Lecture 9
Ordinary Differential Equations
Part I

- ODEs
- Taylor series methods and Euler method
- Midpoint and Heun's method
- Runge-Kutta methods

Derivatives

Derivatives

Ordinary Derivatives

$$\frac{dv}{dt}$$

v is a function of one independent variable

Partial Derivatives

$$\frac{\partial u}{\partial y}$$

u is a function of more than one independent variable

Differential Equations

Differential Equations

Ordinary Differential Equations

$$\frac{d^2 v}{dt^2} + 6tv = 1$$

involve one or more

<u>Ordinary derivatives</u> of

unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more

<u>partial derivatives</u> of

unknown functions

Ordinary Differential Equations

Ordinary Differential Equations (ODEs) involve one or more ordinary derivatives of unknown functions with respect to one independent variable

$Examples:$

$$\frac{dv(t)}{dt} - v(t) = e^t$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

x(t): unknown function

t: independent variable

Example of ODE:

Model of Falling Parachutist

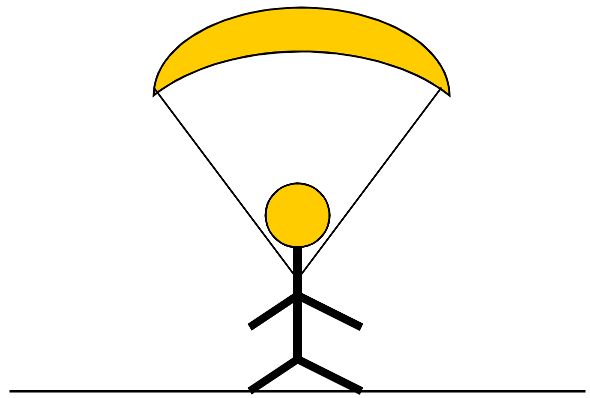
The velocity of a falling parachutist is given by:

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

M : *mass*

c : *drag coefficient*

v : *velocity*



Definitions

$$\frac{dv}{dt} = 9.8 - \frac{c}{M}v$$

Ordinary differential equation

(Dependent variable) unknown function to be determined

(independent variable) the variable with respect to which other variables are differentiated

Order of a Differential Equation

The **order** of an ordinary differential equation is the order of the highest order derivative.

$$Examples:$$

$$\frac{dx(t)}{dt} - x(t) = e^t$$

First order ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

Second order ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1$$

Second order ODE

Solution of a Differential Equation

A **solution** to a differential equation is a function that satisfies the equation.

$$Example:$$

$$\frac{dx(t)}{dt} + x(t) = 0$$

$$Solution \quad x(t) = e^{-t}$$

$$\underline{\text{Proof :}}$$

$$\frac{dx(t)}{dt} = -e^{-t}$$

$$\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

Linear ODE

An ODE is linear if

The unknown function and its derivatives appear to power one

No product of the unknown function and/or its derivatives

Examples:

$$\frac{dx(t)}{dt} - x(t) = e^t$$

Linear ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2x(t) = \cos(t)$$

Linear ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1$$

Non-linear ODE

Nonlinear ODE

An ODE is linear if

The unknown function and its derivatives appear to power one

No product of the unknown function and/or its derivatives

Examples of nonlinear ODE:

$$\frac{dx(t)}{dt} - \cos(x(t)) = 1$$

$$\frac{d^2x(t)}{dt^2} - 5\,\frac{dx(t)}{dt}x(t) = 2$$

$$\frac{d^2x(t)}{dt^2} - \left|\frac{dx(t)}{dt}\right| + x(t) = 1$$

Solutions of Ordinary Differential Equations

$$x(t) = \cos(2t)$$

is a solution to the ODE

$$\frac{d^2x(t)}{dt^2} + 4x(t) = 0$$

Is it unique?

All functions of the form $x(t) = \cos(2t + c)$ (where c is a real constant) are solutions.

Uniqueness of a Solution

In order to uniquely specify a solution to an n^{th} order differential equation we need n conditions.

$$\frac{d^2x(t)}{dt^2} + 4x(t) = 0$$

Second order ODE

$$x(0) = a$$

$$\dot{x}(0) = b$$

Two conditions are needed to uniquely specify the solution

Auxiliary Conditions

Auxiliary Conditions

Initial Conditions

- All conditions are at one point of the independent variable

Boundary Conditions

- The conditions are not at one point of the independent variable

Boundary-Value and Initial value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

Analytic and Numerical Solutions

- Analytical Solutions to ODEs are available for linear ODEs and special classes of nonlinear differential equations.
- Numerical methods are used to obtain a graph or a table of the unknown function.
- Most of the Numerical methods used to solve ODEs are based directly (or indirectly) on the truncated Taylor series expansion.

Classification of the Methods

Numerical Methods for Solving ODE

Single-Step Methods

Estimates of the solution at a particular step are entirely based on information on the previous step

Multiple-Step Methods

Estimates of the solution at a particular step are based on information on more than one step

Taylor Series Method

The problem to be solved is a first order ODE:

$$\frac{dy(x)}{dx} = f(x, y), \quad y(x_0) = y_0$$

Estimates of the solution at different base points:

$$y(x_0 + h),\ y(x_0 + 2h),\ y(x_0 + 3h),\ ....$$

are computed using the truncated Taylor series expansions.

Taylor Series Expansion

Truncated Taylor Series Expansion

$$y(x_0+h) \approx \sum_{k=0}^{n} \frac{h^k}{k!}\left(\left.\frac{d^k y}{dx^k}\right|_{x=x_0,\, y=y_0} \right)$$

$$\approx y(x_0) + h \left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}} + \frac{h^2}{2!} \left.\frac{d^2 y}{dx^2}\right|_{\substack{x=x_0,\\ y=y_0}} + \ldots + \frac{h^n}{n!} \left.\frac{d^n y}{dx^n}\right|_{\substack{x=x_0,\\ y=y_0}}$$

The n^{th} order Taylor series method uses the n^{th} order Truncated Taylor series expansion.

Euler Method

- First order Taylor series method is known as *Euler Method*.
- Only the constant term and linear term are used in the Euler method.
- The error due to the use of the truncated Taylor series is of order $O(h^2)$.

First Order Taylor Series Method

(Euler Method)

$$y(x_0 + h) = y(x_0) + h \left. \frac{dy}{dx} \right|_{\substack{x=x_0,\\ y=y_0}} + O(h^2)$$

$$Notation:$$

$$x_n = x_0 + nh, \qquad y_n = y(x_n),$$

$$\left. \frac{dy}{dx} \right|_{\substack{x=x_i,\\ y=y_i}} = f(x_i, y_i)$$

$$Euler\ Method$$

$$y_{i+1} = y_i + h\ f(x_i, y_i)$$

Euler Method

Problem:

Given the first order ODE: $\dot{y}(x) = f(x, y)$

with the initial condition: $y_0 = y(x_0)$

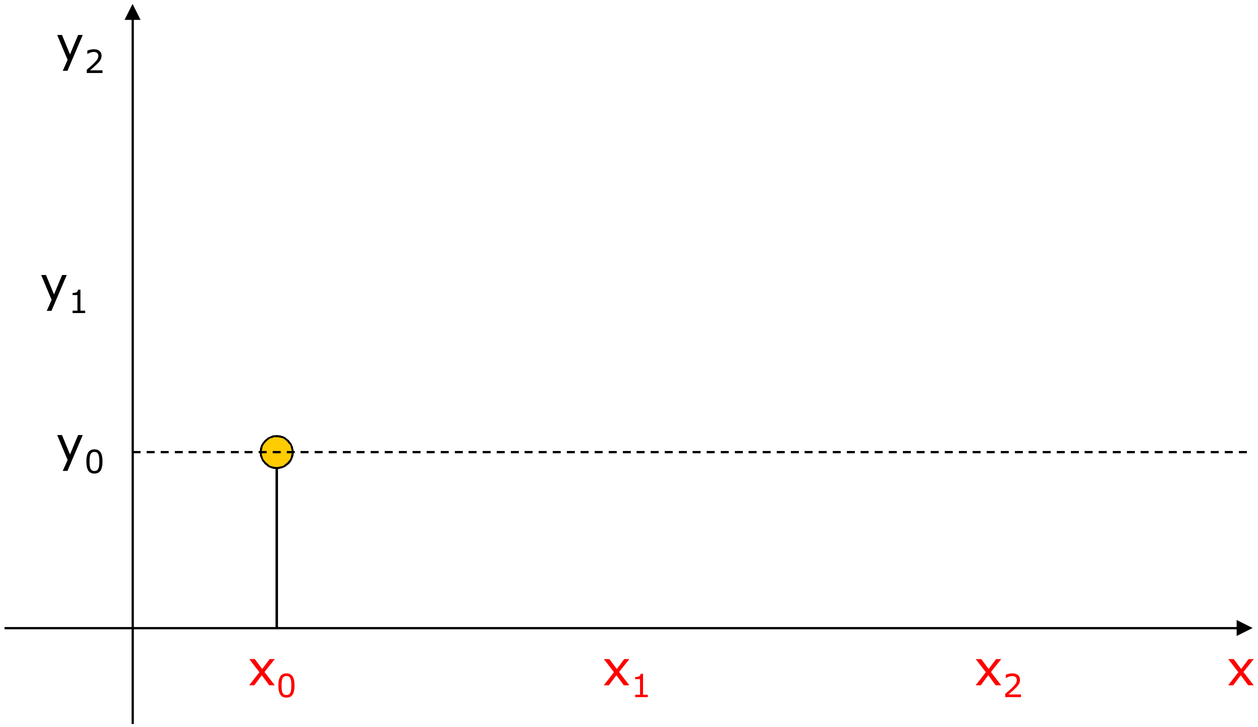
Determine: $y_i = y(x_0 + ih) \quad for\ i = 1,2,\ldots$

Euler Method:

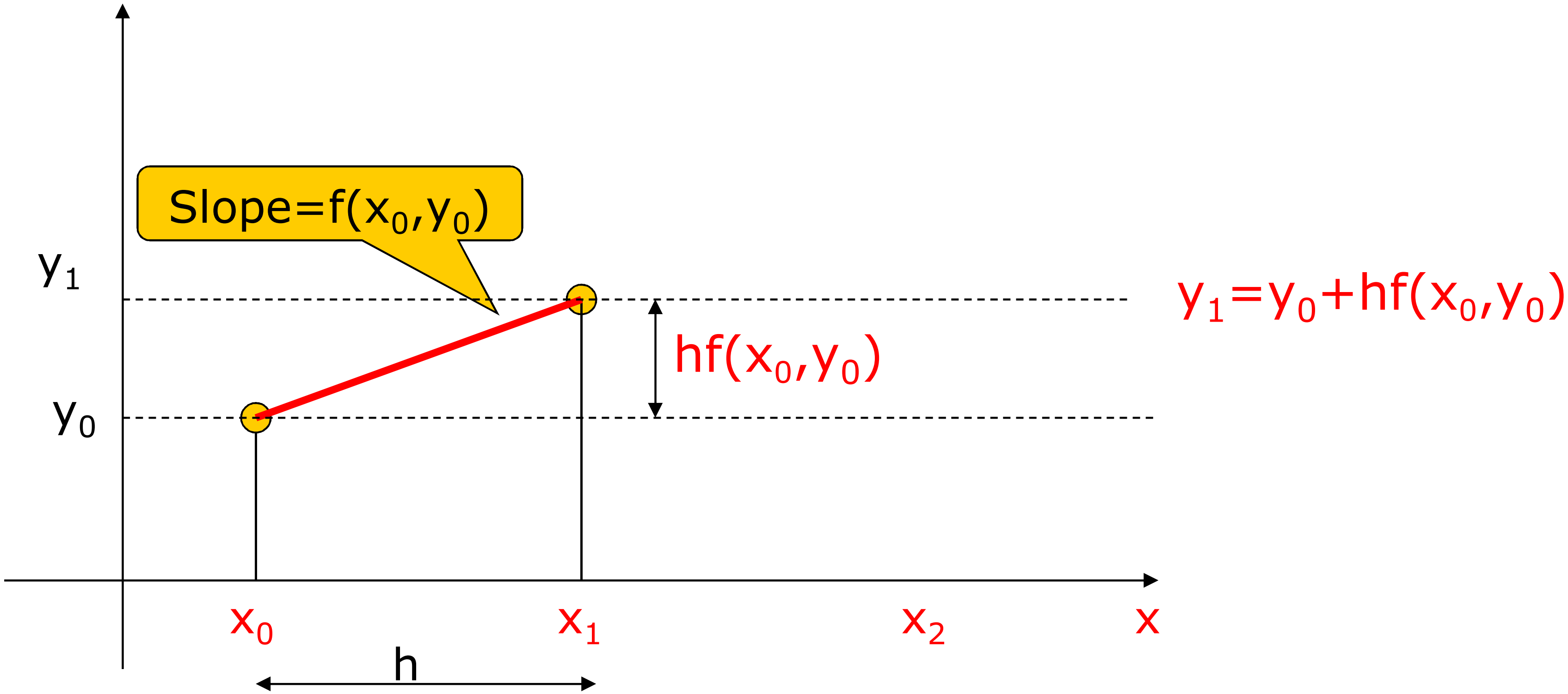
$$y_0 = y(x_0)$$

$$y_{i+1} = y_i + h\ f(x_i, y_i) \quad for\ i = 1,2,\ldots$$

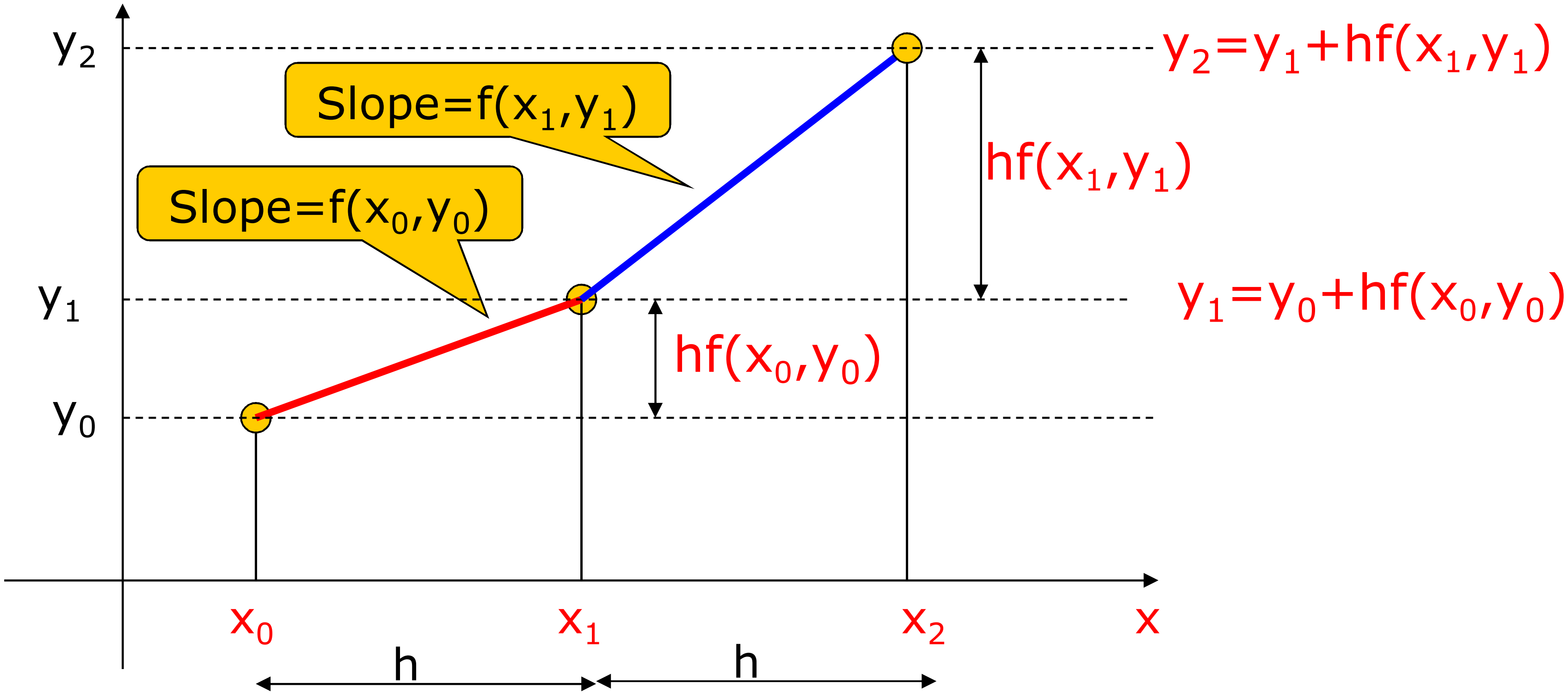
Interpretation of Euler Method



Interpretation of Euler Method



Interpretation of Euler Method



Example 1

Use Euler method to solve the ODE:

$$\frac{dy}{dx} = 1 + x^2 , \qquad y(1) = -4$$

to determine y(1.01), y(1.02) and y(1.03).

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$$

Euler Method

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

$$Step1: \quad y_1 = y_0 + h\, f(x_0, y_0) = -4 + 0.01(1 + (1)^2) = -3.98$$

$$Step2: \quad y_2 = y_1 + h\, f(x_1, y_1) = -3.98 + 0.01\left(1 + (1.01)^2\right) = -3.9598$$

$$Step3: \quad y_3 = y_2 + h\, f(x_2, y_2) = -3.9598 + 0.01\left(1 + (1.02)^2\right) = -3.9394$$

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$$

Summary of the result:

i	xi	yi
0	1.00	-4.00
1	1.01	-3.98
2	1.02	-3.9595
3	1.03	-3.9394

Example 1

$$f(x,y) = 1 + x^2 , \quad x_0 = 1, \quad y_0 = -4 , \quad h = 0.01$$

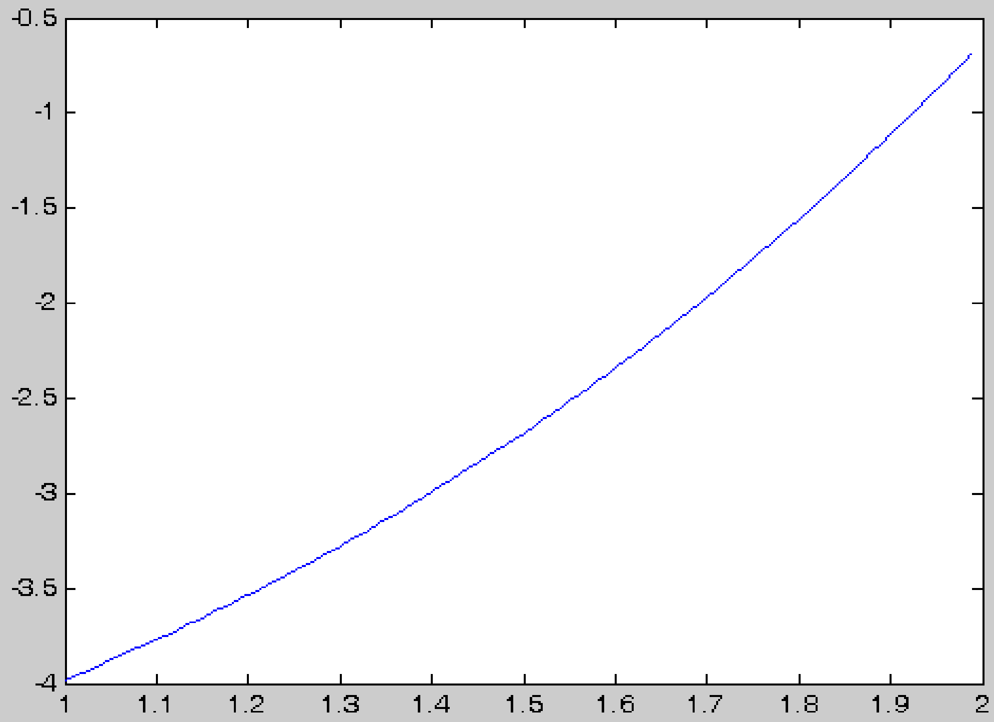
Comparison with true value:

i	x_i	y_i	True value of y_i
0	1.00	-4.00	-4.00
1	1.01	-3.98	-3.97990
2	1.02	-3.9595	-3.95959
3	1.03	-3.9394	-3.93909

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$$

A graph of the solution of the ODE for 1<x<2



Types of Errors

- Local truncation error:

  Error due to the use of truncated Taylor series to compute x(t+h) in one step.

- Global Truncation error:

  Accumulated truncation over many steps.

- Round off error:

  Error due to finite number of bits used in representation of numbers. This error could be accumulated and magnified in succeeding steps.

Second Order Taylor Series Methods

$$\text{Given} \quad \frac{dy(x)}{dx} = f(y, x), \quad y(x_0) = y_0$$

Second order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2 y}{dx^2} + O(h^3)$$

$\dfrac{d^2 y}{dx^2}$ needs to be derived analytically.

Third Order Taylor Series Methods

$$Given \quad \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0$$

Third order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \frac{h^3}{3!}\frac{d^3y}{dx^3} + O(h^4)$$

$\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ need to be derived analytically.

High Order Taylor Series Methods

$$Given \quad \frac{dy(x)}{dx} = f(y, x), \quad y(x_0) = y_0$$

n[th] order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + ... + \frac{h^n}{n!}\frac{d^ny}{dx^n} + O(h^{n+1})$$

$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, ..., \frac{d^ny}{dx^n}$ need to be derived analytically.

Higher Order Taylor Series Methods

- High order Taylor series methods are more accurate than Euler method.

- But, the 2nd, 3rd, and higher order derivatives need to be derived analytically which may not be easy.

Example 2

Second order Taylor Series Method

Use Second order Taylor Series method to solve:

$$\frac{dx}{dt} + 2x^2 + t = 1, \; x(0) = 1, \qquad use \quad h = 0.01$$

What is: $\dfrac{d^2x(t)}{dt^2}$?

Example 2

Use Second order Taylor Series method to solve:

$$\frac{dx}{dt} + 2x^2 + t = 1, \ x(0) = 1, \qquad use \quad h = 0.01$$

$$\frac{dx}{dt} = 1 - 2x^2 - t$$

$$\frac{d^2x(t)}{dt^2} = 0 - 4x\frac{dx}{dt} - 1 = -4x(1 - 2x^2 - t) - 1$$

$$x_{i+1} = x_i + h(1 - 2x_i^{\ 2} - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^{\ 2} - t_i))$$

Example 2

$$f(t,x) = 1 - 2x^2 - t, \quad t_0 = 0, \; x_0 = 1, \; h = 0.01$$

$$x_{i+1} = x_i + h(1 - 2x_i^{\,2} - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^{\,2} - t_i))$$

$Step\,1:$

$$x_1 = 1 + 0.01(1 - 2(1)^2 - 0) + \frac{(0.01)^2}{2}(-1 - 4(1)(1 - 2 - 0)) = 0.9901$$

$Step\,2:$

$$x_2 = 0.9901 + 0.01(1 - 2(0.9901)^2 - 0.01) + \frac{(0.01)^2}{2}(-1 - 4(0.9901)(1 - 2(0.9901)^2 - 0.01)) = 0.9807$$

$Step\,3:$

$x_3 = 0.9716$

Example 2

$$f(t,x) = 1 - 2x^2 - t, \quad t_0 = 0, \; x_0 = 1, \; h = 0.01$$

Summary of the results:

i	t_i	x_i
0	0.00	1
1	0.01	0.9901
2	0.02	0.9807
3	0.03	0.9716

Programming Euler Method

Write a MATLAB program to implement Euler method to solve:

$$\frac{dv}{dt} = 1 - 2v^2 - t. \qquad v(0) = 1$$

$$for \quad t_i = 0.01i, \qquad i = 1,2,...,100$$

Programming Euler Method

```
f=inline('1-2*v^2-t','t','v')
h=0.01
t=0
v=1
T(1)=t;
V(1)=v;
for k=1:100
    v=v+h*f(t,v)
    t=t+h;
    T(k+1)=t;
    V(k+1)=v;
end
```

Programming Euler Method

```
f=inline('1-2*v^2-t','t','v')
h=0.01
t=0
v=1
T(1)=t;
V(1)=v;
for k=1:100
    v=v+h*f(t,v)
    t=t+h;
    T(k+1)=t;
    V(k+1)=v;
end
```

Definition of the ODE

Initial condition

Main loop

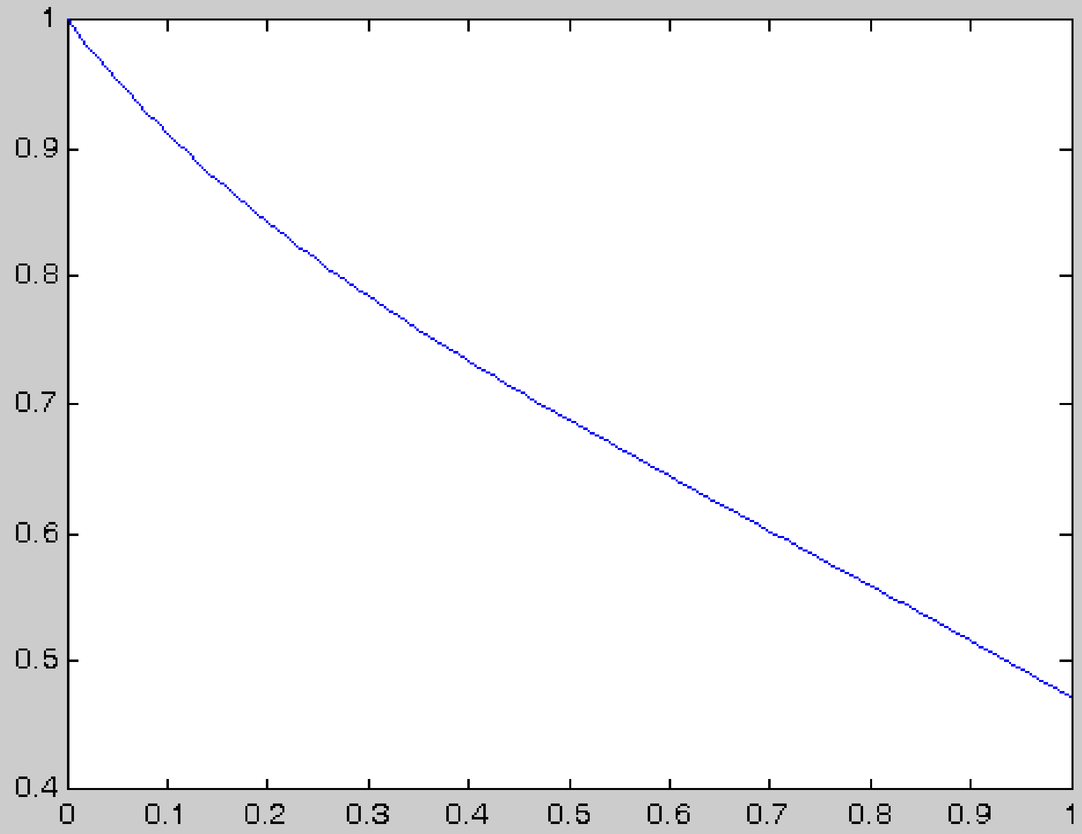
Euler method

Storing information

Programming Euler Method

Plot of the solution

plot(T,V)



Euler Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Euler Method

$$y_0 = y(x_0)$$

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

$$for\ i = 1,2,\ldots$$

Local Truncation Error $O(h^2)$

Global Truncation Error $O(h)$

Midpoint Method Introduction

Problem to be solved is a first order ODE:

$$\dot{y}(x) = f(x, y), \qquad y(x_0) = y_0$$

- The methods proposed in this lesson have the general form:

$$y_{i+1} = y_i + h\,\phi$$

- For the case of Euler: $\phi = f(x_i, y_i)$
- Different forms of ϕ will be used for the Midpoint and Heun's Methods.

Midpoint Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Midpoint Method

$$y_0 = y(x_0)$$

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i)$$

$$y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

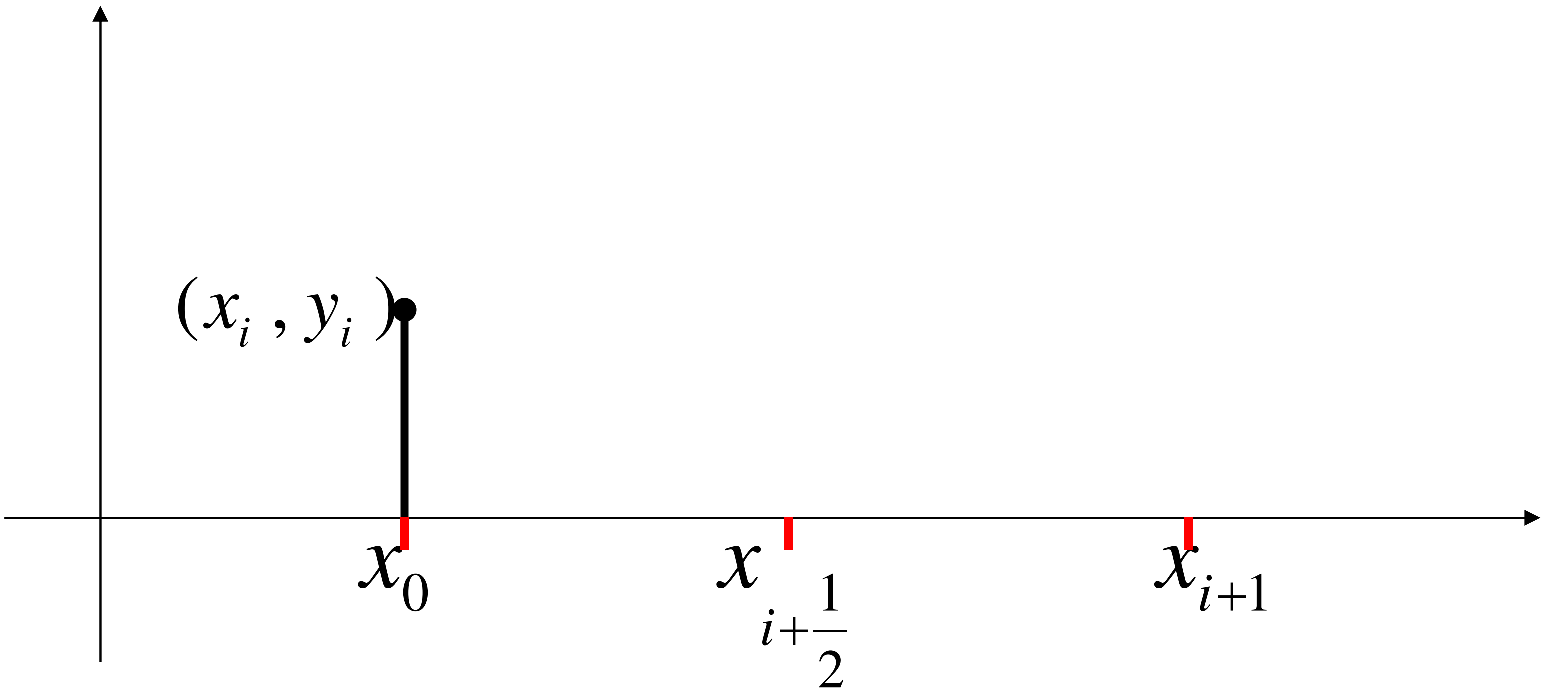
Local Truncation Error $O(h^3)$

Global Truncation Error $O(h^2)$

Motivation

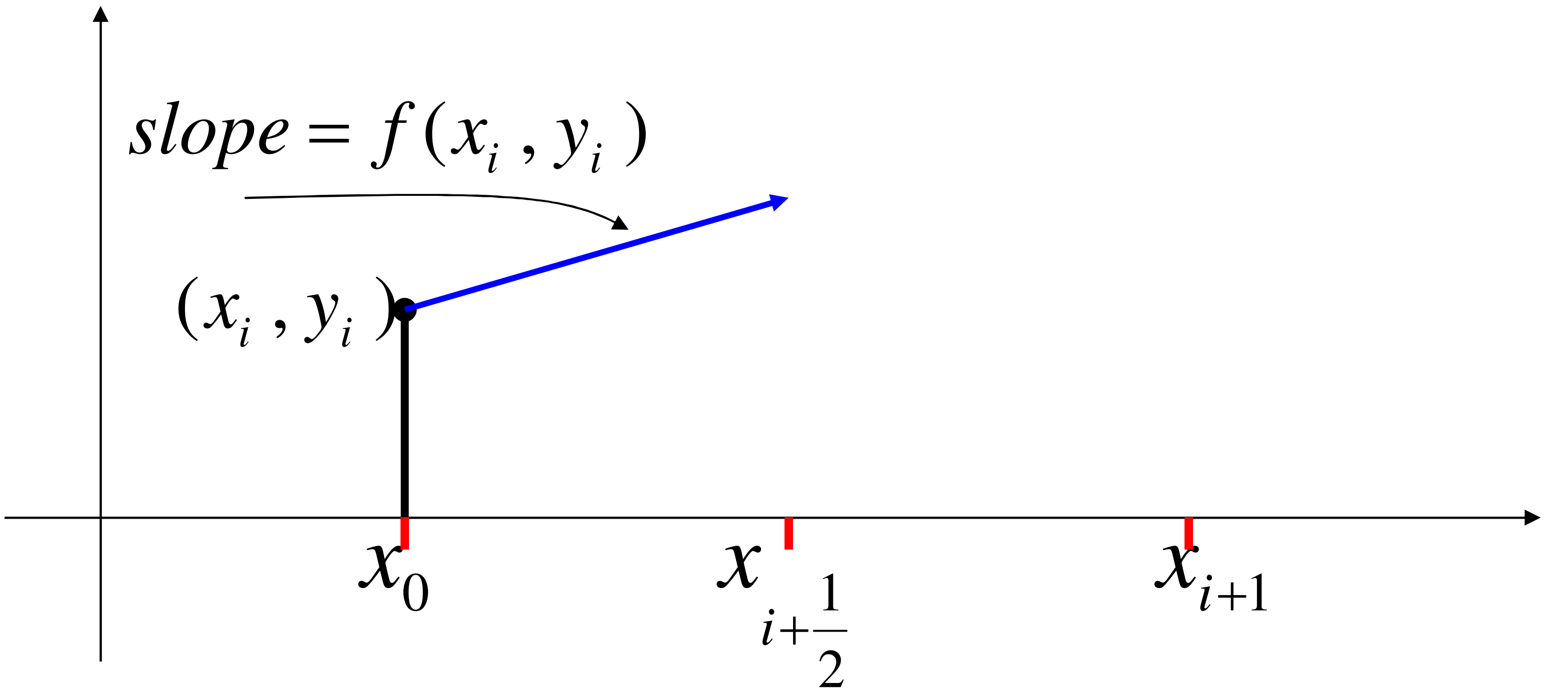
- The midpoint can be summarized as:
  - Euler method is used to estimate the solution at the midpoint.
  - The value of the rate function $f(x,y)$ at the mid point is calculated.
  - This value is used to estimate y_{i+1}.
- Local Truncation error of order $O(h^3)$.
- Comparable to Second order Taylor series method.

Midpoint Method



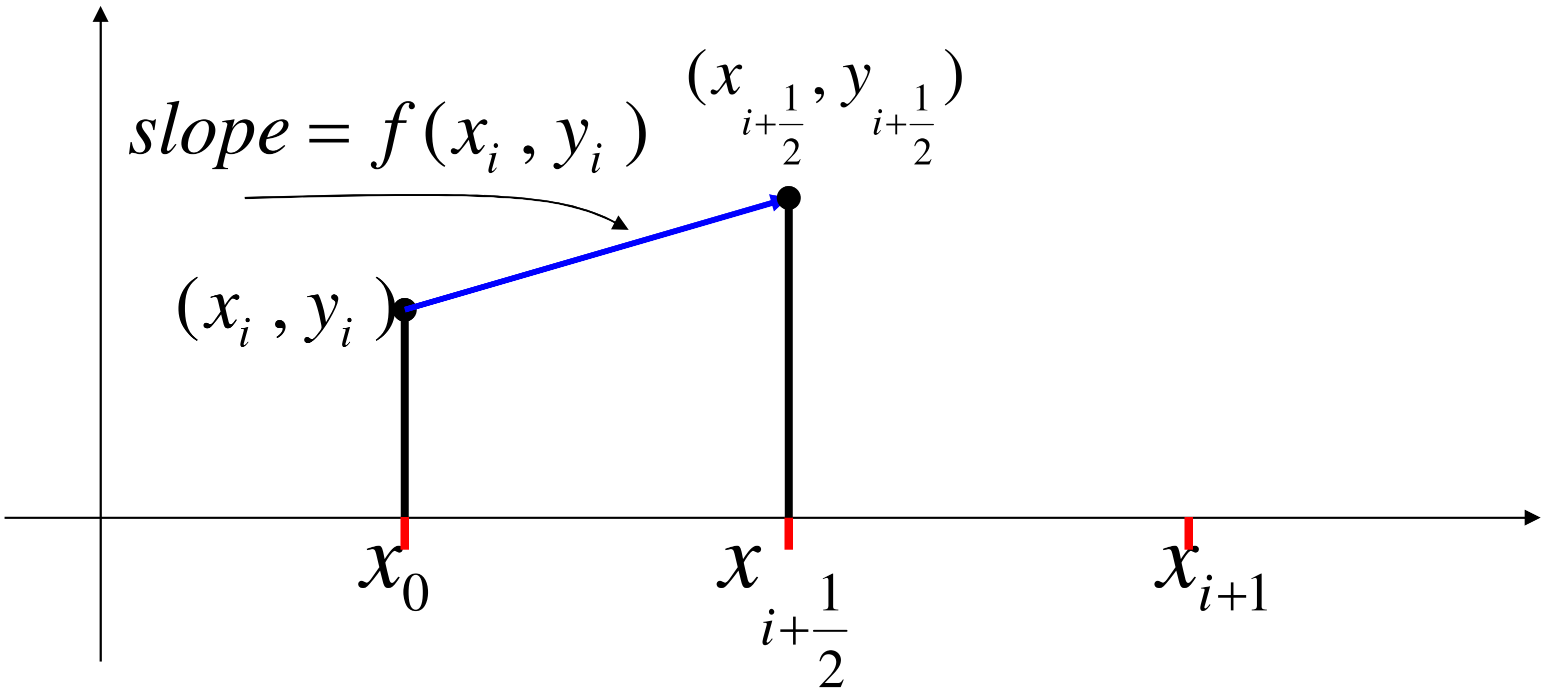
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint Method



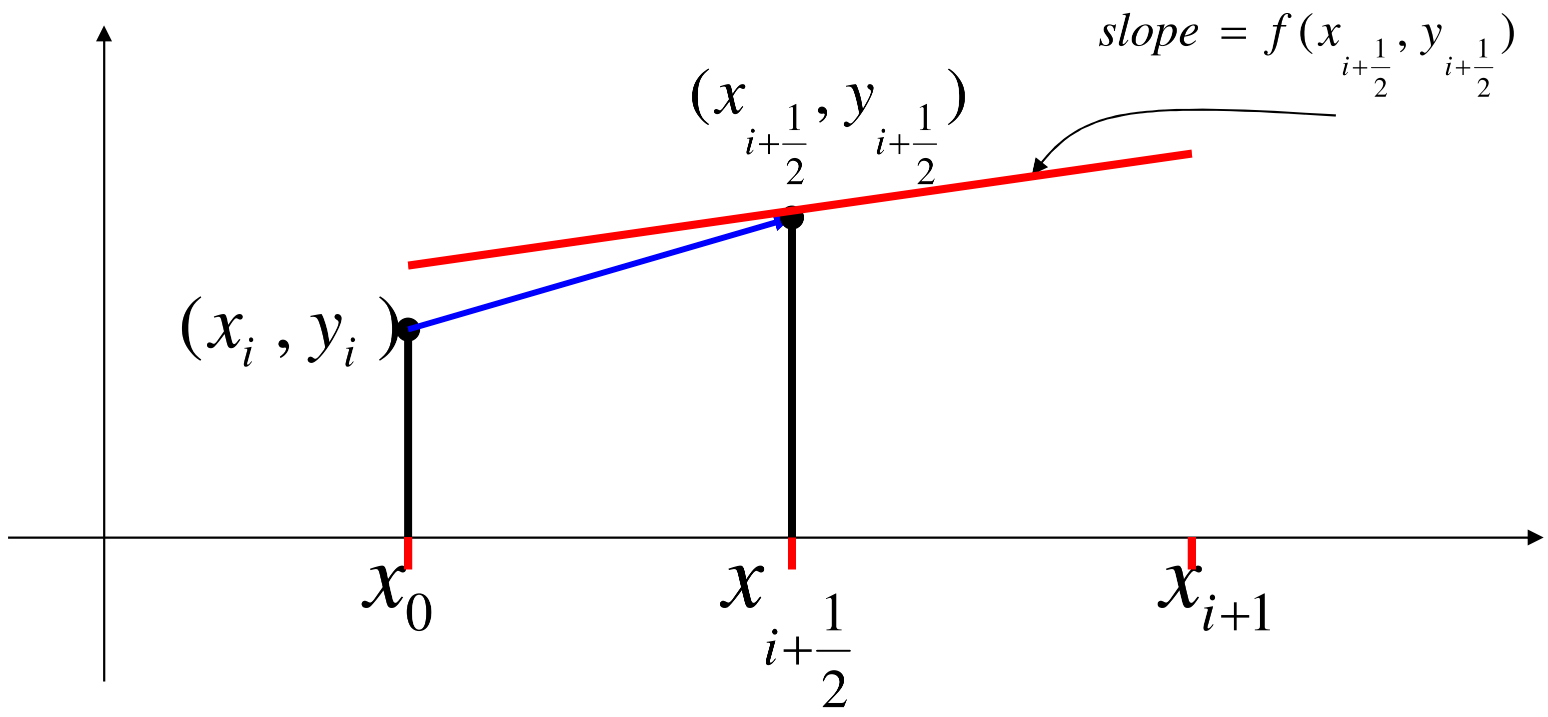
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint Method



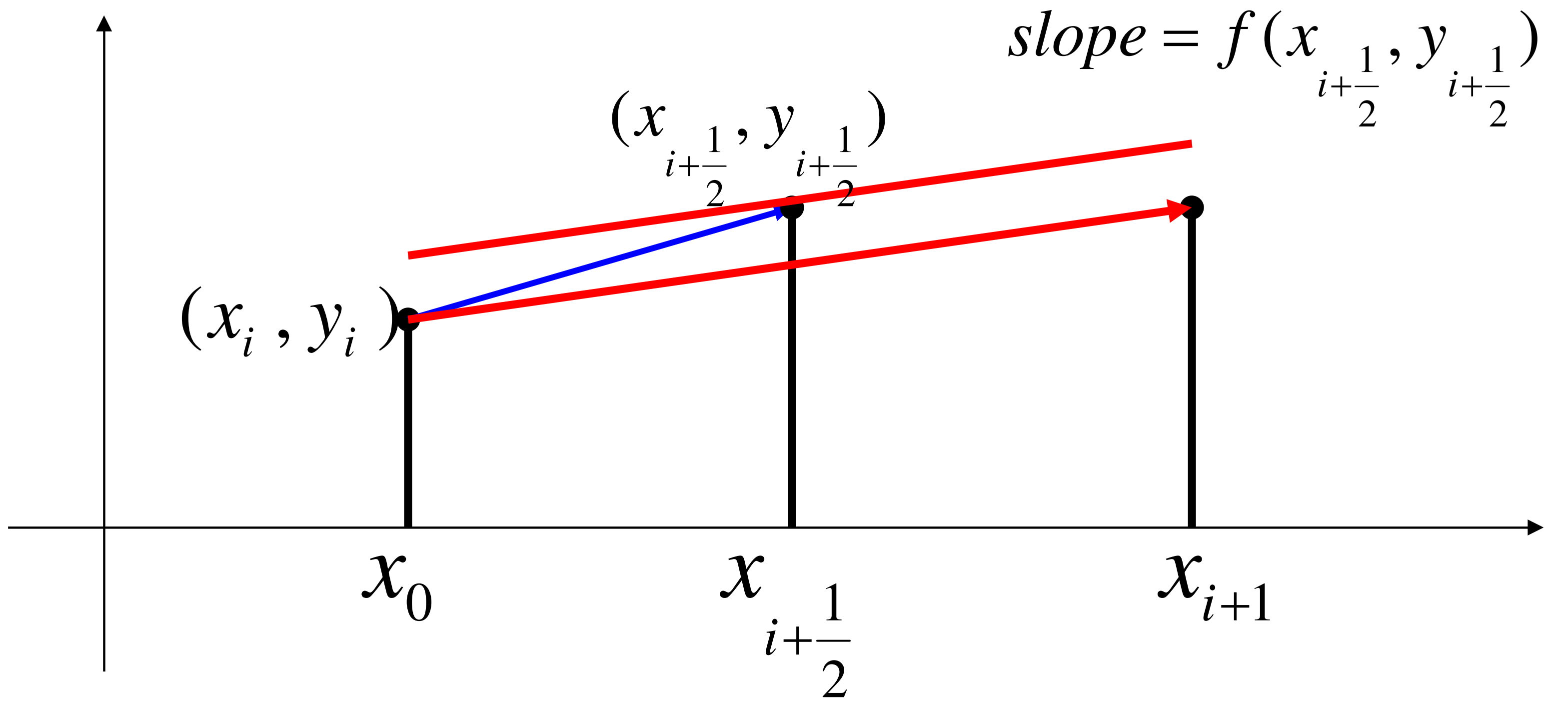
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Example 1

Use the Midpoint Method to solve the ODE

$$\dot{y}(x) = 1 + x^2 + y$$

$$y(0) = 1$$

Use $h = 0.1$. Determine y(0.1) and y(0.2)

Example 1

$$\text{Problem}: f(x,y) = 1 + x^2 + y,\ y_0 = y(0) = 1, h = 0.1$$

$$\text{Step 1}:$$

$$y_{0+\frac{1}{2}} = y_0 + \frac{h}{2} f(x_0, y_0) = 1 + 0.05(1 + 0 + 1) = 1.1$$

$$y_1 = y_0 + h\ f(x_{0+\frac{1}{2}}, y_{0+\frac{1}{2}}) = 1 + 0.1(1 + 0.0025 + 1.1) = 1.2103$$

$$\text{Step 2}:$$

$$y_{1+\frac{1}{2}} = y_1 + \frac{h}{2} f(x_1, y_1) = 1.2103 + .05(1 + 0.01 + 1.2103) = 1.3213$$

$$y_2 = y_1 + h\ f(x_{1+\frac{1}{2}}, y_{1+\frac{1}{2}}) = 1.2103 + 0.1(2.3438) = 1.4446$$

Heun's Predictor Corrector Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Heun's Method

$$y_0 = y(x_0)$$

$$\text{Predictor}: \quad y_{i+1}^0 = y_i + h\ f(x_i, y_i)$$

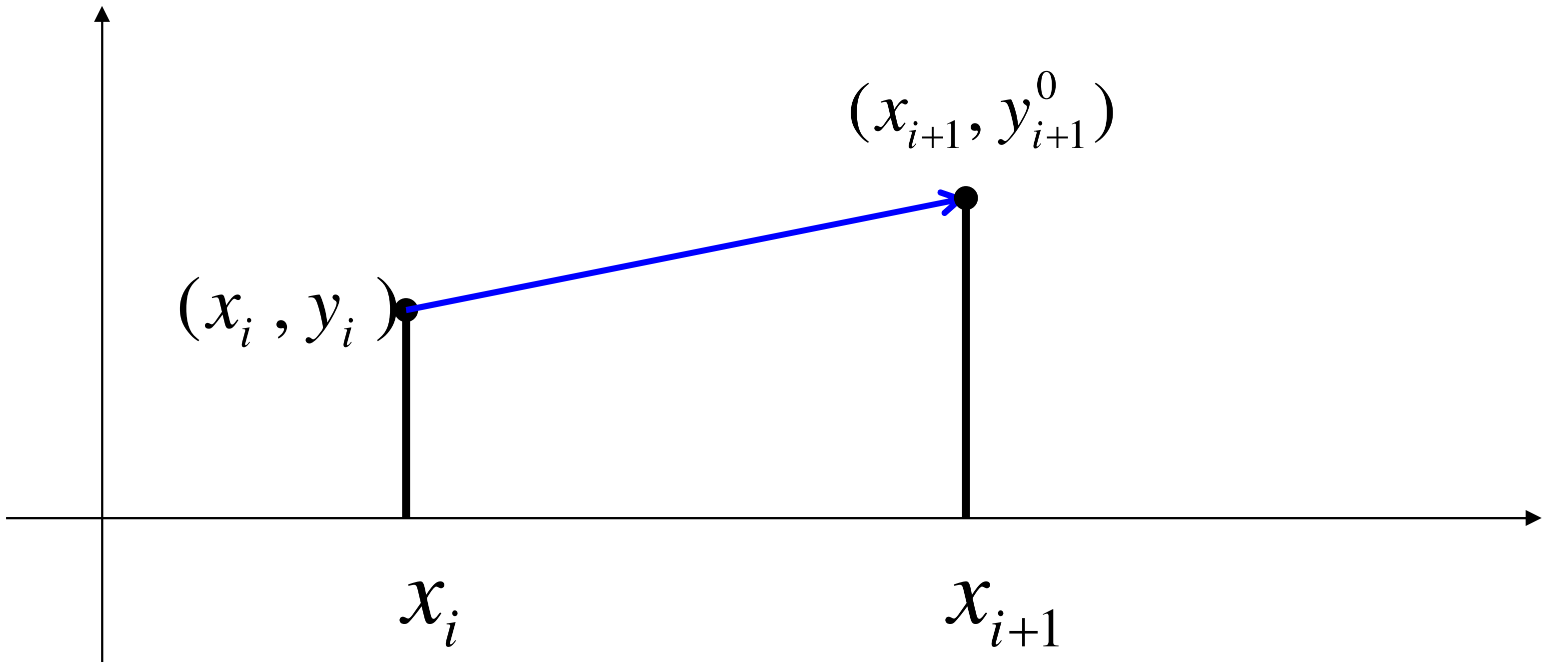
$$\text{Corrector}: \quad y_{i+1}^1 = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)\right)$$

Local Truncation Error $O(h^3)$

Global Truncation Error $O(h^2)$

Heun's Predictor Corrector

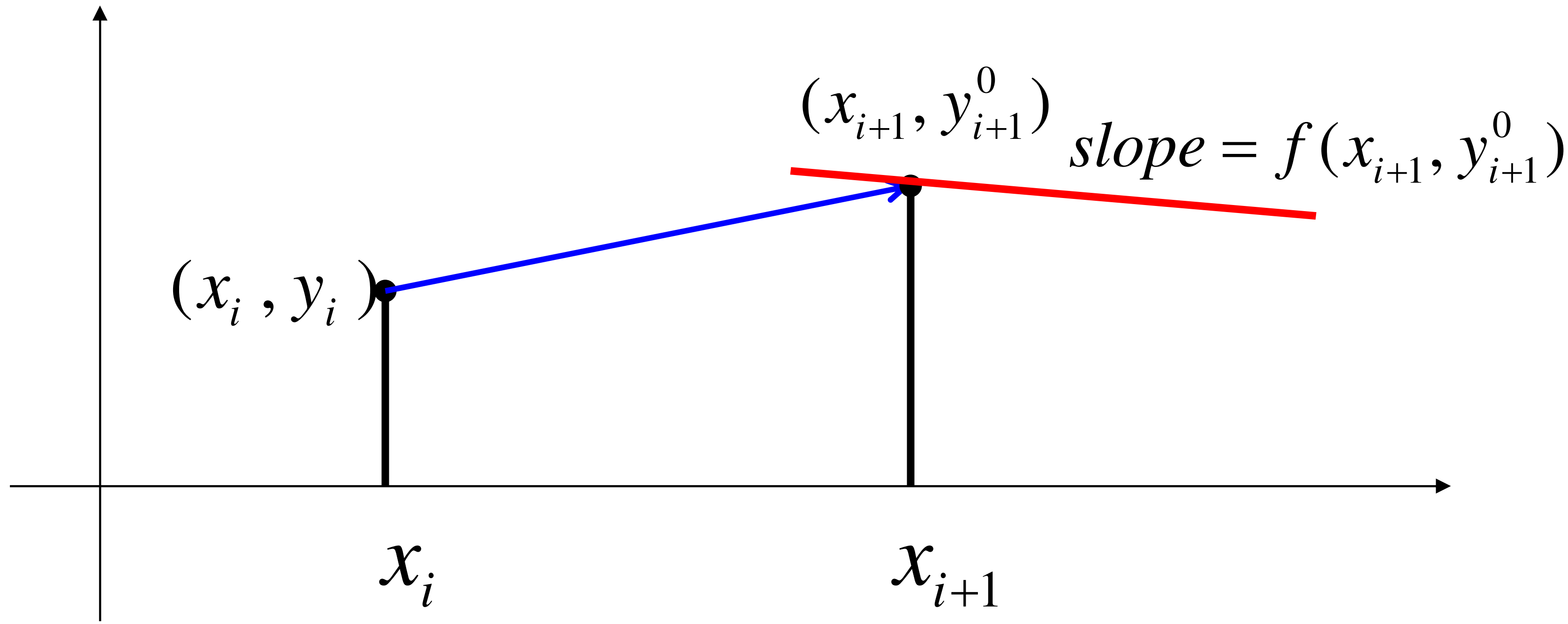
(Prediction)



Prediction $y_{i+1}^0 = y_i + h\ f(x_i, y_i)$

Heun's Predictor Corrector

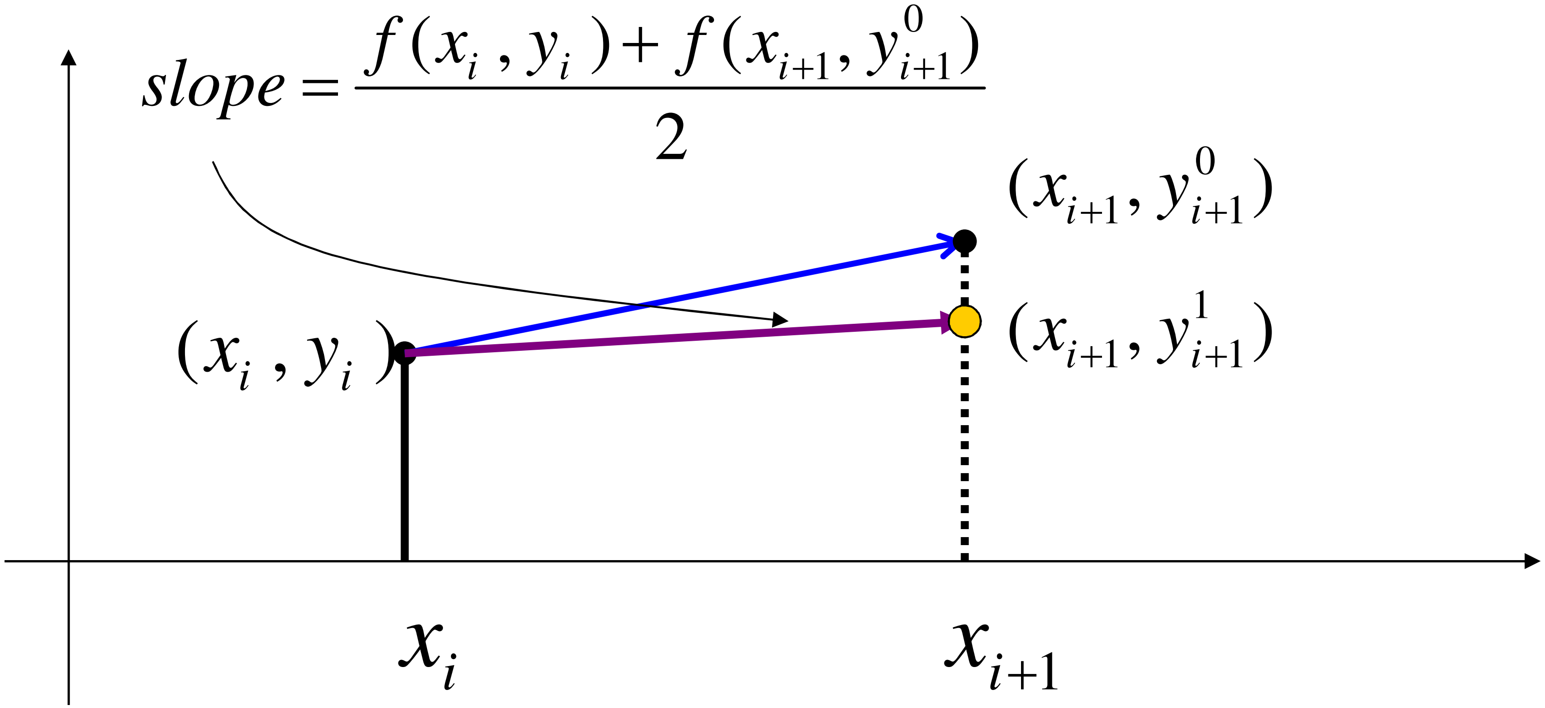
(Prediction)



Prediction $y_{i+1}^0 = y_i + h\ f(x_i, y_i)$

Heun's Predictor Corrector

(Correction)



$$y^1_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})\right)$$

Example 2

Use the Heun's Method to solve the ODE

$$\dot{y}(x) = 1 + x^2 + y$$

$$y(0) = 1$$

Use $h = 0.1$. One correction only

Determine y(0.1) and y(0.2)

Example 2

$$\text{Problem}: f(x, y) = 1 + y + x^2,\ y_0 = y(x_0) = 1, h = 0.1$$

$$\text{Step 1}:$$

$$\text{Predictor}:\ y_1^0 = y_0 + h\ f(x_0, y_0) = 1 + 0.1(2) = 1.2$$

$$\text{Corrector}:\ y_1^1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^0)\right) = 1.2105$$

$$\text{Step 2}:$$

$$\text{Predictor}:\ y_2^0 = y_1 + h\ f(x_1, y_1) = 1.4326$$

$$\text{Corrector}:\ y_2^1 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^0)\right) = 1.4452$$

Summary of Midpoint & Heun's

- Euler, Midpoint and Heun's methods are similar in the following sense:

$$y_{i+1} = y_i + h \times slope$$

  - Different methods use different estimates of the slope.
- Both Midpoint and Heun's methods are comparable in accuracy to the second order Taylor series method.

Comparison

Method	Local truncation error	Global truncation error
Euler Method $y_{i+1} = y_i + h\ f(x_i, y_i)$	$O(h^2)$	$O(h)$
Heun's Method Predictor: $y_{i+1}^0 = y_i + h\ f(x_i, y_i)$ Corrector: $y_{i+1}^{k+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^k)\right)$	$O(h^3)$	$O(h^2)$
Midpoint $y_{i+\frac{1}{2}} = y_i + \frac{h}{2}\ f(x_i, y_i)$ $y_{i+1} = y_i + h\ f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	$O(h^3)$	$O(h^2)$

Runge-Kutta Method : Motivation

- We seek accurate methods to solve ODEs that do not require calculating high order derivatives.
- The approach is to use a formula involving unknown coefficients then determine these coefficients to match as many terms of the Taylor series expansion.

Second Order Runge-Kutta Method

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h, \quad y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Problem :

Find α, β, w_1, w_2

such that y_{i+1} is as accurate as possible.

Taylor Series in One Variable

The n^{th} order Taylor Series expansion of $f(x)$

$$f(x+h) = \underbrace{\sum_{i=0}^{n} \frac{h^i}{i!} f^{(i)}(x)}_{\text{Approximation}} + \underbrace{\frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\bar{x})}_{\text{Error}}$$

where $\bar{x}$ *is between* x *and* $x+h$

Derivation of 2nd Order Runge-Kutta Methods – 1 of 5

Second Order Taylor Series Expansion

Used to solve ODE : $\dfrac{dy}{dx} = f(x, y)$

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2}\frac{d^2y}{dx^2} + O(h^3)$$

which is written as :

$$y_{i+1} = y_i + h\, f(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + O(h^3)$$

Derivation of 2nd Order Runge-Kutta Methods – 2 of 5

$$\text{where } f'(x,y) \text{ is obtained by chain - rule differentiation}$$

$$f'(x,y) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x,y)$$

$$\mathit{Substituting:}$$

$$y_{i+1} = y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x_i, y_i)\right)\frac{h^2}{2} + O(h^3)$$

Taylor Series in Two Variables

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) +$$

$$\frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x^2} + k^2 \frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} \right) + \ldots$$

$$= \underbrace{\sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{i} f(x, y)}_{approximation} + \underbrace{\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{n+1} f(\bar{x}, \bar{y})}_{error}$$

$(\bar{x}, \bar{y})$ is on the line joining between (x, y) and $(x+h, y+k)$

Derivation of 2nd Order Runge-Kutta Methods – 3 of 5

$$\text{Problem : } \textit{Find} \;\; \alpha, \beta, \; w_1, w_2 \;\; \text{such that}$$

$$K_1 = h \; f(x_i, y_i)$$

$$K_2 = h \; f(x_i + \alpha h, \quad y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Substituting :

$$y_{i+1} = y_i + w_1 h f(x_i, y_i) + w_2 h f(x_i + \alpha h, \; y_i + \beta K_1)$$

Derivation of 2nd Order Runge-Kutta Methods – 4 of 5

$$f(x_i + \alpha h,\ \ y_i + \beta K_1) = f(x_i, y_i) + \alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots$$

Substituting :

$$y_{i+1} = y_i + w_1 h\ f(x_i, y_i) + w_2 h \left(f(x_i, y_i) + \alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots \right)$$

$$y_{i+1} = y_i + (w_1 + w_2) h\ f(x_i, y_i) + w_2 h \left(\alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots \right)$$

$$y_{i+1} = y_i + (w_1 + w_2) h\ f(x_i, y_i) + w_2 \alpha h^2 \frac{\partial f}{\partial x} + w_2 \beta h^2 \frac{\partial f}{\partial y} f(x_i, y_i) + \ldots$$

Derivation of 2nd Order Runge-Kutta Methods – 5 of 5

We derived two expansions for y_{i+1} :

$$y_{i+1} = y_i + (w_1 + w_2)h\, f(x_i, y_i) + w_2 \alpha h^2 \frac{\partial f}{\partial x} + w_2 \beta h^2 \frac{\partial f}{\partial y} f(x_i, y_i) + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x_i, y_i) \right) \frac{h^2}{2} + O(h^3)$$

Matching terms, we obtain the following three equations :

$$w_1 + w_2 = 1, \qquad w_2 \alpha = \frac{1}{2}, \quad \text{and} \quad w_2 \beta = \frac{1}{2}$$

3 equations with 4 unknowns $\Rightarrow$ infinite solutions

One possible solution : $\alpha = \beta = 1, \quad w_1 = w_2 = \frac{1}{2}$

2^{nd} Order Runge-Kutta Methods

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h,\quad y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

$$\text{Choose}\quad \alpha, \beta, w_1, w_2 \quad \text{such that:}$$

$$w_1 + w_2 = 1,\ \ w_2 \alpha = \frac{1}{2},\ \text{and}\ \ w_2 \beta = \frac{1}{2}$$

Alternative Form

Second Order Runge Kutta

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h,\ y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Alternative Form

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \alpha h,\ y_i + \beta h k_1)$$

$$y_{i+1} = y_i + h\left(w_1 k_1 + w_2 k_2\right)$$

Choosing α, β, w_1 and w_2

For example, choosing $\alpha = 1$, then $\beta = 1$, $w_1 = w_2 = \frac{1}{2}$

Second Order Runge - Kutta method becomes :

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + h, y_i + K_1)$$

$$y_{i+1} = y_i + \frac{1}{2}\left(K_1 + K_2\right) = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)\right)$$

This is *Heun's Method* with a Single Corrector

Choosing α, β, w_1 and w_2

$$\text{Choosing } \alpha = \frac{1}{2} \text{ then } \beta = \frac{1}{2}, \quad w_1 = 0, \; w_2 = 1$$

Second Order Runge - Kutta method becomes :

$$K_1 = h \; f(x_i, y_i)$$

$$K_2 = h \; f(x_i + \frac{h}{2}, y_i + \frac{K_1}{2})$$

$$y_{i+1} = y_i + K_2 = y_i + h \; f(x_i + \frac{h}{2}, y_i + \frac{K_1}{2})$$

This is the Midpoint Method

2^{nd} Order Runge-Kutta Methods

Alternative Formulas

$$\alpha\, w_2 = \frac{1}{2}, \quad \beta\, w_2 = \frac{1}{2}, \quad w_1 + w_2 = 1$$

$$\text{Pick any nonzero } \alpha \text{ number: } \beta = \alpha, \ w_2 = \frac{1}{2\alpha}, \quad w_1 = 1 - \frac{1}{2\alpha}$$

Second Order Runge Kutta Formulas (select $\alpha \neq 0$)

$$K_1 = h\, f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha\, h, \quad y_i + \alpha\, K_1)$$

$$y_{i+1} = y_i + \left(1 - \frac{1}{2\alpha}\right) K_1 + \frac{1}{2\alpha} K_2$$

Second order Runge-Kutta Method

Example

Solve the following system to find $x(1.02)$ using RK2

$$\dot{x}(t) = 1 + x^2 + t^3, \qquad x(1) = -4, \; h = 0.01, \; \alpha = 1$$

STEP1:

$$K_1 = h \; f(t_0 = 1, x_0 = -4) = 0.01(1 + x_0{}^2 + t_0{}^3) = 0.18$$

$$K_2 = h \; f(t_0 + h, x_0 + K_1)$$

$$= 0.01(1 + (x_0 + 0.18)^2 + (t_0 + .01)^3) = 0.1662$$

$$x(1 + 0.01) = x(1) + \left(K_1 + K_2\right)/2$$

$$= -4 + (0.18 + 0.1662)/2 = -3.8269$$

Second order Runge-Kutta Method

Example

STEP 2

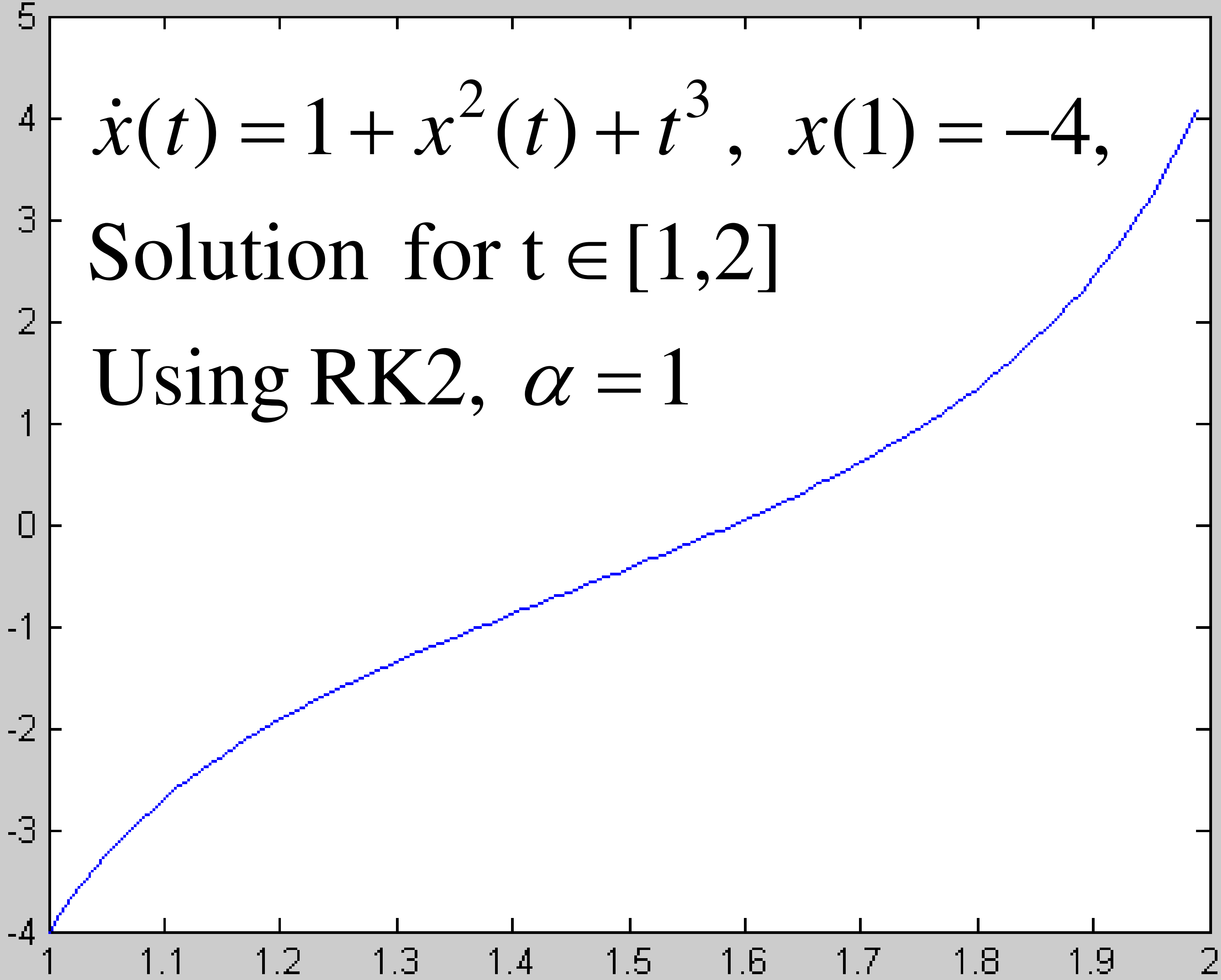
$$K_1 = h\ f(t_1 = 1.01, x_1 = -3.8269) = 0.01(1 + {x_1}^2 + {t_1}^3) = 0.1668$$

$$K_2 = h\ f(t_1 + h, x_1 + K_1)$$

$$= 0.01(1 + (x_1 + 0.1668)^2 + (t_1 + .01)^3) = 0.1546$$

$$x(1.01 + 0.01) = x(1.01) + \frac{1}{2}\left(K_1 + K_2\right)$$

$$= -3.8269 + \frac{1}{2}(0.1668 + 0.1546) = -3.6662$$


ẋ(t) = 1 + x²(t) + t³, x(1) = −4,
Solution for t ∈ [1,2]
Using RK2, α = 1
5
4
3
2
1
0
-1
-2
-3
-4
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2

2nd Order Runge-Kutta

RK2

Typical value of $\alpha = 1$, Known as RK2

Equivalent to Heun's method with a single corrector

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$y_{i+1} = y_i + \frac{h}{2}\left(k_1 + k_2\right)$$

Local error is $O(h^3)$ and global error is $O(h^2)$

Higher-Order Runge-Kutta

Higher order Runge-Kutta methods are available.

Derivation approach similar to that of second-order Runge-Kutta.

Higher order methods are more accurate but require more calculations-> Higher cost.

3rd Order Runge-Kutta

RK3

Know as RK3

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2},\ y_i + \frac{1}{2} k_1 h)$$

$$k_3 = f(x_i + h,\ y_i - k_1 h + 2 k_2 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 4k_2 + k_3\right)$$

Local error is $O(h^4)$ and Global error is $O(h^3)$

4^{th} Order Runge-Kutta

RK4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2}, \quad y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{h}{2}, \quad y_i + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, \quad y_i + k_3 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

Higher-Order Runge-Kutta

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{4}h,\ y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h)$$

$$k_4 = f(x_i + \frac{1}{2}h,\ y_i - \frac{1}{2}k_2 h + k_3 h)$$

$$k_5 = f(x_i + \frac{3}{4}h,\ y_i + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h)$$

$$k_6 = f(x_i + h,\ y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h)$$

$$y_{i+1} = y_i + \frac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)$$

Example

4^{th}-Order Runge-Kutta Method

RK4

$$\frac{dy}{dx} = 1 + y + x^2$$

$$y(0) = 0.5$$

$$h = 0.2$$

$$Use\ RK4\ to\ compute\ \ y(0.2)\ and\ \ y(0.4)$$

4^{th} Order Runge-Kutta

RK4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2}, \quad y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{h}{2}, \quad y_\mathrm{i} + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, \quad y_\mathrm{i} + k_3 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

Example: RK4

See RK4 Formula

Problem:

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

$Use\ RK4\ to\ find\ y(0.2), y(0.4)$

$$h = 0.2$$

$$f(x, y) = 1 + y + x^2$$

$$x_0 = 0, \quad y_0 = 0.5$$

Step 1

$$k_1 = f(x_0, y_0) = (1 + y_0 + x_0^{\ 2}) = 1.5$$

$$k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 h) = 1 + (y_0 + 0.15) + (x_0 + 0.1)^2 = 1.64$$

$$k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h) = 1 + (y_0 + 0.164) + (x_0 + 0.1)^2 = 1.654$$

$$k_4 = f(x_0 + h, y_0 + k_3 h) = 1 + (y_0 + 0.16545) + (x_0 + 0.2)^2 = 1.7908$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.8293$$

Example: RK4

Problem:

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

$$h = 0.2$$

$$f(x, y) = 1 + y + x^2$$

$$x_1 = 0.2, \quad y_1 = 0.8293$$

Step 2

$$k_1 = f(x_1, y_1) = 1.7893$$

$$k_2 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h) = 1.9182$$

$$k_3 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2 h) = 1.9311$$

$$k_4 = f(x_1 + h, y_1 + k_3 h) = 2.0555$$

$$y_2 = y_1 + \frac{0.2}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 1.2141$$

Example: RK4

Problem:

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

$Use\ RK4\ to\ find\ \ y(0.2), y(0.4)$

Summary of the solution

x_i	y_i
0.0	0.5
0.2	0.8293
0.4	1.2141

Analytic Solution: $y(x) = 3.5e^x - (x^2 + 2x + 3)$