Lecture 12

Eigenvalue Problem

- Review of Eigenvalues
- Some properties
- Power method
- Shift method
- Inverse power method
- Deflation
- QR Method

Eigenvalue

- **Eigenvalue** λ

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$$

$$\text{If } \det(\mathbf{A}-\lambda\mathbf{I})\neq 0 \Rightarrow \mathbf{x}=\mathbf{0} \text{ (trivial solution)}$$

To obtain a non-trivial solution, $\det(\mathbf{A}-\lambda\mathbf{I})=0$

➔ $$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & a_{11} \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0$$

➔ **Characteristic equation**

Properties of Eigenvalue

1) $\text{trace}\,\mathbf{A} = tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$

2) $\det \mathbf{A} = \prod_{i=1}^{n} \lambda_i$

3) If A is symmetric, then the **eigenvectors** are orthogonal: $\mathbf{x}_i^T \mathbf{x}_j = \begin{cases} 0, & i \neq j \\ G_{ii} & i = j \end{cases}$

4) Let the eigenvalues of $\mathbf{A} = \lambda_1, \lambda_2, \cdots \lambda_n$

then, the eigenvalues of ($\mathbf{A}$ - $a\mathbf{I}$)

$$= \lambda_1 - a, \lambda_2 - a, \cdots, \lambda_n - a,$$

Application Examples

- If a matrix **A** represents a system, then eigenvalues can be used to system identification, e.g., stability and so on.
- For "infinite"-dimension systems, *eigenvalue-eigenfunction* pairs are used, e.g., resonance frequency of an oscillator or cavity, waveguide modes, etc.
- In image and speech recognition, "eigenface", "eigenvoice" are used for identification.
- In statistics, principal component analysis (PCA) is used to find relationships among variables.

Geometrical Interpretation of Eigenvectors

- Transformation $\mathbf{Ax}$

  $\mathbf{Ax} = \lambda \mathbf{x}$:The transformation of an eigenvector is mapped onto the same line of.

- Symmetric matrix → orthogonal eigenvectors

- Relation to Singular Value

  if **A** is singular → 0 $\in$ {eigenvalues}

Power Method

Assuming "distinct" eigenvalues, any vector **x** can be written as

$$\mathbf{x} = \sum_{m=1}^{n} \alpha_m \phi_m$$

where α_m is a constant

and ϕ_m denotes an eigenvector.

In general, continue the multiplication:

$$\mathbf{A}^k\mathbf{x} = \alpha_1\lambda_1^k\phi_1 + \alpha_2\lambda_2^k\phi_2 + \alpha_3\lambda_3^k\phi_3 + \ldots + \alpha_n\lambda_n^k\phi_n$$

where λ_k denotes an eigenvalue, and

$$\mathbf{A}^k = \underbrace{\mathbf{A}\mathbf{A}\ldots\mathbf{A}}_{k}$$

Power Method

Factor the large λ value term

$$\mathbf{A}^k\mathbf{x} = \lambda_1^{\mathrm{k}}\left[\alpha_1\phi_1 + \alpha_2\left(\frac{\lambda_2}{\lambda_1}\right)^{\mathrm{k}}\phi_2 + \ldots + \alpha_{\mathrm{n}}\left(\frac{\lambda_{\mathrm{n}}}{\lambda_1}\right)^{\mathrm{k}}\phi_n\right]$$

As you continue to multiply the vector by [A]

$$\mathbf{A}^{\mathrm{k}}\mathbf{x} = \lambda_1^{\mathrm{k}}\alpha_1\phi_1 \;\text{ as }\; \mathrm{k} \to \infty$$

Power Method

The basic computation of the power method is summarized as

$$u_k = \frac{A^k x_0}{\left\| A^k x_0 \right\|_\infty} = \frac{\alpha \lambda^k \phi_1}{\left\| \alpha \lambda^k \phi_1 \right\|_\infty} = \frac{\alpha \lambda^k \phi_1}{\alpha \lambda^k \left\| \phi_1 \right\|_\infty} = \frac{\phi_1}{\left\| \phi_1 \right\|_\infty} \quad \text{as} \quad k \to \infty$$

Power Method

The basic computation of the power method is summarized as

$$u_k = \frac{Au_{k-1}}{\|Au_{k-1}\|_\infty} \quad \text{and} \quad \lim_{k\to\infty} u_k = \phi_1$$

The equation can be written as:

$$Au_{k-1} = \lambda_1 u_{k-1} \rightarrow \lambda_1 = \frac{\|Au_{k-1}\|}{\|u_{k-1}\|}$$

The Power Method Algorithm

y=nonzero random vector

Initialize x = A*y vector

for k =1,2,…n

y=x/||x||

x =Ay (x is the approximate eigenvector)

approximate eigenvalue $\mu = (y^T x)/(y^T y)$

r=μy-x

k=k+1

Example of Power Method

Consider the follow matrix **A**

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Assume an arbitrary vector $x_0 = \{\ 1\ 1\ 1\}^T$

Example of Power Method

Multiply the matrix by the matrix [A] by **x**

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 3 \\ -1 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} 5 \\ 3 \\ -1 \end{Bmatrix} = 5 \begin{Bmatrix} 1 \\ 0.6 \\ -0.2 \end{Bmatrix}$$

Normalize the result of the product

Example of Power Method

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.6 \\ -0.2 \end{Bmatrix} = \begin{Bmatrix} 4.6 \\ 1 \\ 0.2 \end{Bmatrix} \quad \Rightarrow \begin{Bmatrix} 4.6 \\ 1 \\ 0.2 \end{Bmatrix} = 4.6 \begin{Bmatrix} 1 \\ 0.217 \\ 0.0435 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.217 \\ 0.0435 \end{Bmatrix} = \begin{Bmatrix} 4.2174 \\ 0.4783 \\ -0.0435 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 4.2174 \\ 0.4783 \\ -0.0435 \end{Bmatrix} = 4.2174 \begin{Bmatrix} 1 \\ 0.1134 \\ -0.0103 \end{Bmatrix}$$

Example of Power Method

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.1134 \\ -0.0103 \end{Bmatrix} = \begin{Bmatrix} 4.1134 \\ 0.2165 \\ 0.0103 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 4.1134 \\ 0.2165 \\ 0.0103 \end{Bmatrix} = 4.1134 \begin{Bmatrix} 1 \\ 0.0526 \\ 0.0025 \end{Bmatrix}$$

As you continue to multiply each successive vector yields $\lambda = 4$ and the vector $u_k = \{1\ 0\ 0\}^T$

Example 2 of Power Method

Consider the follow matrix **A**

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix}$$

Assume an arbitrary vector

$x_0 = \{1\ 1\ 1\}^T$

$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 14 \\ 12 \end{Bmatrix} = 14 \begin{Bmatrix} 0.5714 \\ 1 \\ 0.8571 \end{Bmatrix};$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix} \begin{Bmatrix} 0.5714 \\ 1 \\ 0.8571 \end{Bmatrix} = \begin{Bmatrix} 6.2857 \\ 12.2857 \\ 10.2857 \end{Bmatrix} = 12.2857 \begin{Bmatrix} 0.5116 \\ 1 \\ 0.8372 \end{Bmatrix};$$

Example of Power Method

$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix} \begin{Bmatrix} 0.5116 \\ 1 \\ 0.8372 \end{Bmatrix} = \begin{Bmatrix} 6.0465 \\ 12.0465 \\ 10.0465 \end{Bmatrix} = 12.0465 \begin{Bmatrix} 0.5019 \\ 1 \\ 0.8340 \end{Bmatrix};$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix} \begin{Bmatrix} 0.5019 \\ 1 \\ 0.8340 \end{Bmatrix} = \begin{Bmatrix} 6.0077 \\ 12.0077 \\ 10.0077 \end{Bmatrix} = 12.0077 \begin{Bmatrix} 0.5003 \\ 1 \\ 0.8334 \end{Bmatrix};$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix} \begin{Bmatrix} 0.5003 \\ 1 \\ 0.8334 \end{Bmatrix} = \begin{Bmatrix} 6.0013 \\ 12.0013 \\ 10.0013 \end{Bmatrix} = 12.0013 \begin{Bmatrix} 0.5001 \\ 1 \\ 0.8334 \end{Bmatrix};$$

Result: $\lambda = 12$ and $u_k = \{.5 \ 1 \ 0.8333\}^T$

Power method : Advantages

- Simple, easy to implement.
- The eigenvector corresponding to the *dominant* (i.e., maximum) eigenvalue is generated at the same time.
- The inverse power method solves for the minimal eigenvalue/eigenvector pair.

Power Method : Disadvantages

- The power method provides only one eigenvalue/eigenvector pair, which corresponds to the "maximum" eigenvalue.
- Some modifications must be implemented to find other eigenvalues, e.g., shift method, deflation, etc.
- Also, if the dominant eigenvalue is not sufficiently larger than others, a large number of iterations are required.

Shift method

It is possible to obtain another eigenvalue from the set equations by using a technique known as shifting the matrix.

$$[\mathbf{A}]\mathbf{x} = \lambda \mathbf{x}$$

Subtracting a vector $s\mathbf{x}$ from each side, thereby changing the maximum eigenvalue

$$[\mathbf{A}]\mathbf{x} - s[\mathbf{I}]\mathbf{x} = (\lambda - s)\mathbf{x}$$

Shift method

Let the eigenvalue, λ_{max}, be the maximum value of the matrix **A**, Then the matrix is rewritten in a form:

$$[\mathbf{B}] = [\mathbf{A}] - \lambda_{\max}[\mathbf{I}]$$

The Power method can then be applied to obtain the largest eigenvalue of [**B**].

Example of Power Method

Consider the following matrix **B**

$$B = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

Assume an arbitrary vector $x_0 = \{1\ 1\ 1\}^T$

Example of Power Method

Multiply **x** by the matrix [**B**]

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ -5 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} 1 \\ -1 \\ -5 \end{Bmatrix} = -5 \begin{Bmatrix} -0.2 \\ 0.6 \\ 1 \end{Bmatrix}$$

Normalize the result of the product

Example of Power Method

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{Bmatrix} -0.2 \\ 0.2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.2 \\ 0.6 \\ -5 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} 0.2 \\ 0.6 \\ -5 \end{Bmatrix} = -5 \begin{Bmatrix} -0.04 \\ -0.12 \\ 1 \end{Bmatrix}$$

Continue with the iteration and the final value is λ = -5. However, to get the true you need to shift back by:

$$\lambda = \lambda' + \lambda_{max} = -5 + 4 = -1$$

Inverse Power Method

The inverse method is similar to the power method, except that it finds the smallest eigenvalue. Using the following technique.

$$[\mathbf{A}]\mathbf{x} = \lambda\mathbf{x} \qquad \Rightarrow [\mathbf{A}]^{-1}[\mathbf{A}]\mathbf{x} = \lambda[\mathbf{A}]^{-1}\mathbf{x}$$

$$\frac{1}{\lambda}\mathbf{x} = [\mathbf{A}]^{-1}\mathbf{x} \qquad \Rightarrow \mu\mathbf{x} = [\mathbf{B}]\mathbf{x}$$

Inverse Power Method

The algorithm is the same as the Power method and the “eigenvector” is not the eigenvector for the smallest eigenvalue. To obtain the smallest eigenvalue from the power method.

$$\mu = \frac{1}{\lambda} \quad \Rightarrow \quad \lambda = \frac{1}{\mu}$$

Inverse Power Method

The inverse algorithm use the technique avoids calculating the inverse matrix and uses a LU decomposition to find the **x** vector.

$$[\mathbf{A}]\mathbf{x} = \lambda\mathbf{x} \qquad \Rightarrow \frac{1}{\lambda}[\mathbf{L}][\mathbf{U}]\mathbf{x} = \mathbf{x}$$

Example

$$[\mathbf{A}] = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix} \Rightarrow \lambda = \begin{Bmatrix} 4.9264 \\ 5.2535 \\ 1.82 \end{Bmatrix}$$

Matrix Deflation

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A with associated eigenvectors $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\ldots$, $\mathbf{v}^{(n)}$ and that λ_1 has multiplicity 1. If $\mathbf{x}$ is a vector with $\mathbf{x}^t\mathbf{v}^{(1)} = 1$, then

$$B = A - \lambda_1 \mathbf{v}^{(1)}\mathbf{x}^t$$

has eigenvalues $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ with associated eigenvectors $\mathbf{v}^{(1)}, \mathbf{w}^{(2)}, \mathbf{w}^{(3)}, \ldots, \mathbf{w}^{(n)}$, where $\mathbf{v}^{(i)}$ and $\mathbf{w}^{(i)}$ are related by the equation

$$\mathbf{v}^{(i)} = (\lambda_i - \lambda_1)\mathbf{w}^{(i)} + \lambda_1(\mathbf{x}^t\mathbf{w}^{(i)})\mathbf{v}^{(1)},$$

for each $i = 2, 3, \ldots, n$.

- It is possible to obtain eigenvectors one after another
- Properly assigning the vector x is important.
- e.g. Wielandt's deflation

Example Using Deflation(I)

Example 4 The symmetric matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 6$, $\lambda_2 = 3$, and $\lambda_3 = 1$. Assuming that the dominant eigenvalue $\lambda_1 = 6$ and associated unit eigenvector $\mathbf{v}^{(1)} = (1, -1, 1)^t$ have been calculated, the procedure just outlined for obtaining λ_2 proceeds as follows:

$$\mathbf{x} = \frac{1}{6}\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \left(\frac{2}{3}, -\frac{1}{6}, \frac{1}{6}\right)^t,$$

$$\mathbf{v}^{(1)}\mathbf{x}^t = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\left[\tfrac{2}{3}, \quad -\tfrac{1}{6}, \quad \tfrac{1}{6}\right] = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix},$$

and

$$B = A - \lambda_1\mathbf{v}^{(1)}\mathbf{x}^t = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} - 6\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix}.$$

Example: Using Deflation(II)

Deleting the first row and column gives

$$B' = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which has eigenvalues $\lambda_2 = 3$ and $\lambda_3 = 1$. For $\lambda_2 = 3$, the eigenvector $\mathbf{w}^{(2)'}$ can be obtained by solving the second-order linear system

$$(B' - 3I)\mathbf{w}^{(2)'} = \mathbf{0}, \quad \text{resulting in} \quad \mathbf{w}^{(2)'} = (1, -1)^t.$$

Adding a zero for the first component gives $\mathbf{w}^{(2)} = (0, 1, -1)^t$ and

$$\begin{aligned}\mathbf{v}^{(2)} &= (3-6)(0, 1, -1)^t + 6\left[\left(\frac{2}{3}, -\frac{1}{6}, \frac{1}{6}\right)(0, 1, -1)^t\right](1, -1, 1)^t \\ &= (-2, -1, 1)^t.\end{aligned}$$

Wielandt's Deflation

- Let $\mathbf{x}_1, \lambda_1$ be the eigenvector, eigenvalue of **A**. Consider

$$\mathbf{A}_1 = \mathbf{A} - \mathbf{x}_1 \mathbf{u}^T ; \mathbf{u}^T \mathbf{x}_1 = \lambda_1$$

- Clearly, 0 is an eigenvalue of $\mathbf{A}_1$, since

$$\mathbf{A}_1 \mathbf{x}_1 = \mathbf{A}\mathbf{x}_1 - \mathbf{x}_1 \mathbf{u}^T \mathbf{x}_1 = \lambda_1 \mathbf{x}_1 - \lambda_1 \mathbf{x}_1 = \mathbf{0}$$

- Assume $\mathbf{x}_j, \lambda_j$,j=2,3,... then

$$\mathbf{A}_1 \mathbf{x}_j = \mathbf{A}\mathbf{x}_j - \mathbf{x}_1 \mathbf{u}^T \mathbf{x}_j = \lambda_j \mathbf{x}_j - \mathbf{u}^T \mathbf{x}_j \mathbf{x}_1 \text{, i.e.,}$$

$$\mathbf{A}_1 \left(\mathbf{x}_j - \frac{\mathbf{u}^T \mathbf{x}_j}{\lambda_j} \mathbf{x}_1 \right) = \mathbf{A}_1 \mathbf{y}_j = \lambda_j \mathbf{y}_j$$

Wielandt's Deflation (2)

- Thus, $\mathbf{y}_j, \lambda_j$,j=2,3,... : eigenvector, eigenvalue of $\mathbf{A}_1$.
- To get $\mathbf{u}$, first find $\mathbf{v}$ such that $\mathbf{v}^T\mathbf{x}_1=1$, then

$$\mathbf{v}^T\mathbf{A}\mathbf{x}_1 = \mathbf{v}^T\lambda_1\mathbf{x}_1 = \lambda_1\mathbf{v}^T\mathbf{x}_1 = \lambda_1$$

- Thus,

$$\mathbf{u}^T = \mathbf{v}^T\mathbf{A}$$

- and

$$\mathbf{A}_1 = \mathbf{A} - \mathbf{x}_1\mathbf{v}^T\mathbf{A}$$

$$\mathbf{y}_j = \mathbf{x}_j - \frac{\mathbf{v}^T\mathbf{A}\mathbf{x}_j}{\lambda_j}\mathbf{x}_1 = \mathbf{x}_j - \mathbf{v}^T\mathbf{x}_j\mathbf{x}_1$$

Wielandt's Deflation (3)

- If $\mathbf{x}_1$ has nonzero first element, normalizing it to make its first element to be 1, then we can choose $\mathbf{v}^T = \mathbf{e}_1^T = [1\ 0\ \ldots\ 0]$. Then

$$\mathbf{A}_1 = \mathbf{A} - \mathbf{x}_1\mathbf{e}_1^T\mathbf{A} = \left(\mathbf{I} - \mathbf{x}_1\mathbf{e}_1^T\right)\mathbf{A}$$

Other methods

- Rayleigh quotient iteration
- Bisection method
- Divide-and-conquer
- **QR Algorithm**
- etc.

Similar Matrices

Definition: **A** and **B** are **similar** matrices if and only if there exists a nonsingular matrix **P** such that **B** = $\mathbf{P^{-1}AP}$. (or $\mathbf{PBP^{-1}} = \mathbf{A}$)

Theorem: Similar matrices have the same set of eigenvalues.

Proof: Since $\mathbf{Ap} = \lambda\mathbf{p} \Leftrightarrow \mathbf{PBP^{-1}\,p} = \lambda\mathbf{p}$

$\Leftrightarrow \mathbf{P^{-1}(PBP^{-1}\,p)} = \mathbf{P^{-1}}(\lambda\mathbf{p}) \Leftrightarrow \mathbf{B(P^{-1}\,p)} = \lambda(\mathbf{P^{-1}p})$.

Then, λ is an eigenvalue of **A** (with eigenvector **p**) $\Leftrightarrow$ λ is an eigenvalue of **B** (with eigenvector $\mathbf{P^{-1}p}$). **QED**

QR Algorithm

Given square matrix A we can factor $A = QR$ where Q is orthogonal (i.e., $Q^TQ=I$ or $Q^T=Q^{-1}$) and R is upper triangular.

Algorithm to find eigenvalues:

Start: $A^{(0)} = A = QR$ (note: $R = Q^TA$)

$A^{(1)} = RQ = Q^TAQ$ (A and $A^{(1)}$ are similar)

Factor: $A^{(1)} = Q^{(1)} R^{(1)}$

$A^{(2)} = R^{(1)} Q^{(1)} = Q^{(1)T}Q^TAQQ^{(1)}$ (similar to $A^{(1)}$)

General: Given $A^{(m)}$

Factor: $A^{(m)} = Q^{(m)} R^{(m)}$

$A^{(m+1)} = R^{(m)} Q^{(m)}$ (similar to $A^{(m)}$, …, $A^{(1)}$, A)

QR Algorithm

Note: If the eigenvalues of A satisfy

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$$

Then, the iterates $A^{(m)}$ converge to an upper triangular matrix having the eigenvalues of A on the diagonal. (Proof omitted)

$$\mathbf{A}^{(m)} \to \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

QR Algorithm- Eigenvectors

How do we compute the eigenvectors?

Note: $A^{(m)} = Q^{(m-1)T} \cdots Q^{(1)T} Q^T A\ Q\ Q^{(1)} \cdots Q^{(m-1)}$

Let $Q^* = Q\ Q^{(1)} \cdots Q^{(m-1)}$ Then, $A^{(m)} = Q^{*T} A\ Q^*$

If $A^{(m)}$ becomes diagonal, the eigenvectors of $A^{(m)}$ are e_1, e_1, …, e_n. Since $A^{(m)}$ and A are similar, the eigenvectors of A are $(Q^{*T})^{-1} e_i = Q^* e_i$, i.e., the **eigenvectors of A are the columns of Q^***. For any "symmetric real" matrix, $A^{(m)}$ converges to a diagonal matrix.

For more details, see "Matrix Computations", by Golub, Van Loan, or other books on matrix computations.

Example

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 4 & -2 \\ 2 & -2 & 5 \end{bmatrix} \quad \mathbf{A}^{(20)} = \begin{bmatrix} 6.8191 & 0.0872 & 0.0000 \\ 0.0872 & 6.0093 & -0.0000 \\ 0.0000 & -0.0000 & 1.1716 \end{bmatrix}$$

$$\mathbf{A}^{(50)} = \begin{bmatrix} 6.8284 & 0.0018 & -0.0000 \\ 0.0018 & 6.0000 & 0.0000 \\ 0.0000 & -0.0000 & 1.1716 \end{bmatrix}$$

$$\mathbf{Q}^{(50)} = \begin{bmatrix} 0.5016 & -0.7060 & -0.5000 \\ 0.7071 & 0.0016 & 0.7071 \\ -0.4984 & -0.7082 & 0.5000 \end{bmatrix}$$

Example

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 4 & -2 \\ 1 & -2 & 5 \end{bmatrix} \quad \mathbf{A}^{(10)} = \begin{bmatrix} 6.7200 & 0.2794 & 0.0000 \\ 0.2794 & 6.1084 & 0.0000 \\ 0.0000 & 0.0000 & 1.1716 \end{bmatrix}$$

$$\mathbf{A}^{(100)} = \begin{bmatrix} 6.8284 & 0.0000 & -0.0000 \\ 0.0000 & 6.0000 & 0.0000 \\ 0.0000 & -0.0000 & 1.1716 \end{bmatrix}$$

$$\mathbf{Q}^{(100)} = \begin{bmatrix} 0.5000 & -0.7071 & -0.5000 \\ 0.7071 & 0.0000 & 0.7071 \\ -0.5000 & -0.7071 & 0.5000 \end{bmatrix}$$