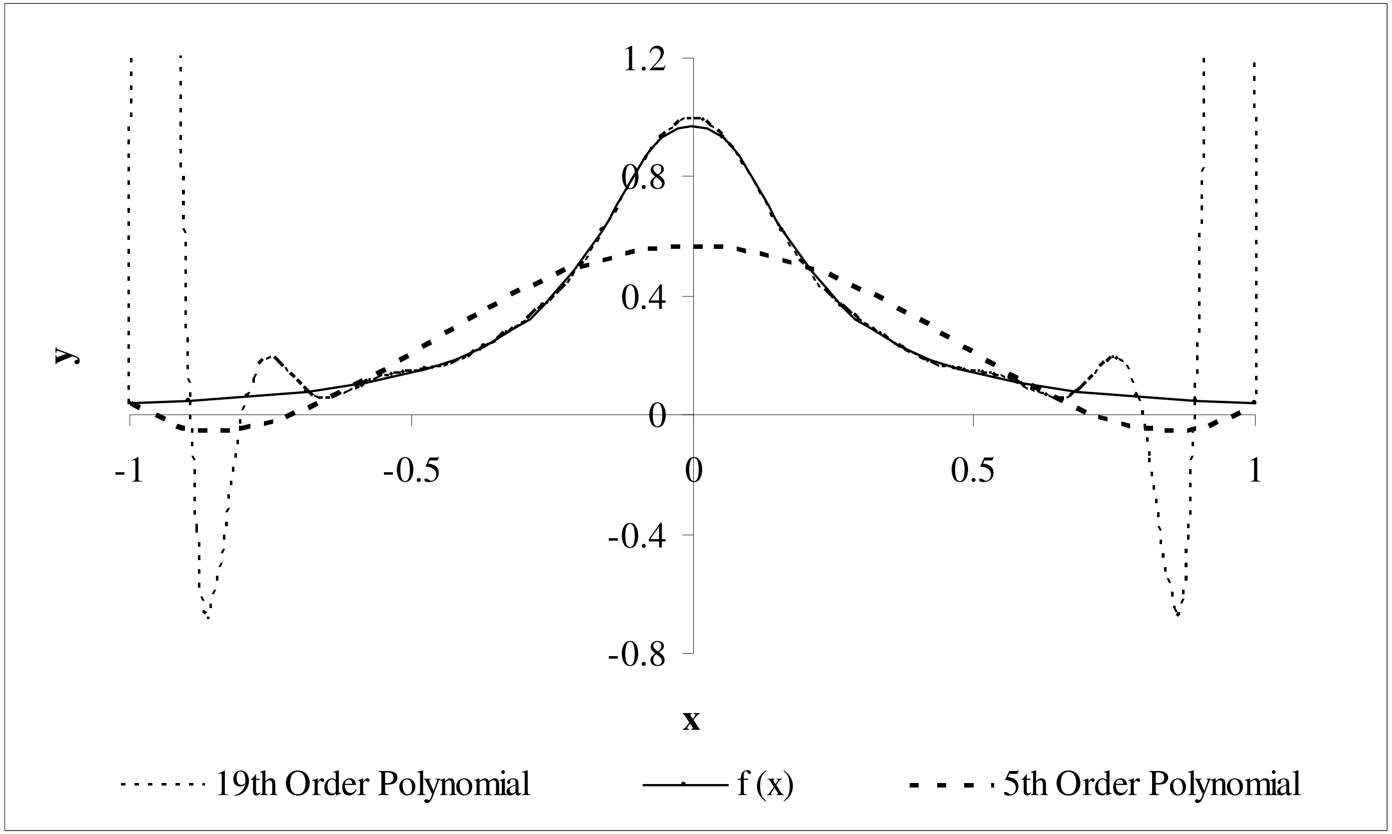
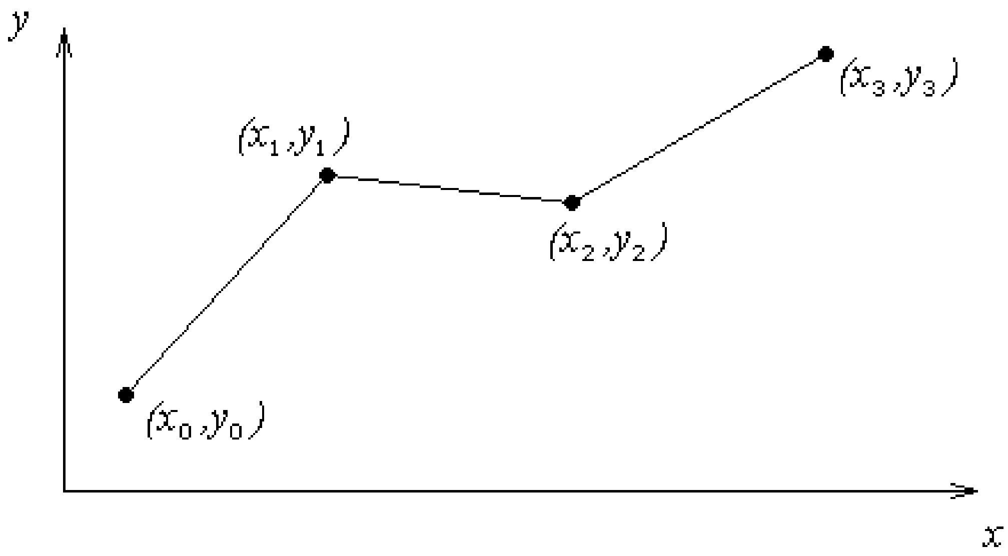


Figure : Higher order polynomial interpolation is a bad idea

Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



Linear Interpolation (contd)

$$
\begin{aligned}
f(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1 \\
&= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n
\end{aligned}
$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using linear splines.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

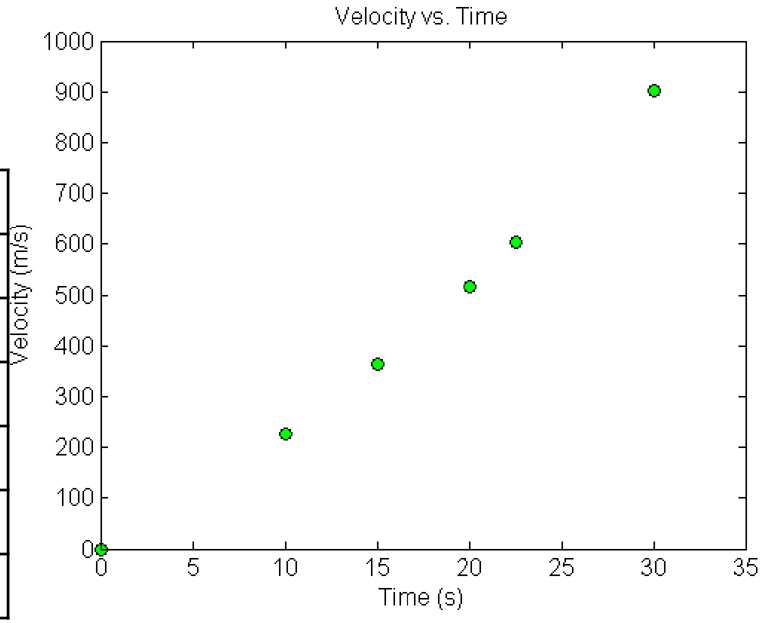


Figure. Velocity vs. time data for the rocket example

Linear Interpolation

$t_0 = 15, \qquad v(t_0) = 362.78$

$t_1 = 20, \qquad v(t_1) = 517.35$

$$v(t) = v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0)$$

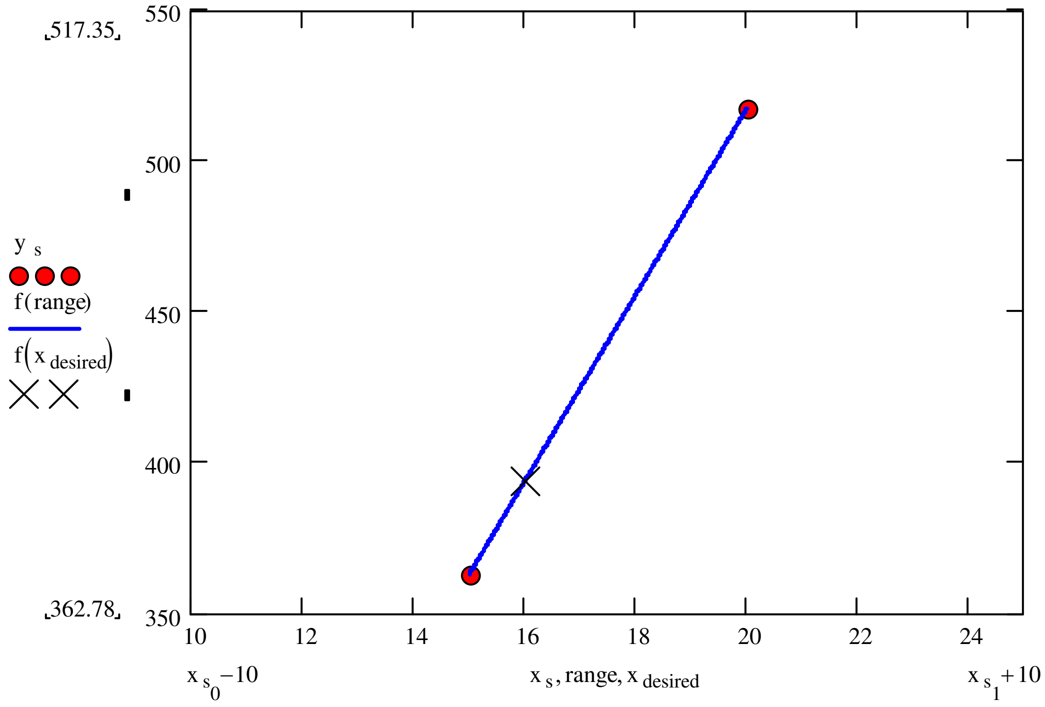
$$= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15)$$

$$v(t) = 362.78 + 30.913(t - 15)$$

At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15)$$

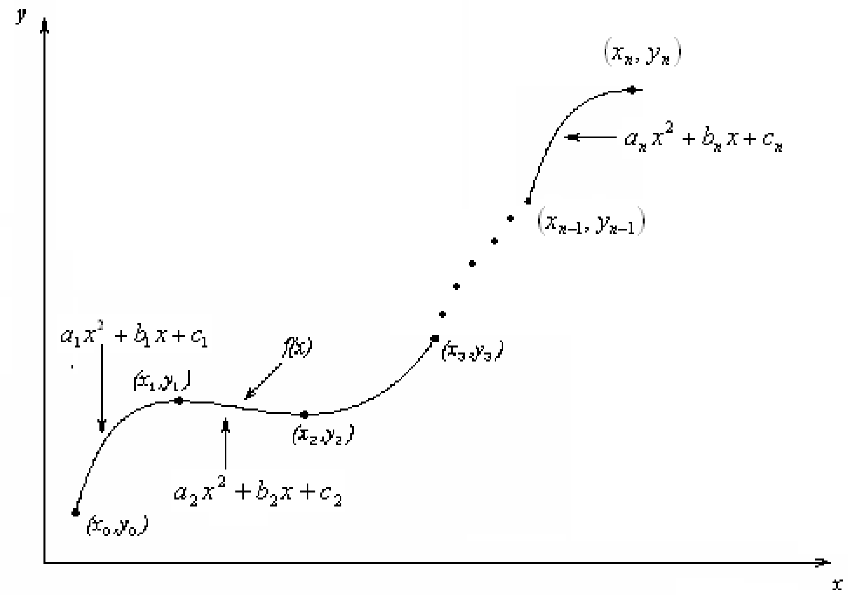
$$= 393.7 \text{ m/s}$$



Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, & x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, & x_1 \le x \le x_2 \\
&\ \ \vdots \\
&= a_nx^2 + b_nx + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$



Find a_i, b_i, c_i, $i = 1, 2, \ldots$, n

Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

.

.

.

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

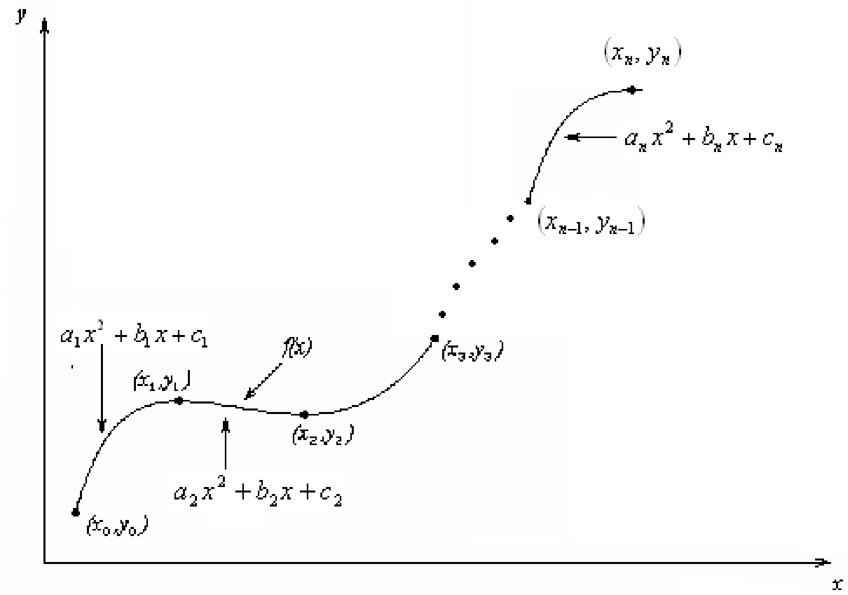
.

.

.

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$



This condition gives 2n equations

Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \quad 2a_1x + b_1$$

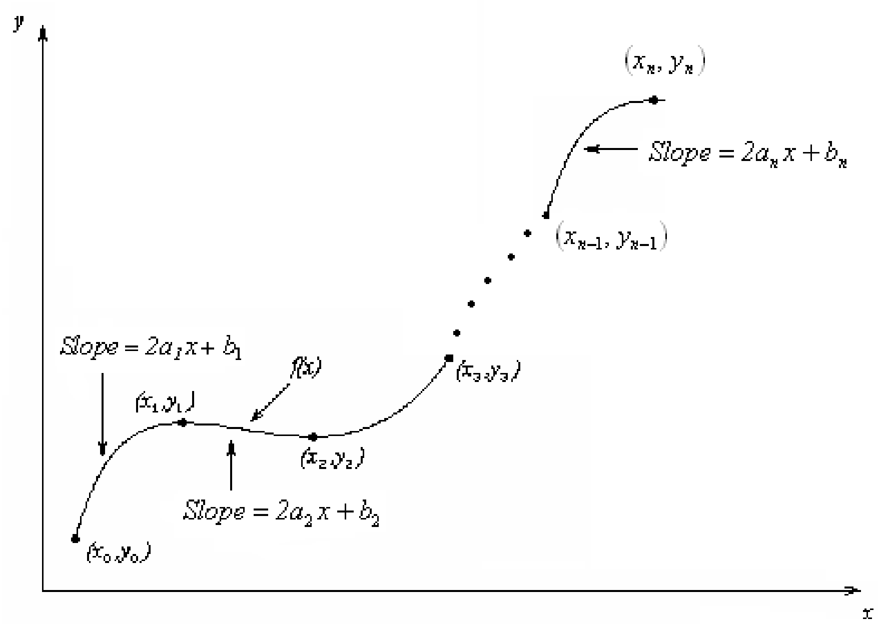
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \quad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2 x_2 + b_2 - 2a_3 x_2 - b_3 = 0$$

.

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$$2a_i x_i + b_i - 2a_{i+1} x_i - b_{i+1} = 0$$

.

.

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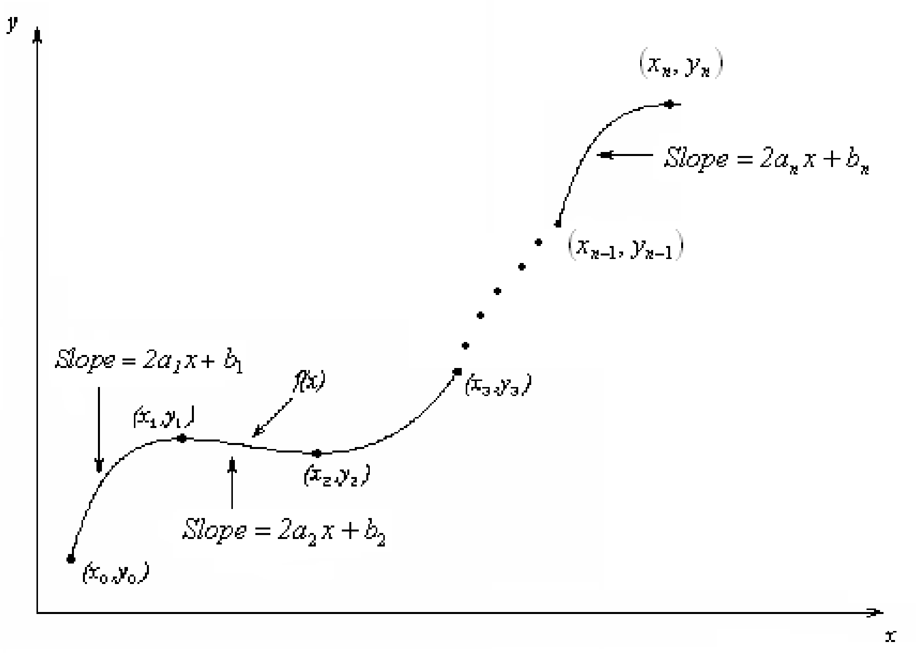
$$2a_{n-1} x_{n-1} + b_{n-1} - 2a_n x_{n-1} - b_n = 0$$



We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$f(x) = a_1 x^2 + b_1 x + c_1, \qquad x_0 \le x \le x_1$$

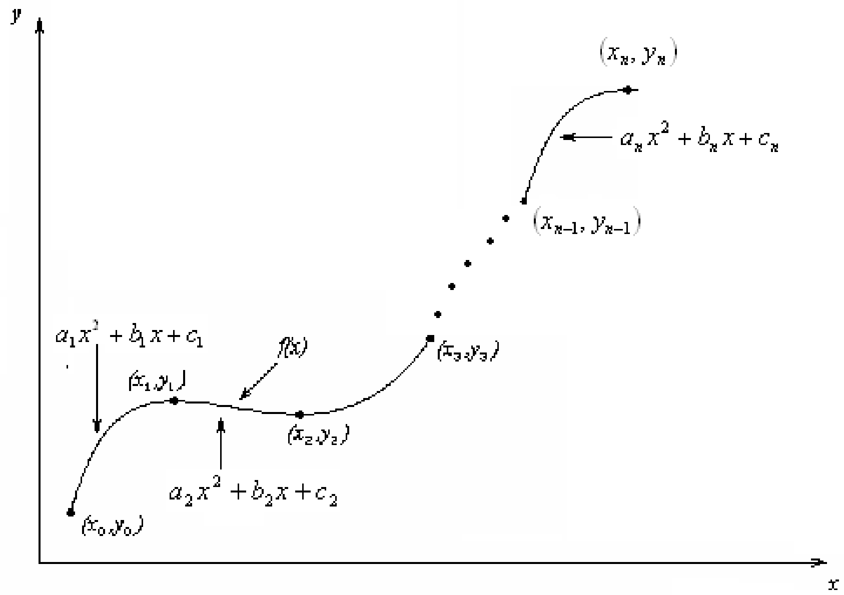
$$= a_2 x^2 + b_2 x + c_2, \qquad x_1 \le x \le x_2$$

.

.

.

$$= a_n x^2 + b_n x + c_n, \qquad x_{n-1} \le x \le x_n$$



Quadratic Spline Example

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

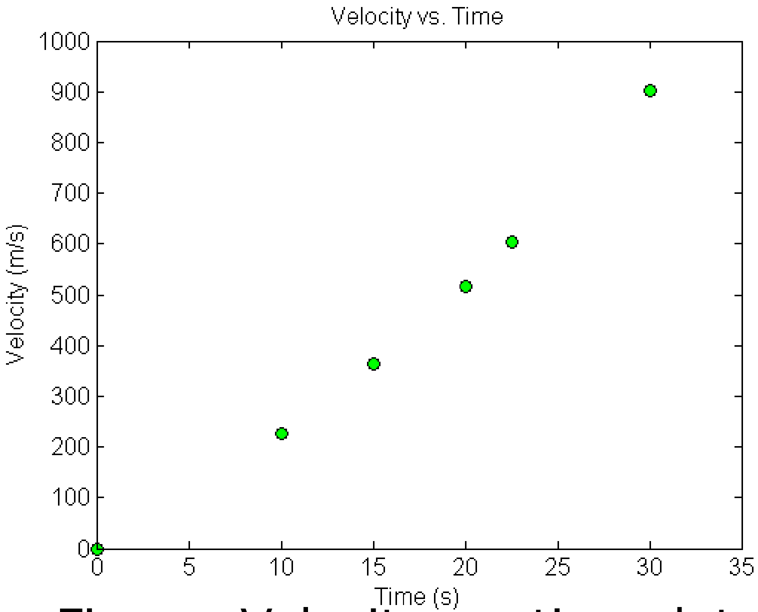


Figure. Velocity vs. time data for the rocket example

Solution

$$
\begin{aligned}
v(t) &= a_1 t^2 + b_1 t + c_1, && 0 \le t \le 10 \\
&= a_2 t^2 + b_2 t + c_2, && 10 \le t \le 15 \\
&= a_3 t^2 + b_3 t + c_3, && 15 \le t \le 20 \\
&= a_4 t^2 + b_4 t + c_4, && 20 \le t \le 22.5 \\
&= a_5 t^2 + b_5 t + c_5, && 22.5 \le t \le 30
\end{aligned}
$$

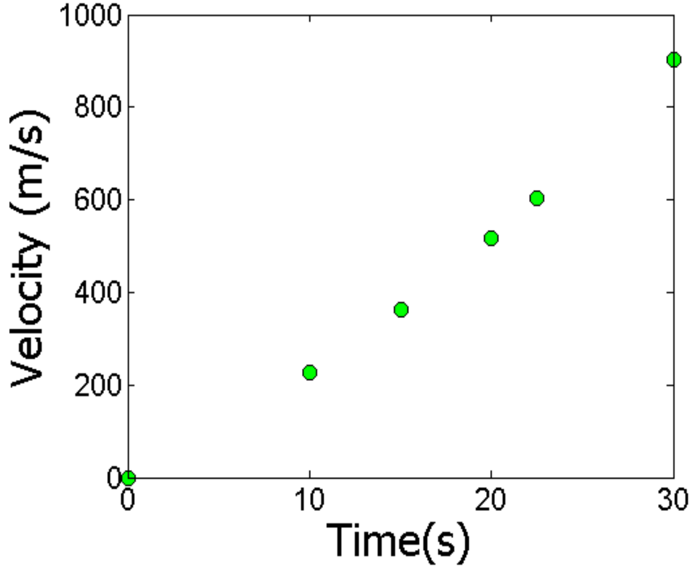
Let us set up the equations

Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \ 0 \le t \le 10$$

$$a_1(0)^2 + b_1(0) + c_1 = 0$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$



Each Spline Goes Through Two Consecutive Data Points

t	v(t)
s	**m/s**
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

Derivatives are Continuous at Interior Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \; 0 \le t \le 10$$

$$= a_2 t^2 + b_2 t + c_2, 10 \le t \le 15$$

$$\frac{d}{dt}\left(a_1 t^2 + b_1 t + c_1\right)\Big|_{t=10} = \frac{d}{dt}\left(a_2 t^2 + b_2 t + c_2\right)\Big|_{t=10}$$

$$\left(2a_1 t + b_1\right)\Big|_{t=10} = \left(2a_2 t + b_2\right)\Big|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

Derivatives are continuous at Interior Data Points

At t=10

$$2a_1(10)+b_1-2a_2(10)-b_2=0$$

At t=15

$$2a_2(15)+b_2-2a_3(15)-b_3=0$$

At t=20

$$2a_3(20)+b_3-2a_4(20)-b_4=0$$

At t=22.5

$$2a_4(22.5)+b_4-2a_5(22.5)-b_5=0$$

Last Equation $a_1=0$

Final Set of Equations

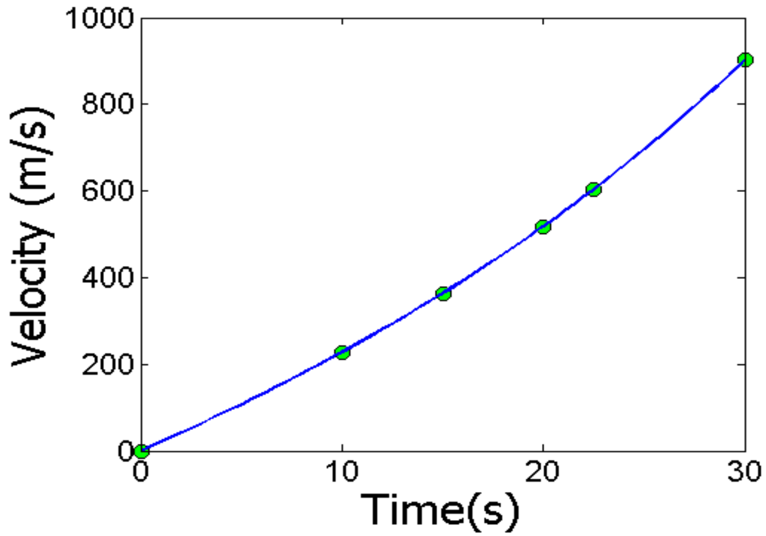
$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Coefficients of Spline

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	−0.1356	35.66	−141.61
4	1.6048	−33.956	554.55
5	0.20889	28.86	−152.13

Final Solution

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$



Velocity at a Particular Point

a) Velocity at t=16

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$

$$
\begin{aligned}
v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\
&= 394.24 \text{ m/s}
\end{aligned}
$$

Acceleration from Velocity Profile

b) The quadratic spline valid at t=16 is given by

$$a(16) = \frac{d}{dt} v(t)\Big|_{t=16}$$

$$v(t) = -0.1356\,t^2 + 35.66t - 141.61, \quad 15 \le t \le 20$$

$$a(t) = \frac{d}{dt}(-0.1356t^2 + 35.66t - 141.61)$$

$$= -0.2712t + 35.66, \quad 15 \le t \le 20$$

$$a(16) = -0.2712(16) + 35.66 = 31.321 \text{m/s}^2$$

Distance from Velocity Profile

c) Find the distance covered by the rocket from t=11s to t=16s.

$$S(16)-S(11)=\int_{11}^{16} v(t)dt$$

$$v(t)=0.8888t^2+4.928t+88.88,\ 10\le t\le 15$$

$$=-0.1356t^2+35.66t-141.61,\ 15\le t\le 20$$

$$S(16)-S(11)=\int_{11}^{16} v(t)dt=\int_{11}^{15} v(t)dt+\int_{15}^{16} v(t)dt$$

$$=\int_{11}^{15}\left(0.8888t^2+4.928t+88.88\right)dt+\int_{15}^{16}\left(-0.1356t^2+35.66t-141.61\right)dt$$

$$=1595.9 \text{ m}$$