# Lecture 3-4
## Solutions of
## System of Linear Equations

- Numeric Linear Algebra
- Review of vectors and matrices
- System of Linear Equations
- Gaussian Elimination (direct solver)
- LU Decomposition
- Gauss-Seidel method (iterative solver)

# VECTORS

Vector : a one dimensional array of numbers

Examples :

row vector $\begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$ column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\textit{Identity vectors} \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

# MATRICES

Matrix : a two dimensional array of numbers

Examples :

zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

diagonal $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$, Tridiagonal $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

# MATRICES

Examples :

$$\text{symmetric}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix}, \quad \text{upper triangular}\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Determinant of a MATRICES

Defined for square matrices only

Examples :

$$\det\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix} = 2\begin{vmatrix} 0 & 5 \\ 5 & 4 \end{vmatrix} -1\begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} -1\begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}$$

$$= 2(-25) - 1(12+5) - 1(15-0) = -82$$

# Adding and Multiplying Matrices

The addition of two matrices A and B

* Defined only if they have the same size
* $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

Multiplication of two matrices $A(n \times m)$ and $B(p \times q)$

* The product $C = AB$ is defined only if $m = p$
* $C = AB \Leftrightarrow c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \quad \forall i, j$

# Systems of Linear Equations

A system of linear equations can be presented in different forms

$$\left.\begin{aligned} 2x_1 + 4x_2 - 3x_3 &= 3 \\ 2.5x_1 - x_2 + 3x_3 &= 5 \\ x_1 \qquad\quad - 6x_3 &= 7 \end{aligned}\right\} \Leftrightarrow \begin{bmatrix} 2 & 4 & -3 \\ 2.5 & -1 & 3 \\ 1 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Standard form Matrix form

# Solutions of Linear Equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{is a solution to the following equations :}$$

$$x_1 + x_2 = 3; x_1 + 2x_2 = 5$$

- A set of equations is **inconsistent** if there exists no solution to the system of equations:

$$x_1 + 2x_2 = 3; 2x_1 + 4x_2 = 5$$

$$\text{These equations are inconsistent}$$

# Solutions of Linear Equations

- Some systems of equations may have **infinite number of solutions**

$$x_1 + 2x_2 = 3$$
$$2x_1 + 4x_2 = 6$$

Both are "same" equations!!!
"linearly dependent" equation

have infinite number of solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ 0.5(3-a) \end{bmatrix} \text{ is a solution for all } a$$

In matrix form:

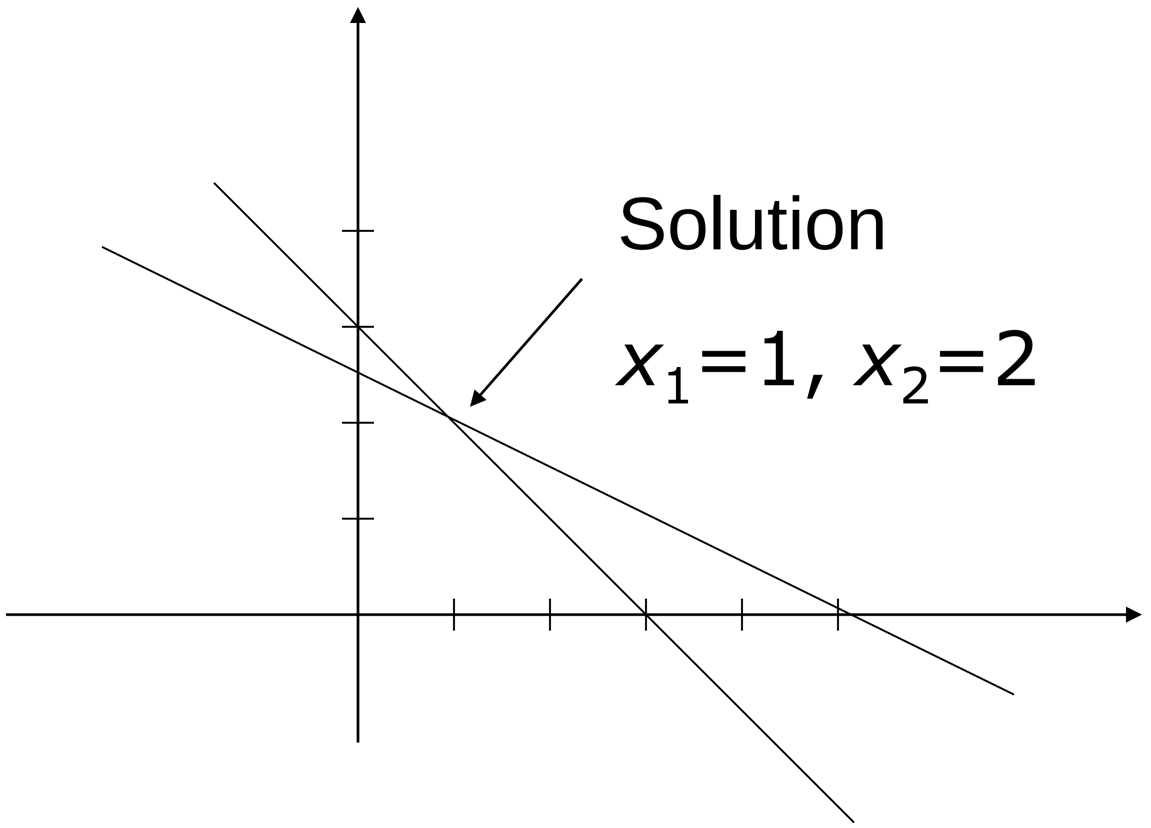
$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}; |\mathbf{A}| = 0$$

→"*Singular*" matrix

# Graphical Solution of Systems of Linear Equations

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$



# Cramer’s Rule is Not Practical

Cramer's Rule can be used to solve the system

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \qquad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

Cramer's Rule is not practical for large systems.

To solve N by N system requires $(N+1)(N-1)N!$ multiplications.

To solve a 30 by 30 system, $2.38 \times 10^{35}$ multiplications are needed.

It can be used if the determinants are computed in efficient way

# Naive Gaussian Elimination

- The method consists of two steps:
  - **Forward Elimination**: the system is reduced to upper triangular form. A sequence of elementary operations is used.
  - **Backward Substitution**: Solve the system starting from the last variable.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

# Elementary Row Operations

- Adding a multiple of one row to another
- Multiply any row by a non-zero constant
- Basically "To eliminate an unknown"

EX:

$$x_1 + x_2 = 3 \ldots (1);\ x_1 + 2x_2 = 5 \ldots (2)$$

$$\text{Add} -2 \times (1) \text{ to } (2) \text{ yields}$$

$$-x_1 = -1 \rightarrow x_1 = 1 \rightarrow x_2 = 2$$

# Example
## Forward Elimination

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

Row 2 – Row 1×2

Row 3 – Row 1×(1/2)

Row 4 – Row 1×(-1)

Part 1: Forward Elimination

Step 1: Eliminate $x_1$ from equations 2, 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

Row 3 – Row 2×3

Row 4 – Row 2×(-1/2)

# Example
## Forward Elimination

$$\text{Step2: Eliminate } x_2 \text{ from equations } 3, 4$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

Row 4 – Row 3×2

$$\text{Step3: Eliminate } x_3 \text{ from equation } 4$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

# Example
## Forward Elimination

Summary of the Forward Elimination:

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

# Example
## Backward Substitution

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Solve for $x_4$, then solve for $x_3$,...solve for $x_1$

$$x_4 = \frac{-3}{-3} = 1, \qquad x_3 = \frac{-9+5}{2} = -2$$

$$x_2 = \frac{-6-2(-2)-2(1)}{-4} = 1, \quad x_1 = \frac{16+2(1)-2(-2)-4(1)}{6} = 3$$

# Forward Elimination

To eliminate $x_1$

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}}\right) a_{1j} \quad (1 \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i1}}{a_{11}}\right) b_1 \end{aligned}\right\} 2 \le i \le n$$

To eliminate $x_2$

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}}\right) a_{2j} \quad (2 \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i2}}{a_{22}}\right) b_2 \end{aligned}\right\} 3 \le i \le n$$

# Forward Elimination

To eliminate $x_k$

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{ik}}{a_{kk}}\right) a_{kj} \quad (k \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{ik}}{a_{kk}}\right) b_k \end{aligned}\right\} k+1 \le i \le n$$

Continue until $x_{n-1}$ is eliminated.

# Backward Substitution

$$x_n = \frac{b_n}{a_{n,n}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n} x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n} x_n - a_{n-2,n-1} x_{n-1}}{a_{n-2,n-2}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j} x_j}{a_{i,i}}$$

# Example 1

Solve using Naive Gaussian Elimination:

Part 1: Forward Elimination ___Step 1: Eliminate $x_1$ from equations 2, 3

$$x_1 + 2x_2 + 3x_3 = 8 \qquad eq1\ unchanged(pivot\ equation)$$

$$2x_1 + 3x_2 + 2x_3 = 10 \qquad eq2 \leftarrow eq2 - \left(\frac{2}{1}\right) eq1$$

$$3x_1 + x_2 + 2x_3 = 7 \qquad eq3 \leftarrow eq3 - \left(\frac{3}{1}\right) eq1$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ -x_2 - 4x_3 &= -6 \\ -5x_2 - 7x_3 &= -17 \end{aligned}$$

# Example 1

Part 1: Forward Elimination Step2: Eliminate $x_2$ from equation 3

$$x_1 + 2x_2 + 3x_3 = 8 \qquad eq1\ unchanged$$

$$-\ x_2 - 4x_3 = -6 \qquad eq2\ unchanged(pivot\ equation)$$

$$-5x_2 - 7x_3 = -17 \qquad eq3 \leftarrow eq3 - \left(\frac{-5}{-1}\right) eq2$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 8 \\ -\ x_2 - 4x_3 = -6 \\ 13x_3 = 13 \end{cases}$$

# Example 1

## Backward Substitution

$$x_3 = \frac{b_3}{a_{3,3}} = \frac{13}{13} = 1$$

$$x_2 = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = \frac{-6 + 4x_3}{-1} = 2$$

$$x_1 = \frac{b_1 - a_{1,2}x_2 - a_{1,3}x_3}{a_{1,1}} = \frac{8 - 2x_2 - 3x_3}{a_{1,1}} = 1$$

$$\text{The solution is } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

# Determinant

The elementary operations do not affect the determinant

Example :

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 2\\ 3 & 1 & 2\end{bmatrix}\xrightarrow{\text{Elementary operations}}\mathbf{A}'=\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -4\\ 0 & 0 & 13\end{bmatrix}$$

$$\det(\mathbf{A})=\det(\mathbf{A}')=-13$$

# How Many Solutions Does a System of Equations **Ax=b** Have?

| Unique | No solution | Infinite |
| --- | --- | --- |
| $\det(\mathbf{A}) \neq 0$ | $\det(\mathbf{A}) = 0$ | $\det(\mathbf{A}) = 0$ |
| reduced matrix has no zero rows | reduced matrix has one or more zero rows | reduced matrix has one or more zero rows |
| | corresponding B elements $\neq 0$ | corresponding B elements $= 0$ |

# Examples

Unique

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$solution:$

$$X = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

No solution

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$No\ solution$

$$0 = -1\ impossible!$$

infinte # of solutions

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$Infinite\ \#\ solutions$

$$X = \begin{bmatrix} \alpha \\ 1 - .5\alpha \end{bmatrix}$$

# Pseudocode: Forward Elimination

```
Do k = 1 to n-1
   Do i = k+1 to n
      factor = a_{i,k} / a_{k,k}
      Do j = k+1 to n
         a_{i,j} = a_{i,j} – factor * a_{k,j}
      End Do
      b_i = b_i – factor * b_k
   End Do
End Do
```

# Pseudocode: Back Substitution

$x_n = b_n / a_{n,n}$

Do i = n-1 downto 1

  sum = $b_i$

  Do j = i+1 to n

    sum = sum – $a_{i,j} * x_j$

  End Do

  $x_i$ = sum / $a_{i,i}$

End Do

# Problems with Naive Gaussian Elimination

- The Naive Gaussian Elimination may fail for very simple cases. (**The pivoting element (diagonal element; "pivot") is zero).**

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

- **Also, very small pivoting element may result in serious computation errors.**

$$\begin{bmatrix} 10^{-10} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

# Remedy : Pivoting & Scaling

- Pivoting
  - *Choose "largest" element in each column for "pivot"*
  - *This can be done by swapping both rows and columns $\rightarrow$ "complete" pivoting.*
  - *Or only swapping rows $\rightarrow$* "Partial" pivoting
- Scaling
  - *Divide all elements in each row by "its largest element"*

# Example 2

Solve the following system using Gaussian Elimination with Partial Pivoting :

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

“Partial” pivoting means row swaps using pivots, i.e., No column swaps.

# Example 2
## Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{Index Vector } \mathrm{L} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

# Why Index Vector?

- Index vectors are used because it is much easier to exchange a single index element compared to exchanging the values of a complete row.
- In practical problems with very large N, exchanging the contents of rows may not be practical, i.e., can take time and memory.

# Example 2

## Forward Elimination-- Step 1: eliminate x1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow L = [1 \ \ 2 \ \ 3 \ \ 4\,]$$

First column $[1\ 3\ 5\ 4]^T \Rightarrow$ max corresponds to $l_3$

equation 3 is the first pivot equation Exchange $l_3$ *and* $l_1$

$L = [3 \ \ 2 \ \ 1 \ \ 4\,]$

# Example 2

## Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ \boxed{5 \quad 8 \quad 6 \quad 3} \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \boxed{1} \\ -1 \end{bmatrix}$$

**First pivot equation**

$$\Rightarrow \begin{bmatrix} 0 & -2.6 & 0.8 & 0.4 \\ 0 & -2.4 & -2.6 & 2.2 \\ 5 & 8 & 6 & 3 \\ \boxed{0 \quad -4.4 \quad 0.2 \quad 0.6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \\ 1 \\ \boxed{-1.8} \end{bmatrix}$$

# Example 2

## Forward Elimination-- Step 2: eliminate $x_2$

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -2.6 & 0.8 & 0.4 \\ 0 & -2.4 & -2.6 & 2.2 \\ 5 & 8 & 6 & 3 \\ 0 & -4.4 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \\ 1 \\ -1.8 \end{bmatrix}$$

$$L = [\ 3 \quad 2 \quad 1 \quad 4\ ]$$

$$2^{nd}\ \text{Column}: \{2.6\ \ 2.4\ \ 4.4\} \Rightarrow L = [\ 3\ 4\ 1\ 2\ ]$$

# Example 2

## Forward Elimination-- Step 3: eliminate x3

**Third pivot equation**

$$\begin{bmatrix} 0 & 0 & 0.6818 & 0.0455 \\ 0 & 0 & \boxed{-2.7273 \quad 1.8182} \\ 5 & 8 & 6 & 3 \\ 0 & -4.4 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.8636 \\ 1.5455 \\ 1 \\ -1.8 \end{bmatrix}$$

$L = [\ 3\ 4\ 2\ 1]$

$$\begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 0 & 0 & -2.7273 & 1.8182 \\ 5 & 8 & 6 & 3 \\ 0 & -4.4 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.25 \\ 1.5455 \\ 1 \\ -1.8 \end{bmatrix}$$

# Example 2
## Backward Substitution

$$\begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 0 & 0 & -2.7273 & 1.8182 \\ 5 & 8 & 6 & 3 \\ 0 & -4.4 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.25 \\ 1.5455 \\ 1 \\ -1.8 \end{bmatrix} \qquad L = [\ 3\ 4\ 2\ 1]$$

$$x_4 = \frac{b_1}{a_{1,4}} = \frac{2.25}{0.5} = 4.5, \quad x_3 = \frac{b_2 - a_{1,4}x_4}{a_{2,3}} = \frac{1.5455 - 1.8182x_4}{-2.7273} = 2.4327$$

$$x_2 = \frac{b_4 - a_{4,4}x_4 - a_{4,3}x_3}{a_{4,2}} = \frac{-1.8 - 0.6x_4 - 0.2x_3}{-4.4} = 1.1333$$

$$x_1 = \frac{b_3 - a_{3,4}x_4 - a_{3,3}x_3 - a_{3,2}x_2}{a_{3,1}} = \frac{1 - x_4 - 6x_3 - 8x_2}{5} = -7.2333$$

# Example 2 with Scaling

Solve the following system using Gaussian Elimination with Scaled Partial Pivoting:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

“Scaled Partial” pivoting means row swaps using scaled pivots.

# Example 2
## Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Scale vector:

disregard sign

find largest in magnitude in each row

$$\text{Scale vector} \quad S = \begin{bmatrix} 2 & 4 & 8 & 5 \end{bmatrix}$$

$$\text{Index Vector} \quad L = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

# Example 2

## Forward Elimination-- Step 1: eliminate x1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 \quad 4 \quad 8 \quad 5] \\ L = [1 \quad 2 \quad 3 \quad 4\ ] \end{cases}$$

$$Ratios = \left\{ \frac{\left|a_{l_i,1}\right|}{S_{l_i}} \quad i = 1,2,3,4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \boxed{\frac{|4|}{5}} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange $l_4$ *and* $l_1$

$L = [\,4 \quad 2 \quad 3 \quad 1\,]$

# Example 2

## Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ \boxed{4 \quad 2 \quad 5 \quad 3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \boxed{-1} \end{bmatrix}$$

**First pivot equation**

$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ \boxed{4 \quad 2 \quad 5 \quad 3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ \boxed{-1} \end{bmatrix}$$

# Example 2

## Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ \ 4 \ \ 8 \ \ 5\ ] \qquad L = [\ \ 4 \quad 2 \quad 3 \quad 1\ ]$$

$$\text{Ratios}: \left\{ \frac{|a_{l_i,2}|}{S_{l_i}} \ i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \quad \frac{5.5}{8} \quad \boxed{\frac{1.5}{2}} \right\} \Rightarrow L = [\ 4\ 1\ \ 3\ \ 2\ \ ]$$

# Example 2

## Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & \boxed{-2.5 \quad 1.8333} & \\ 0 & 0 & 0.25 & 1.6667 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 6.8333 \\ -1 \end{bmatrix}$$

**Third pivot equation**

$L = [\ 4\ 1\ 2\ \ 3]$

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix}$$

# Example 2
## Backward Substitution

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix} \qquad L = [\ 4\ 1\ 2\ \ 3]$$

$$x_4 = \frac{b_3}{a_{3,4}} = \frac{9}{2} = 4.5, \quad x_3 = \frac{b_2 - a_{2,4}x_4}{a_{2,3}} = \frac{2.1667 - 1.8333x_4}{-2.5} = 2.4327$$

$$x_2 = \frac{b_1 - a_{1,4}x_4 - a_{1,3}x_3}{a_{1,2}} = \frac{1.25 - 0.25x_4 - 0.75x_3}{-1.5} = 1.1333$$

$$x_1 = \frac{b_4 - a_{4,4}x_4 - a_{4,3}x_3 - a_{4,2}x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -7.2333$$

# Example 3 : Scaled Partial Pivoting

Solve the following system using Gaussian elimination with scaled partial pivoting

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

# Example 3

## Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{Scale vector} \quad S = \begin{bmatrix} 2 & 4 & 8 & 5 \end{bmatrix}$$

$$\text{Index Vector} \quad L = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

# Example 3

## Forward Elimination-- Step 1: eliminate x1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 \quad 4 \quad 8 \quad 5] \\ L = [1 \quad 2 \quad 3 \quad 4\,] \end{cases}$$

$$Ratios = \left\{ \frac{|a_{l_i,1}|}{S_{l_i}} \quad i = 1,2,3,4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \boxed{\frac{|4|}{5}} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange $l_4$ *and* $l_1$

$L = [\,4 \quad 2 \quad 3 \quad 1\,]$

# Example 3

## Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 3 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ \boxed{4 \quad 2 \quad 5 \quad 3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \boxed{-1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ \boxed{4 \quad 2 \quad 5 \quad 3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

# Example 3

## Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \;\; 4 \;\; 8 \;\; 5\,] \qquad L = [\;\; 4 \quad 2 \quad 3 \quad 1\;]$$

$$\text{Ratios}: \left\{ \frac{\left| a_{l_i,2} \right|}{S_{l_i}} \;\; i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \quad \boxed{\frac{10.5}{8}} \quad \frac{1.5}{2} \right\} \Rightarrow L = [\;4 \; 3 \; 2 \; 1]$$

# Example 3

## Forward Elimination-- Step 2: eliminate x2

Updating A and B

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & \boxed{-10.5 & -0.25 & -0.75} \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$L = [\ 4\ 1\ 3\ 2]$

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

# Example 3

## Forward Elimination-- Step 3: eliminate x3

Selection of the third pivot equation

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \;\; 4 \;\; 8 \;\; 5\,] \qquad L = [\;\; 4 \quad 3 \quad 2 \quad 1\;]$$

$$\text{Ratios}: \left\{ \frac{\left| a_{l_i,3} \right|}{S_{l_i}} \;\; i = 3,4 \right\} = \left\{ \boxed{\frac{2.7619}{4}} \;\; \frac{0.7857}{2} \right\} \Rightarrow L = [\;4\;3\;2\;1]$$

# Example 3

## Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$$L = [\ 4\ 3\ 2\ \ 1]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

# Example 3
## Backward Substitution

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix} \qquad L = [\ 4\ 3\ 2\ \ 1]$$

$$x_4 = \frac{b_{l_4}}{a_{l_4,4}} = \frac{1.4569}{0.8448} = 1.7245, \quad x_3 = \frac{b_{l_3} - a_{l_3,4}x_4}{a_{l_3,3}} = \frac{1.8571 - 1.7143x_4}{-2.7619} = 0.3980$$

$$x_2 = \frac{b_{l_2} - a_{l_2,4}x_4 - a_{l_2,3}x_3}{a_{l_2,2}} = -0.3469$$

$$x_1 = \frac{b_{l_1} - a_{l_1,4}x_4 - a_{l_1,3}x_3 - a_{l_1,2}x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -1.8673$$

# How Do We Know If a Solution is Good or Not

Given **Ax**=**b** (**A** : matrix, **b** : right-hand-side (RHS) vector)

- **x** is a solution if **Ax**-**b**=0
- Compute the residual vector **r**= **Ax**-**b**
- Due to rounding error, **r** may not be zero

The solution is acceptable if $\max_i |r_i| \le \varepsilon$

- $\varepsilon$ is usually called "tolerance".

# How Good is the Solution?

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad solution \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1.8673 \\ -0.3469 \\ 0.3980 \\ 1.7245 \end{bmatrix}$$

$$\text{Residues}: \ R = \begin{bmatrix} 0.005 \\ 0.002 \\ 0.003 \\ 0.001 \end{bmatrix}$$

# Remarks:

- We use index vector to avoid the need to move the rows which may not be practical for large problems.
- If we order the equation as in the last value of the index vector, we have a triangular form.
- Scale vector is formed by taking maximum in magnitude in each row.
- Scale vector does not change.
- The original matrix **A** and vector **b** are used in checking the residuals.

# Tridiagonal Systems

Tridiagonal Systems:

- The non-zero elements are in the main diagonal, super diagonal and subdiagonal.
- $a_{ij}=0 \quad if \quad |i-j| > 1$

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 0 & 2 & 6 & 2 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

# Tridiagonal Systems

- Occur in many applications
- Needs <u>less storage</u> (4n-2 compared to $n^2$ +n for the general cases)
- Selection of pivoting rows is unnecessary (under some conditions)
- <u>Efficiently</u> solved by Gaussian elimination

# Algorithm to Solve Tridiagonal Systems

- Based on Naive Gaussian elimination.
- As in previous Gaussian elimination algorithms
  - Forward elimination step
  - Backward substitution step
- Elements in the super diagonal are not affected.
- Elements in the main diagonal, and **b** need updating

# Tridiagonal System

All the $a$ elements will be zeros, need to update the $d$ and $b$ elements

The $c$ elements are not updated

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \begin{bmatrix} d_1 & c_1 & & & \\ & d_2' & c_2 & & \\ & & d_3' & \ddots & \\ & & & \ddots & c_{n-1} \\ & & & & d_n' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \\ \vdots \\ b_n' \end{bmatrix}$$

# Diagonal Dominance

A matrix **A** is diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \qquad \textit{for } (1 \le i \le n)$$

The magnitude of each diagonal element is larger than the sum of elements in the corresponding row.

Examples:

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 6 & 1 \\ 1 & 2 & -5 \end{bmatrix} \qquad\qquad \begin{bmatrix} -3 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Diagonally dominant        Not Diagonally dominant

# Diagonally Dominant Tridiagonal System

- A tridiagonal system is diagonally dominant if

$$|d_i| > |c_i| + |a_{i-1}| \qquad (1 \le i \le n)$$

- Forward Elimination preserves diagonal dominance

# Solving Tridiagonal System

Forward Eliminatio n

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}}\right)c_{i-1}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}}\right)b_{i-1} \qquad 2 \le i \le n$$

Backward Substitution

$$x_n = \frac{b_n}{d_n}$$

$$x_i = \frac{1}{d_i}\left(b_i - c_i x_{i+1}\right) \qquad \text{for } i = n-1, n-2, \ldots, 1$$

# Example

Solve

$$\begin{bmatrix} 5 & 2 & & \\ 1 & 5 & 2 & \\ & 1 & 5 & 2 \\ & & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix} \qquad \Rightarrow D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \\ \end{bmatrix}, B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Eliminatio n

$$d_i \leftarrow d_i - \left( \frac{a_{i-1}}{d_{i-1}} \right) c_{i-1}, \quad b_i \leftarrow b_i - \left( \frac{a_{i-1}}{d_{i-1}} \right) b_{i-1} \qquad 2 \le i \le 4$$

Backward Substitution

$$x_n = \frac{b_n}{d_n}, \quad x_i = \frac{1}{d_i}\left( b_i - c_i x_{i+1} \right) \qquad \text{for } i = 3,2,1$$

# Example

$$D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \\ \end{bmatrix}, B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Eliminatio n

$$d_2 = d_2 - \left(\frac{a_1}{d_1}\right) c_1 = 5 - \frac{1 \times 2}{5} = 4.6, \quad b_2 = b_2 - \left(\frac{a_1}{d_1}\right) b_1 = 9 - \frac{1 \times 12}{5} = 6.6$$

$$d_3 = d_3 - \left(\frac{a_2}{d_2}\right) c_2 = 5 - \frac{1 \times 2}{4.6} = 4.5652, \quad b_3 = b_3 - \left(\frac{a_2}{d_2}\right) b_2 = 8 - \frac{1 \times 6.6}{4.6} = 6.5652$$

$$d_4 = d_4 - \left(\frac{a_3}{d_3}\right) c_3 = 5 - \frac{1 \times 2}{4.5652} = 4.5619, \quad b_4 = b_4 - \left(\frac{a_3}{d_3}\right) b_3 = 6 - \frac{1 \times 6.5652}{4.5652} = 4.5619$$

# Example
## Backward Substitution

- After the Forward Elimination:

$$\mathbf{d}^T = \begin{bmatrix} 5 & 4.6 & 4.5652 & 4.5619 \end{bmatrix}, \mathbf{b}^T = \begin{bmatrix} 12 & 6.6 & 6.5652 & 4.5619 \end{bmatrix}$$

- Backward Substitution:

$$x_4 = \frac{b_4}{d_4} = \frac{4.5619}{4.5619} = 1,$$

$$x_3 = \frac{b_3 - c_3 x_4}{d_3} = \frac{6.5652 - 2 \times 1}{4.5652} = 1$$

$$x_2 = \frac{b_2 - c_2 x_3}{d_2} = \frac{6.6 - 2 \times 1}{4.6} = 1$$

$$x_1 = \frac{b_1 - c_1 x_2}{d_1} = \frac{12 - 2 \times 1}{5} = 2$$

# LU Decomposition

Method

For most non-singular matrix **A** that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$\mathbf{A} = \mathbf{L}\mathbf{U}$$

where

$\mathbf{L}$ = lower triangular matrix

$\mathbf{U}$ = upper triangular matrix

# How does LU Decomposition work?

If solving a set of linear equations $\mathbf{Ax} = \mathbf{b}$

If $\mathbf{A} = \mathbf{LU}$ then $\mathbf{LUx} = \mathbf{b}$

Multiply by $\mathbf{L}^{-1}$ which gives $\mathbf{L}^{-1}\mathbf{LUx} = \mathbf{L}^{-1}\mathbf{b}$

Remember $\mathbf{L}^{-1}\mathbf{L} = \mathbf{I}$ which leads to $\mathbf{IUx} = \mathbf{L}^{-1}\mathbf{b}$

Now, since $\mathbf{IU} = \mathbf{U}$ then $\mathbf{Ux} = \mathbf{L}^{-1}\mathbf{b}$

Now, let $\mathbf{L}^{-1}\mathbf{b} = \mathbf{z}$

Which ends with $\mathbf{Lz} = \mathbf{b}$ (1)

And $\mathbf{Ux} = \mathbf{z}$ (2)

Thus, given $\mathbf{Ax} = \mathbf{b}$,

1. Decompose $\mathbf{A} = \mathbf{LU}$
2. Solve $\mathbf{Lz} = \mathbf{b}$ for $\mathbf{z}$
3. Solve $\mathbf{Ux} = \mathbf{z}$ for $\mathbf{x}$

# Is LU Decomposition better than Gaussian Elimination?

## Solve **Ax = b**

T = clock cycle time and n×n = size of the matrix

**Forward Elimination**

$$CT|_{FE} = T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right)$$

**Back Substitution**

$$CT|_{BS} = T\left(4n^2 + 12n\right)$$

**Decomposition**

$$CT|_{DE} = T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right)$$

**Forward Substitution**

$$CT|_{FS} = T\left(4n^2 - 4n\right)$$

**Back Substitution**

$$CT|_{BS} = T\left(4n^2 + 12n\right)$$

# Is LU Decomposition better than Gaussian Elimination?

To solve **Ax** = **b**

**Time taken by methods**

| Gaussian Elimination | LU Decomposition |
|---|---|
| $T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$ | $T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$ |

T = clock cycle time and n×n = size of the matrix

So both methods are equally efficient.

# To find inverse of **A**

Time taken by Gaussian Elimination

$$= n\left(CT|_{FE} + CT|_{BS}\right)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$= CT|_{LU} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{32n^3}{3} + 12n^2 + \frac{20n}{3}\right)$$

# To find inverse of **A**

<u>Time taken by Gaussian Elimination</u>

$$T\left(\frac{8n^4}{3}+12n^3+\frac{4n^2}{3}\right)$$

<u>Time taken by LU Decomposition</u>

$$T\left(\frac{32n^3}{3}+12n^2+\frac{20n}{3}\right)$$

**Table 1** Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

| $n$ | 10 | 100 | 1000 | 10000 |
|---|---|---|---|---|
| $CT\|_{\text{inverse GE}} / CT\|_{\text{inverse LU}}$ | 3.28 | 25.83 | 250.8 | 2501 |

# Method: **A** Decomposes to **L** and **U**

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

**U** is the same as the coefficient matrix at the end of the forward elimination step.

**L** is obtained using the *multipliers* that were used in the forward elimination process

# Finding the **U** matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $$\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

# Finding the U Matrix

Matrix after Step 1: $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Step 2: $\dfrac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

$$\mathbf{U} = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

# Finding the **L** matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

# Finding the **L** Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \quad \ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

# Does **LU** = **A**?

$$\mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = ?$$

# General Formula (Doolittle's method)

$$u_{1k} = a_{1k} \qquad k = 1, \ldots, n$$

$$u_{jk} = a_{jk} - \sum_{s=1}^{j-1} \ell_{js} u_{sk} \qquad k = j, \ldots, n;\ j \geq 2$$

$$\ell_{j1} = \frac{a_{j1}}{u_{11}} \qquad j = 2, \ldots, n$$

$$\ell_{jk} = \frac{1}{u_{kk}} \left( a_{jk} - \sum_{s=1}^{k-1} \ell_{js} u_{sk} \right) \qquad j = k+1, \ldots, n;\ k \geq 2$$

# Previous Example Revisited

$$\mathbf{A} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$u_{11} = 25; u_{12} = 5; u_{13} = 1; \ell_{21} = 64/25; \ell_{31} = 144/25;$$

$$u_{22} = 8 - 64 \cdot 5/25 = -4.8; u_{23} = 1 - 64/25 = -1.56;$$

$$\ell_{32} = (12 - 144 \cdot 5/25)/(-4.8) = 3.5$$

$$u_{33} = 1 - 144/25 - 3.5(-1.56) = 0.7$$

# Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the **L** and **U** matrices

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

# Example

Set $\mathbf{Lz} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $\mathbf{z}$

$$z_1 = 106.8$$

$$2.56 z_1 + z_2 = 177.2$$

$$5.76 z_1 + 3.5 z_2 + z_3 = 279.2$$

$$\begin{aligned} z_2 &= 177.2 - 2.56 z_1 \\ &= 177.2 - 2.56(106.8) \\ &= -96.2 \end{aligned}$$

$$\begin{aligned} z_3 &= 279.2 - 5.76 z_1 - 3.5 z_2 \\ &= 279.2 - 5.76(106.8) - 3.5(-96.21) \\ &= 0.735 \end{aligned}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

# Example

Set **Ux = z**

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for **x**

The 3 equations become

$$25x_1 + 5x_2 + x_3 = 106.8$$

$$-4.8x_2 - 1.56x_3 = -96.21$$

$$0.7x_3 = 0.735$$

# Example

From the 3rd equation

$$0.7x_3 = 0.735$$

$$x_3 = \frac{0.735}{0.7}$$

$$x_3 = 1.050$$

Substituting in $x_3$ and using the second equation

$$-4.8x_2 - 1.56x_3 = -96.21$$

$$x_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$x_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$x_2 = 19.70$$

# Example

Substituting in $x_3$ and $x_2$ using the first equation

$$25x_1 + 5x_2 + x_3 = 106.8$$

$$x_1 = \frac{106.8 - 5x_2 - x_3}{25}$$

$$= \frac{106.8 - 5(19.70) - 1.050}{25}$$

$$= 0.2900$$

Hence the Solution Vector is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

# Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Let **B** = $\mathbf{A}^{-1}$ and assume the first column of **B** to be $[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \ldots \; \mathbf{b}_n]^T$

Using this and the definition of matrix multiplication

First column of **B**, $\mathbf{b}_1$

$$\mathbf{A}\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of **B**, $\mathbf{b}_2$

$$\mathbf{A}\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in **B** can be found in the same manner

# Example: Inverse of a Matrix

Find the inverse of a square matrix **A**

$$\mathbf{A} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the **L** and **U** matrices are found to be

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

# Example: Inverse of a Matrix

Solving for the each column of **B** requires two steps

1) Solve **Lz = b** for **z**

2) Solve **Ux** = **z** for **x**

Step 1: $$\mathbf{Lz} = \mathbf{b} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This generates the equations:

$$z_1 = 1$$

$$2.56 z_1 + z_2 = 0$$

$$5.76 z_1 + 3.5 z_2 + z_3 = 0$$

# Example: Inverse of a Matrix

Solving for **z**

$$
\begin{aligned}
z_1 &= 1 \\
z_2 &= 0 - 2.56 z_1 \\
&= 0 - 2.56(1) \\
&= -2.56 \\
z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\
&= 0 - 5.76(1) - 3.5(-2.56) \\
&= 3.2
\end{aligned}
$$

$$
\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}
$$

# Example: Inverse of a Matrix

Solving **U**x = **z** for **x**

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$

$$-4.8b_{21} - 1.56b_{31} = -2.56$$

$$0.7b_{31} = 3.2$$

# Example: Inverse of a Matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

So the first column of the inverse of **A** is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

# Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

**Second Column**

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

**Third Column**

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

# Example: Inverse of a Matrix

The inverse of **A** is

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$\mathbf{AA^{-1} = I = A^{-1}A}$$

# Gauss-Jordan Method

- The method reduces the general system of equations **Ax**=**b** to **Ix**=**b'** where **I** is an identity matrix.
- Only Forward elimination is done and no backward substitution is needed.
- It has the same problems as Naive Gaussian elimination and can be modified to do scaled partial pivoting.
- It takes 50% more time than Naive Gaussian method.
- It is often used to find inverse matrices.

# Gauss-Jordan Method

## Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

*Step* 1 Eleminate $x_1$ from *equations* 2*and* 3

$$\left.\begin{aligned} eq1 &\leftarrow eq1/2 \\ eq2 &\leftarrow eq2 - \left(\frac{4}{1}\right) eq1 \\ eq3 &\leftarrow eq3 - \left(\frac{2}{1}\right) eq1 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

# Gauss-Jordan Method

## Example

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

$Step\ \ 2\ \ \text{Eleminate}\ \ \mathrm{x}_2\ \ \text{from}\ \ equations\ 1\ and\ 3$

$$\left.\begin{aligned} eq2 &\leftarrow eq2/6 \\ eq1 &\leftarrow eq1 - \left(\frac{-1}{1}\right) eq2 \\ eq3 &\leftarrow eq3 - \left(\frac{0}{1}\right) eq2 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

# Gauss-Jordan Method

## Example

$$\begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

*Step* 3 Eleminate $x_3$ from *equations* 1 *and* 2

$$\left.\begin{aligned} eq3 &\leftarrow eq3/2 \\ eq1 &\leftarrow eq1 - \left(\frac{0.1667}{1}\right) eq3 \\ eq2 &\leftarrow eq2 - \left(\frac{-0.8333}{1}\right) eq3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

# Gauss-Jordan Method

## Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

*is transformed to*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow solution\ is \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

# Gauss-Jordan Method

## Finding inverse matrix (1)

Find the inverse matrix of

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix}$$

First construct the augmented matrix:

$$\mathbf{B} = [\mathbf{A} \,|\, \mathbf{I}] = \begin{bmatrix} 2 & -2 & 2 & 1 & 0 & 0 \\ 4 & 2 & -1 & 0 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{bmatrix}$$

# Gauss-Jordan Method

## Finding inverse matrix (2)

Applying the Gauss-Jordan method:

$$\begin{bmatrix} 2 & -2 & 2 & 1 & 0 & 0 \\ 4 & 2 & -1 & 0 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1/2 & 0 & 0 \\ 4 & 2 & -1 & 0 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 1/2 & 0 & 0 \\ 0 & 6 & -5 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & -5/6 & -1/3 & 1/6 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1/6 & 1/6 & 1/6 & 0 \\ 0 & 1 & -5/6 & -1/3 & 1/6 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/6 & 1/6 & 1/6 & 0 \\ 0 & 1 & -5/6 & -1/3 & 1/6 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{bmatrix}$$

# Gauss-Jordan Method

## Finding inverse matrix (3)

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1/4 & 1/6 & -1/12 \\ 0 & 1 & 0 & -3/4 & 1/6 & 5/12 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{bmatrix}$$

Therefore,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/4 & 1/6 & -1/12 \\ -3/4 & 1/6 & 5/12 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

# QR Decomposition (or Factorization)

Method

For most non-singular matrix **A**, one can factorize it as $\mathbf{A} = \mathbf{QR}$

where

$\mathbf{Q}$ = orthonormal matrix, i.e., $\mathbf{QQ}^T=\mathbf{Q}^T\mathbf{Q}=\mathbf{I}$

$\mathbf{R}$ = upper triangular matrix

NOTE: Given set of **orthonormal** vectors, $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$, then

$$\underline{\mathbf{a}}_i^T \underline{\mathbf{a}}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & else \end{cases}$$

# How can QR Decomposition be used?

To solve a set of linear equations,

If $\mathbf{A} = \mathbf{QR}$ then

$$\mathbf{Ax} = \mathbf{QRx} = \mathbf{b} \rightarrow \mathbf{Rx} = \mathbf{Q}^T\mathbf{b} = \mathbf{z}$$

Thus, given $\mathbf{Ax} = \mathbf{b}$,

1. Factorize $\mathbf{A} = \mathbf{QR}$
2. Find $\mathbf{z} = \mathbf{Q}^T\mathbf{b}$
3. Solve $\mathbf{Rx} = \mathbf{z}$ for $\mathbf{x}$

# Computation Methods

- Gram-Schmidt orthogonalization
  - Straightforward, but numerically unstable
- Using Householder reflection
  - More stable, but parallel computing (parallerization) is not available
- Using Givens rotation
  - Enable parallel computing

# Gram-Schmidt Orthogonalization

Method

Gram-Schmidt orthogonalization process is a way to find a set of orthonormal bases (or vectors) for column space of **A**.

Let

$$\mathbf{A} = \begin{bmatrix} \underline{\mathbf{a}}_1 & \underline{\mathbf{a}}_2 & \cdots & \underline{\mathbf{a}}_n \end{bmatrix}$$

where

$\underline{\mathbf{a}}_k$ = $k^{th}$ column of **A**

# G-S Process Algorithm

Let

$$\mathbf{Q} = \begin{bmatrix} \underline{\mathbf{q}}_1 & \underline{\mathbf{q}}_2 & \cdots & \underline{\mathbf{q}}_n \end{bmatrix}$$

where

$\underline{\mathbf{q}}_k$ = $k^{th}$ column of **Q**

1. Normalize $\underline{\mathbf{a}}_1$ : $\underline{\mathbf{q}}_1 = \underline{\mathbf{a}}_1/||\underline{\mathbf{a}}_1||$; $R_{11} = ||\underline{\mathbf{a}}_1||$
2. For k = 2:n

   a) Compute

$$\underline{\mathbf{q}}_k = \underline{\mathbf{a}}_k - \sum_{m=1}^{k-1} \left(\underline{\mathbf{a}}_k{}^T \underline{\mathbf{q}}_m\right)\underline{\mathbf{q}}_m;$$

$$R_{mk} = \underline{\mathbf{a}}_k{}^T \underline{\mathbf{q}}_m$$

Projection of $\underline{a}_k$ onto $\underline{q}_m$

   b) Normalize $\underline{\mathbf{q}}_k = \underline{\mathbf{q}}_k/||\underline{\mathbf{q}}_k||$; $R_{kk} = ||\underline{\mathbf{q}}_k||$.

# QR using G-S example

- Consider

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

- Normalizing $\underline{\mathbf{a}}_1$ yields $[4\ 1\ -1]^T / \sqrt{18}$

$$\underset{\mathbf{Q}}{\begin{bmatrix} 0.9428 & * & * \\ 0.2357 & * & * \\ -0.2357 & * & * \end{bmatrix}} \underset{\mathbf{R}}{\begin{bmatrix} 4.2426 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}}$$

# QR using G-S example (2)

- Calculate $\underline{\mathbf{a}}_2^T\underline{\mathbf{q}}_1=1.4142$;

$\underline{\mathbf{q}}_2=\underline{\mathbf{a}}_2-(\underline{\mathbf{a}}_2^T\underline{\mathbf{q}}_1)\underline{\mathbf{q}}_1=[-0.3333\ 2.6667\ 1.3333]^T$

Normalize $\underline{\mathbf{q}}_2=\underline{\mathbf{q}}_2/||\underline{\mathbf{q}}_2||$; $||\underline{\mathbf{q}}_2||=3$

$$\underbrace{\begin{bmatrix} 0.9428 & -0.1111 & * \\ 0.2357 & 0.8889 & * \\ -0.2357 & 0.4444 & * \end{bmatrix}}_{\mathbf{Q}}\underbrace{\begin{bmatrix} 4.2426 & 1.4142 & * \\ 0 & 3 & * \\ 0 & 0 & * \end{bmatrix}}_{\mathbf{R}}$$

# QR using G-S example (3)

- Calculate $\underline{\mathbf{a}}_3^T\underline{\mathbf{q}}_1$=-1.6499; $\underline{\mathbf{a}}_3^T\underline{\mathbf{q}}_2$=2.7778

$\underline{\mathbf{q}}_3=\underline{\mathbf{a}}_3-(\underline{\mathbf{a}}_3^T\underline{\mathbf{q}}_1)\underline{\mathbf{q}}_1-(\underline{\mathbf{a}}_3^T\underline{\mathbf{q}}_2)\underline{\mathbf{q}}_2=[0.8642\ -1.0802\ 2.3765]^T$

Normalize $\underline{\mathbf{q}}_3=\underline{\mathbf{q}}_3/||\underline{\mathbf{q}}_3||$; $||\underline{\mathbf{q}}_3||$= 2.7499

$$\underbrace{\begin{bmatrix} 0.9428 & -0.1111 & 0.3143 \\ 0.2357 & 0.8889 & -0.3928 \\ -0.2357 & 0.4444 & 0.8642 \end{bmatrix}}_{\mathbf{Q}}\underbrace{\begin{bmatrix} 4.2426 & 1.4142 & -1.6499 \\ 0 & 3 & 2.7778 \\ 0 & 0 & 2.7499 \end{bmatrix}}_{\mathbf{R}}$$

# Using QR to solve SLE

- Consider

$$\mathbf{Ax} = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{z} = \underbrace{\begin{bmatrix} 0.9428 & 0.2357 & -0.2357 \\ -0.1111 & 0.8889 & 0.4444 \\ 0.3143 & -0.3928 & 0.8642 \end{bmatrix}}_{\mathbf{Q}^T} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{b}} = \begin{bmatrix} 0.7071 \\ 3.0000 \\ 2.1213 \end{bmatrix}$$

$$\begin{bmatrix} 4.2426 & 1.4142 & -1.6499 \\ 0 & 3 & 2.7778 \\ 0 & 0 & 2.7499 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 3.0000 \\ 2.1213 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3714 \\ 0.2857 \\ 0.7714 \end{bmatrix}$$

# Gauss-Seidel Method

- Gauss-Seidel is the simplest *iterative* method for solving matrix equation.
- Basic Procedure:
  - Algebraically solve each linear equation for $x_i$
  - Assume an initial guess solution array
  - Solve for each $x_i$ and repeat
- Use absolute approximate error after each iteration to check if error is within a pre-specified tolerance.

# Advantages

- If only few iterations are required, computational cost is lower compared to Gauss elimination or LU decomposition.
- If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

However, if the matrix is "ill-conditioned", the solution might not converge.

# Algorithm

Input : matrix **A**, vector **b**, initial guess **x0**, tolerance

1. Update $x_j$, j=1,…,n using

$$x_j^{(m+1)} = \frac{1}{A_{jj}}\left( b_j - \sum_{k=1}^{j-1} A_{jk} x_k^{(m+1)} - \sum_{k=j+1}^{n} A_{jk} x_k^{(m)} \right)$$

2. Check if the convergence criterion is met.

$$\max_j \left| x_j^{(m+1)} - x_j^{(m)} \right| < \varepsilon$$

Repeat until convergence.

# Example

- Solve $\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2.5 \\ 7 \end{bmatrix}; \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

1st iteration: **x**=[1.0000 -0.3714 0.6626]

2nd iteration: **x**=[1.0318 -0.3435 0.6622]

3rd iteration: **x**=[1.0327 -0.3435 0.6621]

4th iteration: **x**=[1.0327 -0.3435 0.6621]

NOTE: Only few iterations are required for *diagonally dominant* matrices (well-conditioned, or small condition number).

# Example (2)

$$\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}; \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

1st: $\mathbf{x}$=[0.5000 2.1364 -0.8864 0.9631]
2nd: $\mathbf{x}$=[0.9909 2.0196 -0.9999 0.9927]
3rd: $\mathbf{x}$=[1.0019 2.0022 -1.0009 0.9991]
4th: $\mathbf{x}$=[1.0004 2.0002 -1.0002 0.9999]