Lecture 14

Graph Theory and Circuit Analysis

- Basic Concepts of Graph Theory
- Cut-set
- Incidence Matrix
- Circuit Matrix
- Cut-set Matrix

Definition of Graph

- Definition: In a connected graph G of *n* nodes (vertices), the subgraph T that satisfies the following properties is called a tree.
  - T is connected
  - T contains all the vertices of G
  - T contains no circuit,
  - T contains exactly *n*-1 number of edges.

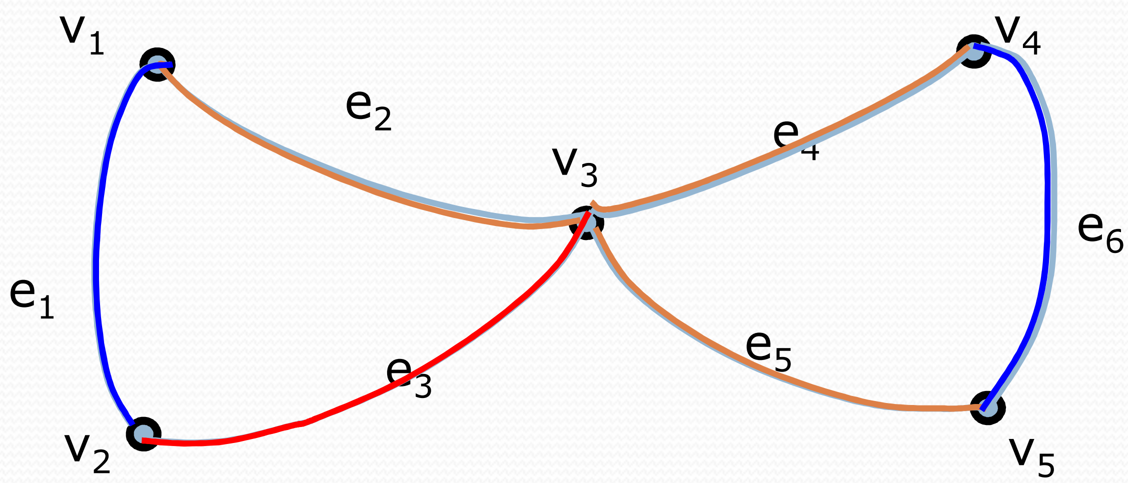
In every connected graph G there exists at least one tree.

Tree & Co-tree

- Let G have p separated parts $G_1, G_2, \ldots, G_p$, that is $G=G_1\cup G_2\cup\ldots\cup G_p$, and let T_i be a tree in G_i, $i=1,2,\ldots,p$, then, $T=T_1 \cup T_2\ldots \cup T_p$ is called a forest of G.

- DEFINITION: The complement of a tree is called a co-tree and the complement of a forest is called a co-forest. The edges of a tree or a forest are called branches and the edges of a co-tree or co-forest are called links (chords).

Tree & Co-tree Examples



9 possible trees and corresponding co-trees:

$T_1 = \{e_2, e_3, e_4, e_5\}$	$T_4 = \{e_1, e_2, e_5, e_6\}$	$T_7 = \{e_2, e_3, e_5, e_6\}$
$T_1' = \{e_1, e_6\}$	$T_4' = \{e_3, e_4\}$	$T_7' = \{e_1, e_4\}$
$T_2 = \{e_1, e_2, e_4, e_6\}$	$T_5 = \{e_1, e_3, e_4, e_6\}$	$T_8 = \{e_1, e_2, e_4, e_5\}$
$T_2' = \{e_3, e_5\}$	$T_5' = \{e_2, e_5\}$	$T_8' = \{e_3, e_6\}$
$T_3 = \{e_1, e_3, e_5, e_6\}$	$T_6 = \{e_2, e_3, e_4, e_6\}$	$T_9 = \{e_1, e_3, e_4, e_5\}$
$T_4' = \{e_2, e_4\}$	$T_6' = \{e_1, e_5\}$	$T_9' = \{e_2, e_6\}$

Rank & Nullity

- DEFINITION: Let G be a graph and let b and l be respectively the number of branches and chords of G, then b and l are called respectively the rank and the nullity of the graph.
- THEOREM: Let G have n nodes, e edges and p connected parts, then its rank and nullity are given respectively by

$$b = n - p$$

and

$$l = e - n + p$$

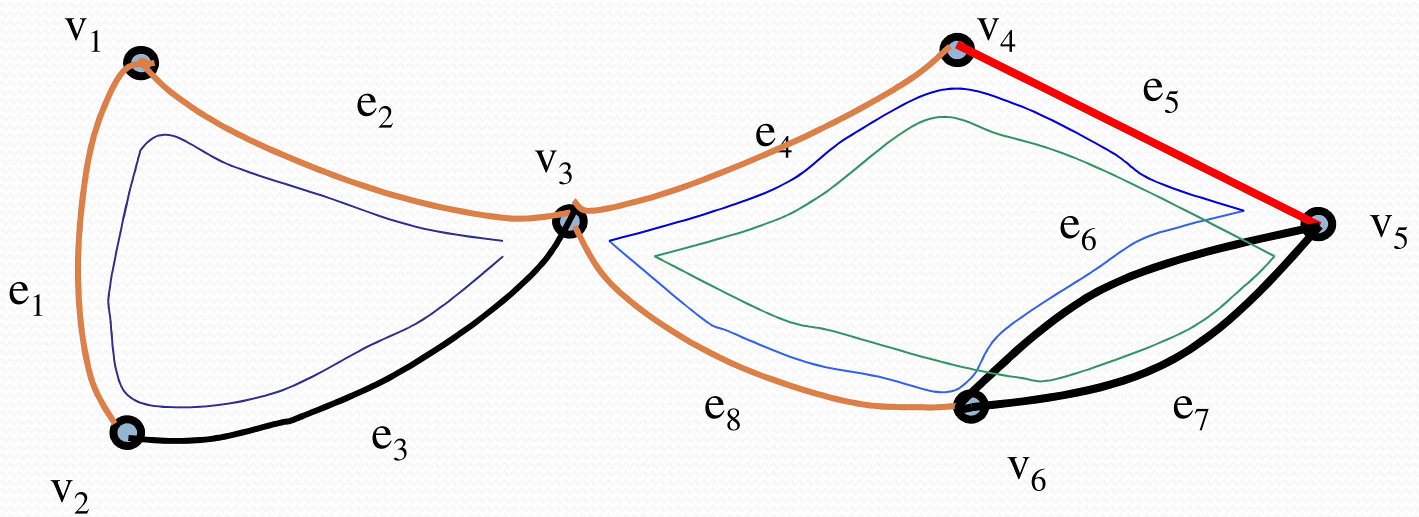
Fundamental Circuit (f-circuit)

- DEFINITION: Let G be a connected graph and let T and T' be tree and co-tree respectively, that is $G=T\cup T'$. Let a link $e'\subseteq T'$ and its unique tree path (a path which is formed by the branches of T) define a circuit. This circuit is called the fundamental circuit (f-circuit) of G. All such circuits defined by all the chords of T' are called the fundamental circuits (f-circuits) of G. If G is not connected, then the f-circuits are defined with respect to a forest.

f-circuit Example

- Note that the number of f-circuits is given by the nullity of G and that, with respect to a chosen tree T of G, each f-circuit contains one and only link.

Consider the following graph



f-circuits:

$c_{f1}=\{e_3,e_1,e_2\}$,
$c_{f2}=\{e_6,e_8,e_4,e_5\}$,
$c_{f3}=\{e_7,e_8,e_4,e_5\}$

Nullity of G

$l=e-n+p=8-6+1=3$

Cut-Set

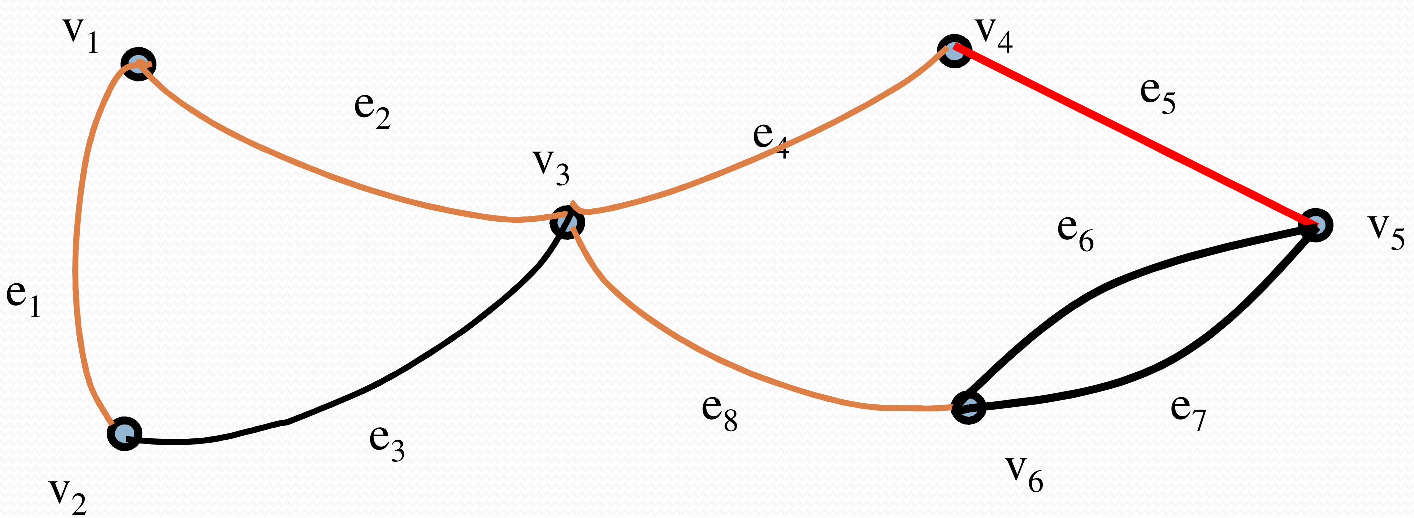
- DEFINITION: The cut-set of a graph G is the subgraph G_x of G consisting of the set of edges satisfying the following properties:
  - The removal of G_x from G reduces the rank of G exactly by one.
  - No proper subgraph of G_x has this property.

If G is connected, then the first property in the above definition can be replaced by the following phrase.

  - The removal of G_x from G separates the given connected graph G into exactly two connected subgraphs.

Cut-set example

Consider the following graph and the following set of edges



$G_1=\{e_1,e_2\}$ → is also a cut-set

$G_2=\{e_4,e_6,e_7\}$ → Cut-set

$G_3=\{e_2,e_3,e_4,e_8\}$ → is not a cut-set, because the removal of G_3 from G results in three connected subgraphs

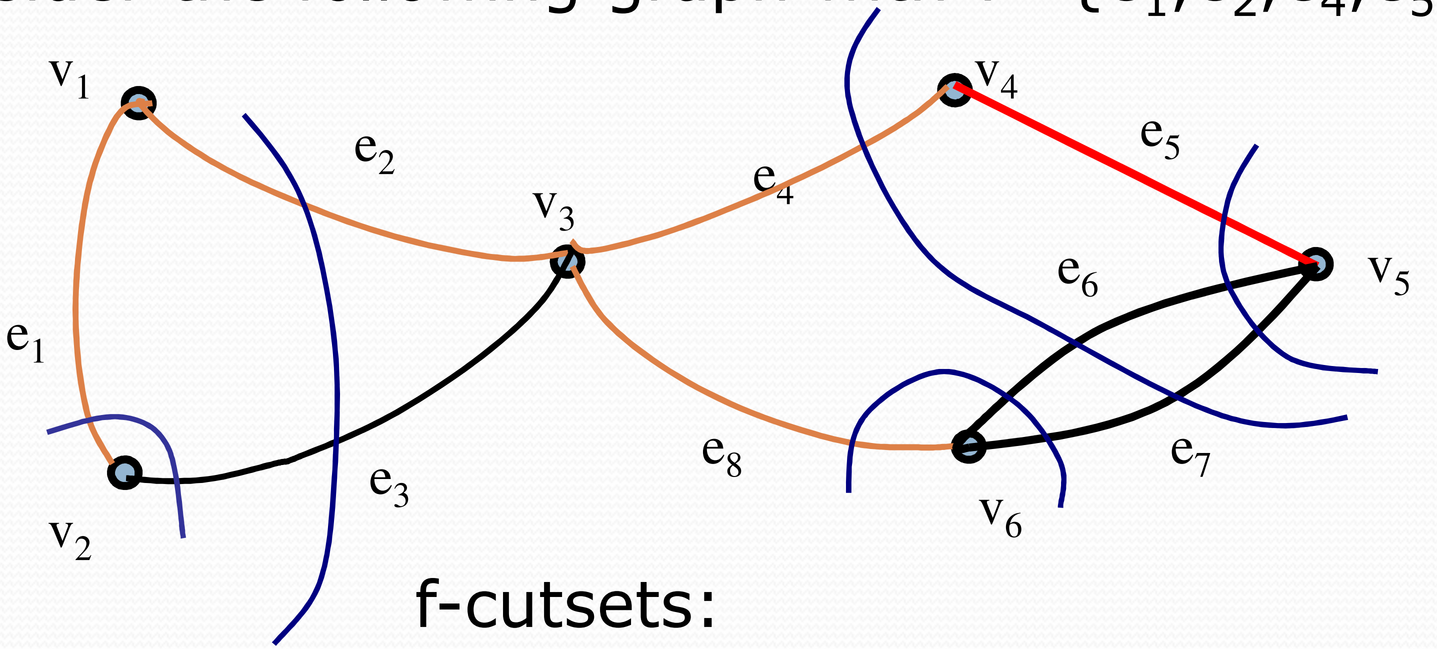
$G_4=\{e_2,e_3,e_6\}$ → is not a cut-set, because a subset of G_4 is cut-set

Fundamental cut-set (f-cutset)

- DEFINITION: Let G be a connected graph and let T be its tree. The branch $e_t \subseteq T$ defines a unique cut-set (a cut-set which is formed by e_t and the links of G). This cut-set is called the fundamental cut-set (f-cutset) of G. All such cut-sets defined by all the branches of T are called the fundamental cut-sets (f-cutsets) of G. If G is not connected then the f-cut sets are defined with respect to a forest.
- Note that the number of fundamental cut-sets is given by the rank of G and with respect to a chosen tree T of G, each fundamental cut-set contains one and only one branch.

f-cutset example

Consider the following graph with $T=\{e_1,e_2,e_4,e_5,e_8\}$



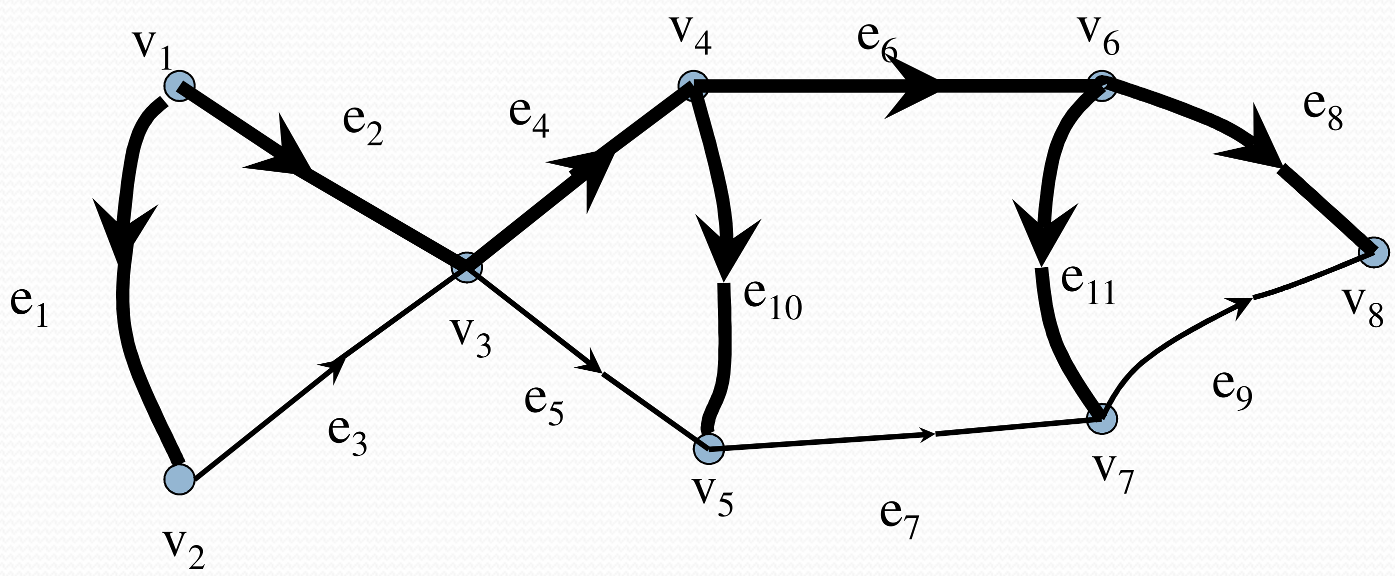
f-cutsets:

$x_{f1}=\{e_1,e_3\}$ $x_{f2}=\{e_2,e_3\}$

$x_{f3}=\{e_4,e_6,e_7\}$ $x_{f4}=\{e_5,e_6,e_7\}$

$x_{f5}=\{e_8,e_6,e_7\}$

Matrices of Directed Graphs



- The edge e_1 which has a direction from node v_1 to node v_2 simply indicates that any transmission from v_1 to v_2 along e_1 is assumed to be positive.
- Any transmission from v_2 to v_1 along e_1 is assumed to be negative.

Incidence Matrix

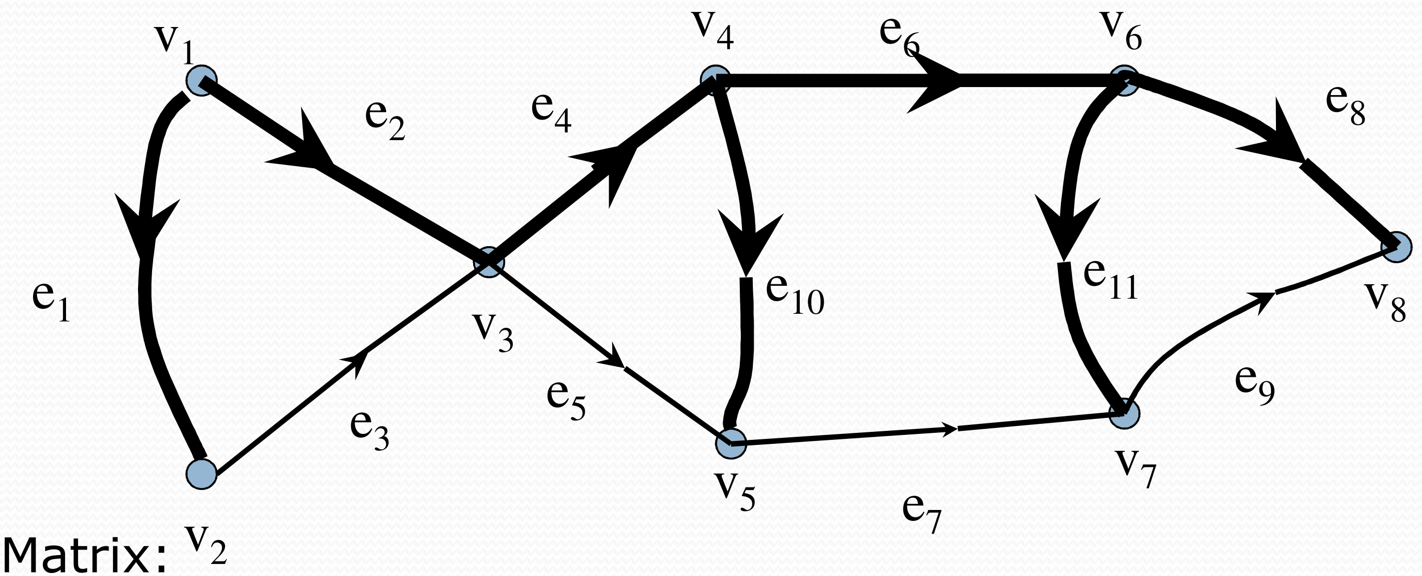
- DEFINITION: Let *e* and *n* represent respectively the number of edges and nodes of a graph G. The **incidence matrix**

$$\underset{n\times e}{\mathbf{A}_a} = [a_{ij}]$$

having *n* rows and *e* columns with its elements are defined as

$$a_{ij} = \begin{cases} 1 & \text{if edge } j \text{ incident to node } i \text{ and oriented "outward"} \\ -1 & \text{if edge } j \text{ incident to node } i \text{ and oriented "inward"} \\ 0 & \text{if edge } j \text{ not incident to node } i \end{cases}$$

Incidence Matrix (2)



Incident Matrix:

A =

edge	1	2	3	4	5	6	7	8	9	10	11	node
	1	1	0	0	0	0	0	0	0	0	0	1
	−1	0	1	0	0	0	0	0	0	0	0	2
	0	−1	−1	1	1	0	0	0	0	0	0	3
	0	0	0	−1	0	1	0	0	0	1	0	4
	0	0	0	0	−1	0	1	0	0	−1	0	5
	0	0	0	0	0	−1	0	1	0	0	1	6
	0	0	0	0	0	0	−1	0	1	0	−1	7
	0	0	0	0	0	0	0	−1	−1	0	0	8

Property:

Any column of **A** contains exactly two nonzero entries of opposite sign.

Reduced Incidence Matrix

- DEFINITION: For a connected graph G, the matrix **A**, obtained by deleting any one of the rows of the incidence matrix $\mathbf{A_a}$ is called the **reduced incidence matrix**.
- Note that since any column of $\mathbf{A_a}$ contains exactly two nonzero entries of opposite sign, one can uniquely determine the incident matrix when the reduced incident matrix is given.
- Note also that the rank of $\mathbf{A_a}$ is n-1.

Circuit Matrix

- In a graph G, let k be the number of circuits and let an arbitrary circuit orientation be assigned to each one of these circuits.
- DEFINITION: The circuit matrix

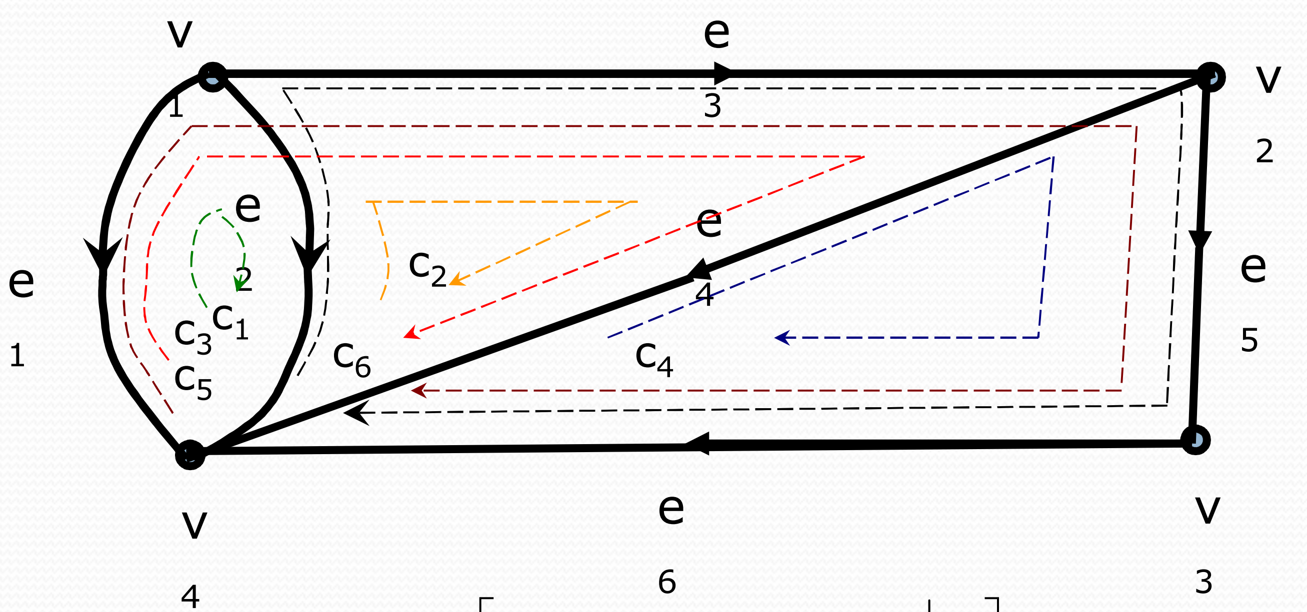
$$\underset{k\times e}{\mathbf{B}} = [b_{ij}]$$

for a graph G of e edges and k circuits is defined as

$$b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ incident to circuit } i \text{ with "same" orientation} \\ -1 & \text{if edge } j \text{ incident to circuit } i \text{ with "opposite" orientation} \\ 0 & \text{if edge } j \text{ not incident to circuit } i \end{cases}$$

Circuit Matrix

- Consider the following graph



$$
\mathbf{B} = \left[\begin{array}{cccccc|c}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & \\
\hline
-1 & 1 & 0 & 0 & 0 & 0 & c_1 \\
0 & -1 & 1 & 1 & 0 & 0 & c_2 \\
-1 & 0 & 1 & 1 & 0 & 0 & c_3 \\
0 & 0 & 0 & -1 & 1 & 1 & c_4 \\
-1 & 0 & 1 & 0 & 1 & 1 & c_5 \\
0 & -1 & 1 & 0 & 1 & 1 & c_6
\end{array}\right]
$$

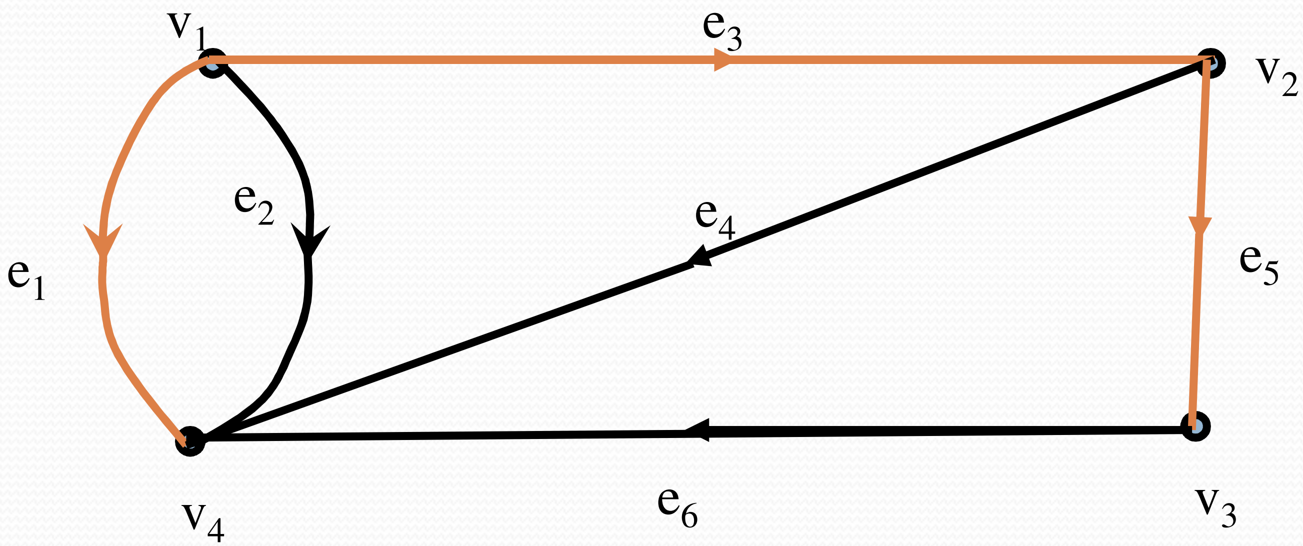
f-Circuit Matrix

- Let $\mathbf{b}_i$ represent the row of **B** that corresponds to circuit c_i. The circuits $c_i,\ldots,c_j$ are independent if the rows $\mathbf{b}_i,\ldots\ \mathbf{b}_j$ are independent.
- DEFINITION: The f-circuit matrix $\mathbf{B_f}$ of a graph G with respect to some tree T is defined as the circuit matrix consisting of the fundamental circuits of G only whose orientations are chosen in the same direction as that of defining links.
- The fundamental circuit matrix $\mathbf{B_f}$ of a graph G with respect to some tree T can always be written as

$$\underset{l\times e}{\mathbf{B}_f} = [\mathbf{U}\ \mathbf{B}_{f12}]$$

f-Circuit Matrix (2)

- Consider the following graph with T' = {e_1,e_3,e_5}



$$\begin{array}{c} \textit{edge} \quad 1 \quad 3 \quad 5 \quad\; 2 \quad 4 \quad 6 \end{array}$$

$$\mathbf{B}_f = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{U} & \mathbf{B}_{f12} \end{bmatrix}$$

Incidence & Circuit Matrices

- THEOREM: If the column orderings of the circuit and incident matrices are identical then

$$\mathbf{A}_a \mathbf{B}_f^T = \mathbf{0}$$

$$\mathbf{B}_f \mathbf{A}_a^T = \mathbf{0}$$

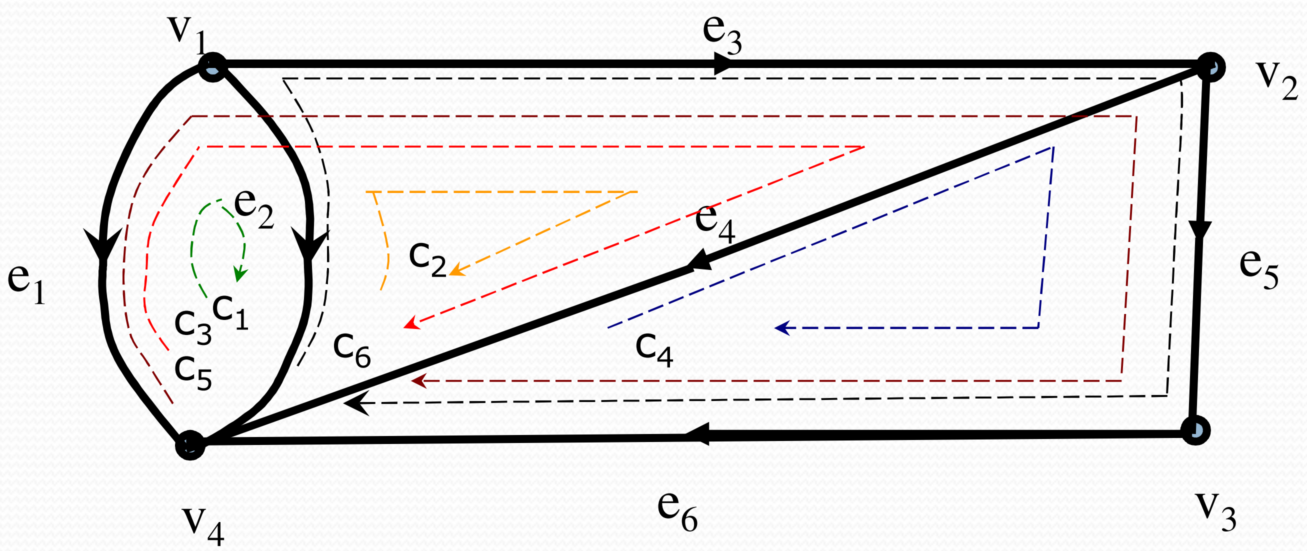
- Also

$$\mathbf{A}_a \mathbf{B}^T = \mathbf{0}$$

$$\mathbf{B} \mathbf{A}_a^T = \mathbf{0}$$

Matrices of Oriented Graphs

- Consider the following graph



$$\begin{array}{cccccc} edge & 1 & 3 & 5 & 2 & 4 & 6 \end{array}$$

$$\mathbf{B}_f = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}; \quad \mathbf{A}_a = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

Cut-set Matrix

- In a graph G let x be the number of cut-sets having arbitrary orientations. Then, we have the following definition.
- DEFINITION: The cut-set matrix

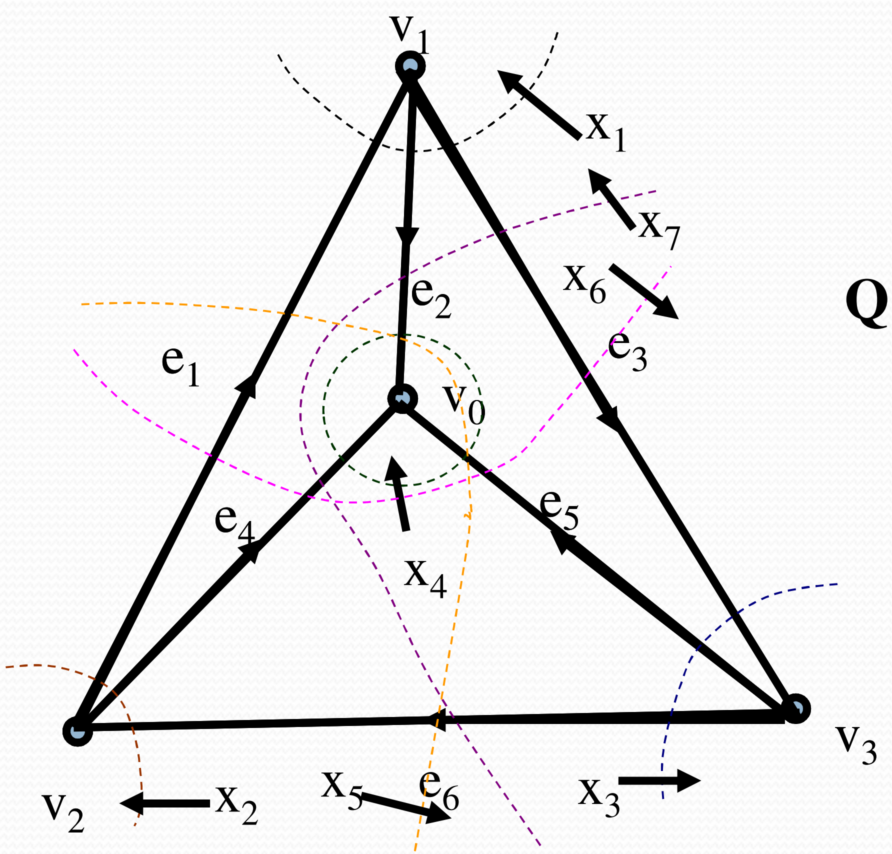
$$\underset{x\times e}{\mathbf{Q}} = [q_{ij}]$$

for a graph G of e edges and x cut-sets is defined as

$$q_{ij} = \begin{cases} 1 & \text{if edge } j \text{ in cut-set } i \text{ with } e_j, x_i \text{ "same" orientation} \\ -1 & \text{if edge } j \text{ in cut-set } i \text{ with } e_j, x_i \text{ "opposite" orientation} \\ 0 & \text{if edge } j \text{ not in cut-set } i \end{cases}$$

Cut-set Matrix (2)

- Consider the following graph and its seven possible cut-sets



$$Q = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix}$$

f-Cut-set Matrix

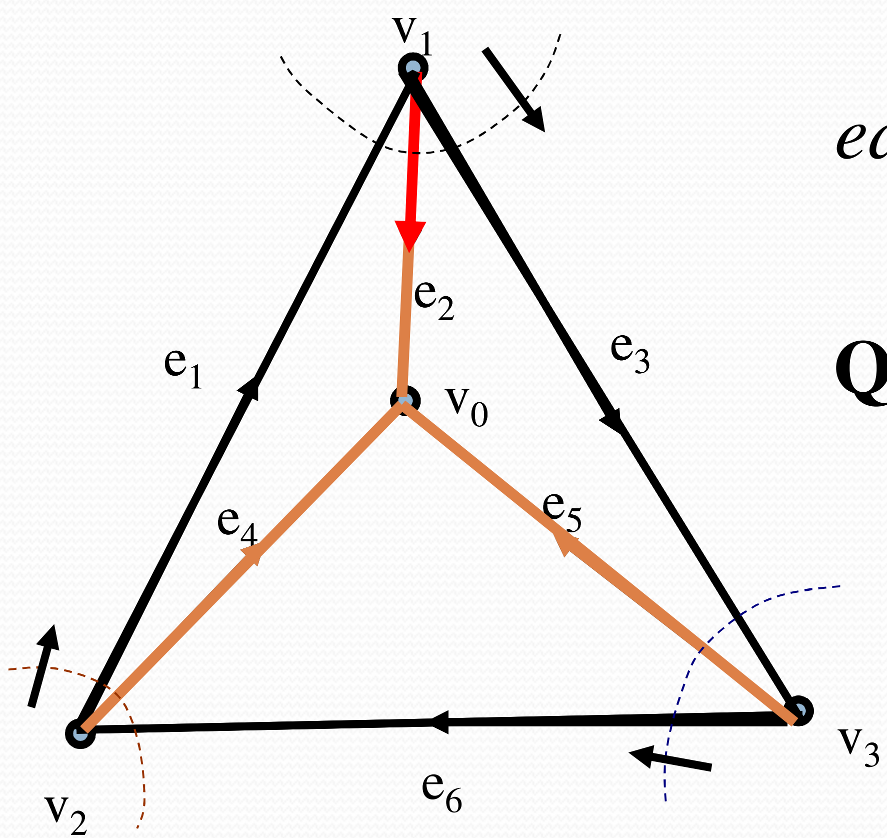
- DEFINITION: The f-cutset matrix $\mathbf{Q_f}$ of a graph G with respect to some tree T is defined as the cut-set matrix consisting of the fundamental cut-set of G only whose orientations are chosen in the same direction as that of defining branches.
- The fundamental cut-set matrix $\mathbf{A_f}$ of a graph G with respect to some tree T can always be written as

$$\underset{b\times e}{\mathbf{Q}_f} = [\underset{b\times b}{\mathbf{U}} \;\; \underset{b\times(e-b)}{\mathbf{Q}_{f11}}]$$

- Recall that $b = n$-1

f-Cut-set Matrix (2)

- Consider the following graph with T = {e_2,e_4,e_5}



$$
\begin{array}{ccccccc}
edge & 2 & 4 & 5 & 1 & 3 & 6
\end{array}
$$

$$
\mathbf{Q}_f = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}
$$

Cut-set & Circuit Matrices

- THEOREM: If the column orderings of the circuit and incident matrices are identical then

$$\mathbf{Q}\mathbf{B}_f^T = \mathbf{0}$$

$$\mathbf{B}_f\mathbf{Q}^T = \mathbf{0}$$

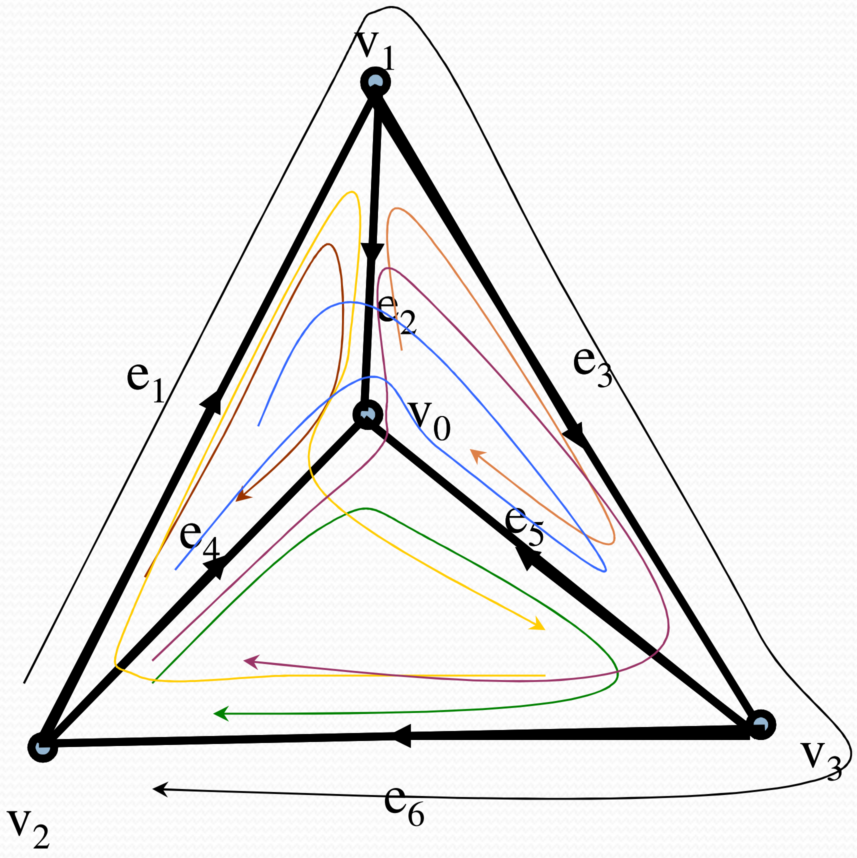
- Also

$$\mathbf{Q}\mathbf{B}^T = \mathbf{0}$$

$$\mathbf{B}\mathbf{Q}^T = \mathbf{0}$$

Cut-set & Circuit Matrices

- Consider the following graph



$$\mathbf{Q} = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

FUNDAMENTAL POSTULATES

- Now, Let G be a connected graph having e edges and let

$$\mathbf{x}^T = \left[x_1(t), x_2(t), \cdots x_e(t) \right]$$

and

$$\mathbf{y}^T = \left[y_1(t), y_2(t), \cdots y_e(t) \right]$$

be two vectors where x_i and y_i, i=1,...,e, correspond to the across and through variables associated with the edge *i* respectively.

FUNDAMENTAL POSTULATES

- 2. POSTULATE Let **B** be the circuit matrix of the graph G having e edges then we can write the following algebraic equation for the across variables of G (e.g., edge voltage)

$$\mathbf{Bx} = \mathbf{0} \Rightarrow \text{KVL}$$

- 3. POSTULATE Let **Q** be the cut-set matrix of the graph G having e edges then we can write the following algebraic equation for the through variables of G (e.g., edge current)

$$\mathbf{Qy} = \mathbf{0} \Rightarrow \text{KCL}$$

Fundamental Circuit & Cut-set Equations

- Consider a graph G and a tree T in G. Let the vectors **v** and **i** partitioned as

$$\mathbf{v} = [\mathbf{v}_{link} \;\; \mathbf{v}_{branch}]^T ; \mathbf{i} = [\mathbf{i}_{branch} \;\; \mathbf{i}_{link}]^T$$

- Then

$$\mathbf{B}_f \mathbf{v} = [\mathbf{U} \;\; \mathbf{B}_{f12}] \begin{bmatrix} \mathbf{v}_{link} \\ \mathbf{v}_{branch} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{v}_{link} = -\mathbf{B}_{f12} \mathbf{v}_{branch}$$

$$\mathbf{Q}_f \mathbf{i} = [\mathbf{U} \;\; \mathbf{Q}_{f11}] \begin{bmatrix} \mathbf{i}_{branch} \\ \mathbf{i}_{link} \end{bmatrix} = \mathbf{0}$$

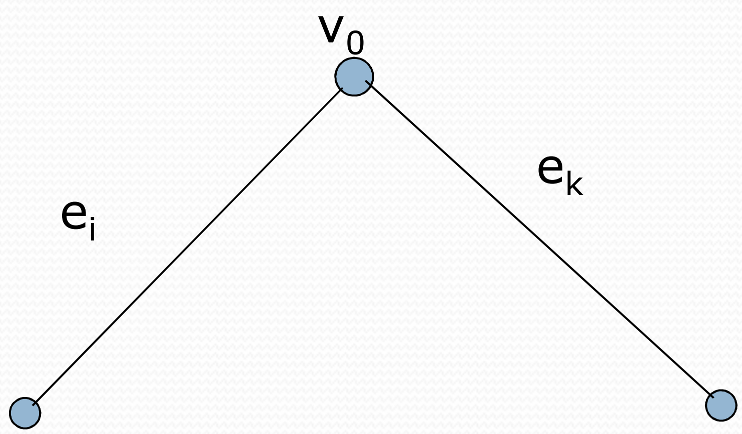
$$\mathbf{i}_{branch} = -\mathbf{Q}_{f11} \mathbf{i}_{link}$$

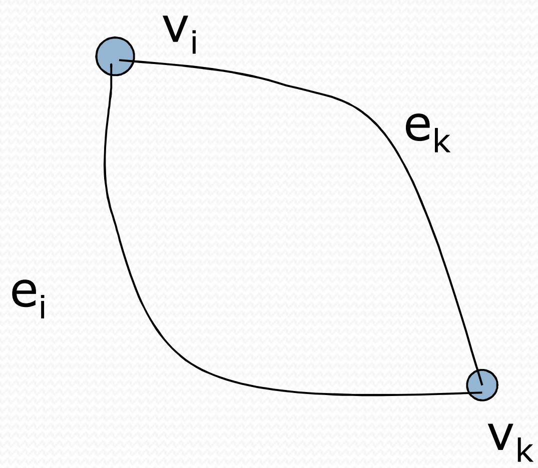
fundamental circuit equation

fundamental cut-set equation

Series & Parallel Edges

- Definition: Two edges e_i and e_k are said to be connected in series if they have exactly one common vertex of degree two.
- Definition: Two edges e_i and e_k are said to be connected in parallel if they are incident at the same pair of vertices v_i and v_k.





General Procedure

1. Draw a graph and then identify a tree.
2. Place all control-voltage branches for voltage-controlled dependent sources in the tree, if possible.
3. Place all control-current branches for current-controlled dependent sources in the cotree, if possible.
4. Find incidence, f-circuit, or f-cutset matrix.
5. Replace voltage, current sources with short, open circuits, respectively.
6. Formulate the matrix equation.