Lecture 10
Ordinary Differential Equations
Part II

- Solving systems of ODEs
- Multiple step Methods
- Boundary value Problems

Solving a System of First Order ODEs

- Methods discussed earlier such as Euler, Runge-Kutta,… are used to solve first order ordinary differential equations.
- The same formulas will be used to solve a system of first order ODEs.
  - In this case, the differential equation is a vector equation and the dependent variable is a vector variable.

Euler Method for Solving a System of First Order ODEs

Recall Euler method for solving a first order ODE:

$$Given \quad \frac{dy(x)}{dx} = f(y, x), \quad y(a) = y_a$$

$$Euler\,Method:$$

$$y(a+h) = y(a) + h\,f(y(a), a)$$

$$y(a+2h) = y(a+h) + h\,f(y(a+h), a+h)$$

$$y(a+3h) = y(a+2h) + h\,f(y(a+2h), a+2h)$$

Example - Euler Method

Euler method to solve a system of *n* first order ODEs.

$$Given \quad \frac{dY(x)}{dx} = F(Y,x) = \begin{bmatrix} f_1(Y,x) \\ f_2(Y,x) \\ \dots \\ f_n(Y,x) \end{bmatrix}, \quad Y(a) = \begin{bmatrix} y_1(a) \\ y_2(a) \\ \dots \\ y_n(a) \end{bmatrix}$$

$$Euler\ Method:$$

$$Y(a+h) = Y(a) + h\ F(Y(a),a)$$

$$Y(a+2h) = Y(a+h) + h\,F(Y(a+h),a+h)$$

$$Y(a+3h) = Y(a+2h) + h\,F(Y(a+2h),a+2h)$$

Solving a System of *n* First Order ODEs

- Exactly the same formula is used but the scalar variables and functions are replaced by vector variables and vector values functions.
- Y is a vector of length *n.*
- F(Y,x) is a vector valued function.

$$Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \dots \\ y_n(x) \end{bmatrix} \qquad Y \ \ is \ \ n \times 1 \ vector$$

$$\frac{d\,Y(x)}{dx} = \begin{bmatrix} \frac{d\,y_1}{dx} \\ \frac{d\,y_2}{dx} \\ \dots \\ \frac{d\,y_n}{dx} \end{bmatrix} = \begin{bmatrix} f_1(Y,x) \\ f_2(Y,x) \\ \dots \\ f_n(Y,x) \end{bmatrix} = F(Y,x)$$

$$Y(a+h) = Y(a) + h \ \ F(Y(a),a)$$

$$Y(a+2h) = Y(a+h) + h\,F(Y(a+h),a+h)$$

$$Y(a+3h) = Y(a+2h) + h\,F(Y(a+2h),a+2h)$$

Example :

Euler method for solving a system of first order ODEs.

$$\begin{bmatrix} \dot{y}_1(x) \\ \dot{y}_2(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ 1 - y_1 \end{bmatrix} = F(Y, x), \quad Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Two steps of Euler Method with $h = 0.1$

STEP 1 :

$$Y(0+h) = Y(0) + h \ F(Y(0), 0)$$

$$\begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + 0.1 \begin{bmatrix} y_2(0) \\ 1 - y_1(0) \end{bmatrix} = \begin{bmatrix} -1 + 0.1 \\ 1 + 0.1(1+1) \end{bmatrix} = \begin{bmatrix} -0.9 \\ 1.2 \end{bmatrix}$$

STEP 2 :

$$Y(0+2h) = Y(h) + h \, F(Y(h), h)$$

$$\begin{bmatrix} y_1(0.2) \\ y_2(0.2) \end{bmatrix} = \begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} + 0.1 \begin{bmatrix} y_2(0.1) \\ 1 - y_1(0.1) \end{bmatrix} = \begin{bmatrix} -0.9 + 0.12 \\ 1.2 + .1(1 + 0.9) \end{bmatrix} = \begin{bmatrix} -0.78 \\ 1.39 \end{bmatrix}$$

Example :

RK2 method for solving a system of first order ODEs

$$\begin{bmatrix} \dot{y}_1(x) \\ \dot{y}_2(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ 1 - y_1 \end{bmatrix} = F(Y, x), \quad Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Two steps of second order Runge – Kutta Method *with* $h = 0.1$

STEP 1:

$$K1 = h \;\; F(Y(0), 0) = 0.1 \begin{bmatrix} y_2(0) \\ 1 - y_1(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$$K2 = h \;\; F(Y(0) + K1, 0 + h) = 0.1 \begin{bmatrix} y_2(0) + 0.2 \\ 1 - (y_1(0) + 0.1) \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.19 \end{bmatrix}$$

$$Y(0 + h) = Y(0) + 0.5(K1 + K2)$$

$$\begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 0.12 \\ 0.19 \end{bmatrix} \right) = \begin{bmatrix} -0.89 \\ 1.195 \end{bmatrix}$$

Example :

RK2 method for solving a system of first order ODEs

$$\begin{bmatrix} \dot{y}_1(x) \\ \dot{y}_2(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ 1-y_1 \end{bmatrix} = F(Y,x), \quad Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$STEP\ 2:$

$$K1 = h\ \ F(Y(0.1),0.1) = 0.1\begin{bmatrix} y_2(0.1) \\ 1-y_1(0.1) \end{bmatrix} = \begin{bmatrix} 0.1195 \\ 0.1890 \end{bmatrix}$$

$$K2 = h\ \ F(Y(0.1)+K1,0.1+h) = 0.1\begin{bmatrix} y_2(0.1)+0.189 \\ 1-(y_1(0.1)+0.1195) \end{bmatrix} = \begin{bmatrix} 0.1384 \\ 0.1771 \end{bmatrix}$$

$$Y(0.1+h) = Y(0.1) + 0.5(K1+K2)$$

$$\begin{bmatrix} y_1(0.2) \\ y_2(0.2) \end{bmatrix} = \begin{bmatrix} -0.89 \\ 1.195 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 0.1195 \\ 0.1890 \end{bmatrix} + \begin{bmatrix} 0.1384 \\ 0.1771 \end{bmatrix}\right) = \begin{bmatrix} -0.7611 \\ 1.3780 \end{bmatrix}$$

Methods for Solving a System of First Order ODEs

- We have extended Euler and RK2 methods to solve systems of first order ODEs.

- Other methods used to solve first order ODE can be easily extended to solve systems of first order ODEs.

High Order ODEs

- How do we solve a second order ODE?

$$\ddot{x} + 3\dot{x} + 6x = 1$$

- How do we solve high order ODEs?

The General Approach to Solve ODEs

High order ODE → Convert → System of first order ODEs → Solve

$$\ddot{x} + 3\dot{x} + 6x = 1$$

$$\dot{x}(0) = 1; x(0) = 4$$

Second order ODE

Convert →

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 1 - 3z_2 - 6z_1 \end{bmatrix},$$

$$Z(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Two first order ODEs

Solve →

Conversion Procedure

High order ODE → Convert → System of first order ODEs → Solve

1. **<u>Select the dependent variables</u>**

   One way is to take the original dependent variable and its derivatives up to one degree less than the highest order derivative.

2. **<u>Write the Differential Equations</u>** in terms of the new variables. The equations come from the way the new variables are defined or from the original equation.

3. **<u>Express the equations in a matrix form.</u>**

Remarks on the Conversion Procedure

High order ODE → Convert → System of first order ODE → Solve

1. Any n^{th} order ODE is converted to a system of n first order ODEs.
2. There are an infinite number of ways to select the new variables. As a result, for each high order ODE there are an infinite number of set of equivalent first order systems of ODEs.
3. Use a table to make the conversion easier.

Example of Converting a High Order ODE to First Order ODEs

Convert $\ddot{x} + 3\dot{x} + 6x = 1, \quad \dot{x}(0) = 1; \quad x(0) = 4$

to a system of first order ODEs

1. Select a new set of variables

(Second order ODE $\Rightarrow$ We need two variables)

$$z_1 = x$$

$$z_2 = \dot{x}$$

One degree less than the highest order derivative

Example of Converting a High Order ODE to First Order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	1	$\dot{z}_2 = 1 - 3z_2 - 6z_1$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 1 - 3z_2 - 6z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Example of Converting a High Order ODE to First Order ODEs

Convert

$$\dddot{x} + 2\ddot{x} + 7\dot{x} + 8x = 0$$

$$\ddot{x}(0) = 9,\ \dot{x}(0) = 1;\quad x(0) = 4$$

1. Select a new set of variables (3 of them)

$$z_1 = x$$

$$z_2 = \dot{x}$$

$$z_3 = \ddot{x}$$

One degree less than the highest order derivative

Example of Converting a High Order ODE to First Order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	1	$\dot{z}_2 = z_3$
$\ddot{x}$	z_3	9	$\dot{z}_3 = -2z_3 - 7z_2 - 8z_1$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ -2z_3 - 7z_2 - 8z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$$

Conversion Procedure for Systems of High Order ODEs

System of high order ODEs → Convert → System of first order ODE → Solve

1. **Select the dependent variables**

   Take the original dependent variables and their derivatives up to one degree less than the highest order derivative for each variable.

2. **Write the Differential Equations** in terms of the new variables. The equations come from the way the new variables are defined or from the original equation.

3. **Express the equations in a matrix form.**

Example of Converting a High Order ODE to First Order ODEs

Convert

$$\dddot{x} + 5\ddot{x} + 2\dot{x} + 8y = 0$$

$$\ddot{y} + 2xy + \dot{x} = 2$$

$$x(0) = 4;\ \dot{x}(0) = 2;\ \ddot{x}(0) = 9;\ y(0) = 1;\ \dot{y}(0) = -3$$

1. Select a new set of variables ((3 + 2) variables)

$$z_1 = x$$

$$z_2 = \dot{x}$$

$$z_3 = \ddot{x}$$

$$z_4 = y$$

$$z_5 = \dot{y}$$

One degree less than the highest order derivative

Example of Converting a High Order ODE to First Order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	2	$\dot{z}_2 = z_3$
$\ddot{x}$	z_3	9	$\dot{z}_3 = -5z_3 - 2z_2 - 8z_4$
y	z_4	1	$\dot{z}_4 = z_5$
$\dot{y}$	z_5	-3	$\dot{z}_5 = 2 - z_2 - 2z_1 z_4$

Solution of a Second Order ODE

- Solve the equation using Euler method. Use h=0.1

$$\ddot{x} + 2\dot{x} + 8x = 2$$

$$x(0) = 1; \dot{x}(0) = -2$$

Select a new set of variables: $z_1 = x, z_2 = \dot{x}$

The second order equation is expressed as:

$$\dot{Z} = F(Z) = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 2 - 2z_2 - 8z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\text{Analytic Solution}: x(t) = \frac{u(t)}{4} + e^{-t}\left(\frac{3\cos\sqrt{7}t}{4} - \frac{5\sqrt{7}\sin\sqrt{7}t}{28}\right)$$

Solution of a Second Order ODE

$$F(Z) = \begin{bmatrix} z_2 \\ 2 - 2z_2 - 8z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, h = 0.1$$

$$Z(0+0.1) = Z(0) + hF(Z(0))$$

$$= \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 0.1 \begin{bmatrix} -2 \\ 2 - 2(-2) - 8(1) \end{bmatrix} = \begin{bmatrix} 0.8 \\ -2.2 \end{bmatrix}$$

$$Z(0.2) = Z(0.1) + hF(Z(0.1))$$

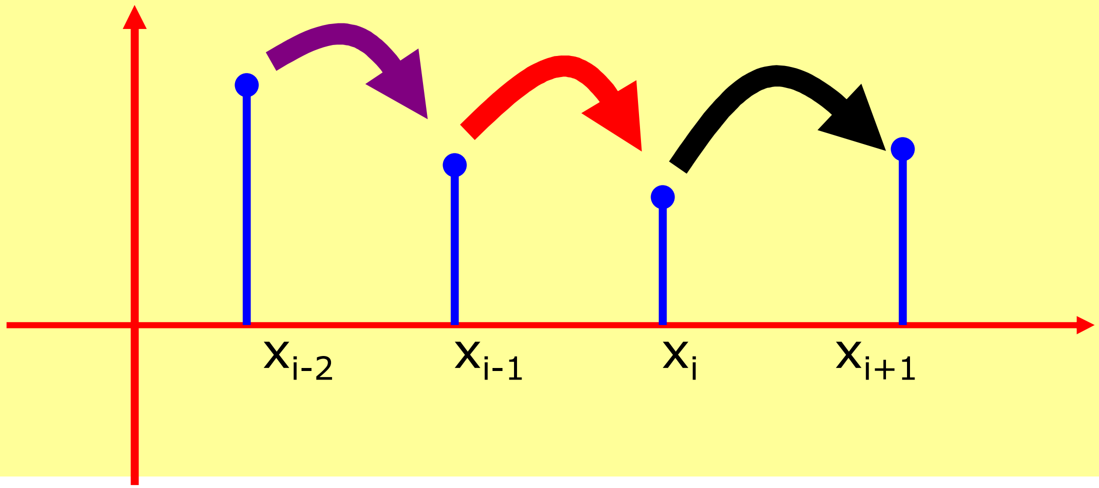
$$= \begin{bmatrix} 0.8 \\ -2.2 \end{bmatrix} + 0.1 \begin{bmatrix} -2.2 \\ 2 - 2(-2.2) - 8(0.8) \end{bmatrix} = \begin{bmatrix} 0.58 \\ -2.2 \end{bmatrix}$$

Adams Moulton multi-step method

- All the methods discussed so far are so-called “single-step” method.
- In multi-step methods, estimates y_{i+1} from more than one y_i and x_i.

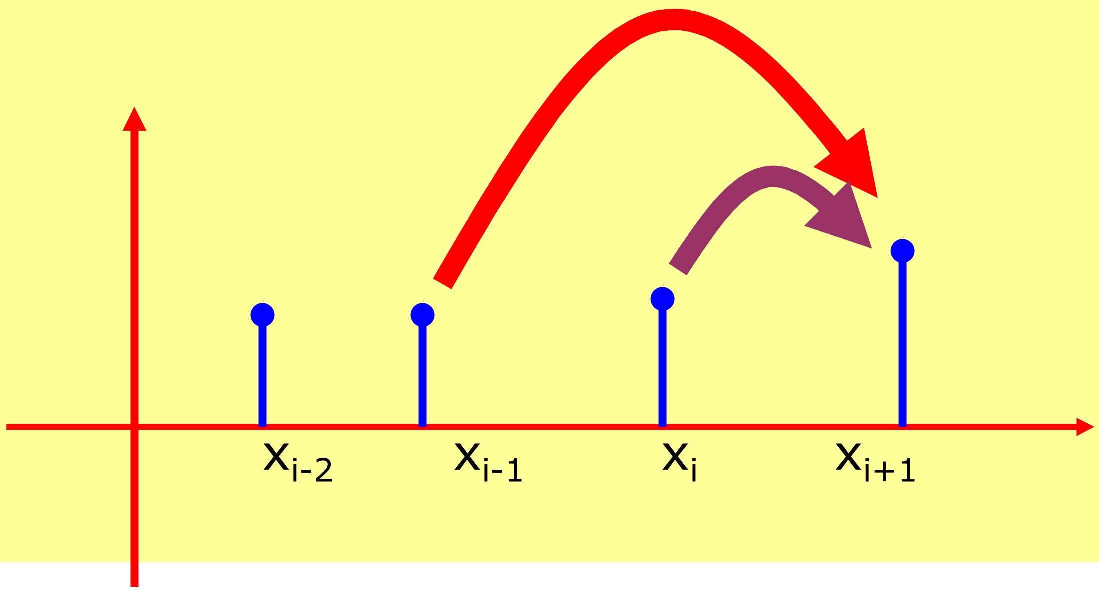
Single Step Methods

- Single Step Methods:
  - Euler, Heun's method and Runge-Kutta are single step methods.
  - Estimates of $\boldsymbol{y_{i+1}}$ depends only on $\boldsymbol{y_i}$ and $\boldsymbol{x_i}$.



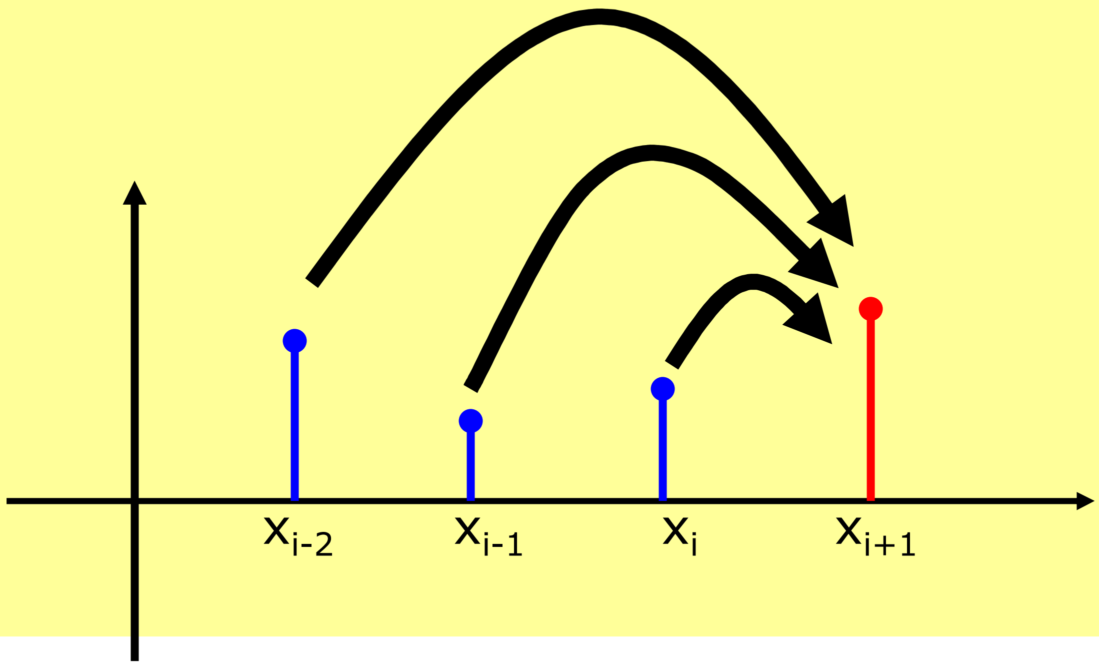
Multi-Step Methods

- 2-Step Methods
  - In a two-step method, estimates of y_{i+1} depends on y_i, y_{i-1}, x_i, and x_{i-1}



Multi-Step Methods

- 3-Step Methods
  - In an 3-step method, estimates of y_{i+1} depends on y_i ,y_{i-1} ,y_{i-2}, x_i , x_{i-1}, and x_{i-2}



2-Step Predictor-Corrector

$$\text{Predictor} \;:\; y_{i+1}^{0} = y_i + h\left(\frac{3}{2} f\left(x_i, y_i\right) - \frac{1}{2} f\left(x_{i-1}, y_{i-1}\right)\right)$$

$$\text{Corrector} \;:\; y_{i+1}^{k} = y_i + h\left(\frac{1}{2} f\left(x_{i+1}, y_{i+1}^{k-1}\right) + \frac{1}{2} f\left(x_i, y_i\right)\right)$$

- At each iteration one prediction step is done and as many correction steps as needed.
- y_{i+1}^{k} is the estimate of the solution at x_{i+1} after k correction steps.

3-Step Predictor-Corrector

Predictor :

$$y_{i+1}^{0} = y_i + h\left(\frac{23}{12} f(x_i, y_i) - \frac{16}{12} f(x_{i-1}, y_{i-1}) + \frac{5}{12} f(x_{i-2}, y_{i-2})\right)$$

Corrector :

$$y_{i+1}^{k} = y_i + h\left(\frac{5}{12} f(x_{i+1}, y_{i+1}^{k-1}) + \frac{8}{12} f(x_i, y_i) - \frac{1}{12} f(x_{i-1}, y_{i-1})\right)$$

4-Step Adams-Moulton Predictor-Corrector

Predictor: (Adams - Bashforth Predictor)

$$y_{i+1}^{0} = y_i + \frac{h}{24}\begin{pmatrix} 55f(x_i, y_i) - 59f(x_{i-1}, y_{i-1}) \\ +37f(x_{i-2}, y_{i-2}) - 9f(x_{i-3}, y_{i-3}) \end{pmatrix}$$

Corrector: (Adams - Moulton Corrector)

$$y_{i+1}^{k} = y_i + \frac{h}{24}\begin{pmatrix} 9f(x_{i+1}, y_{i+1}^{k-1}) + 19f(x_i, y_i) \\ -5f(x_{i-1}, y_{i-1}) + f(x_{i-2}, y_{i-2}) \end{pmatrix}$$

Next slide: Predictor (Top), Corrector (Bottom)

Order	β_0	β_1	β_2	β_3	β_4	β_5	Local Truncation Error
1	1						$\frac{1}{2}h^2f'(\xi)$
2	3/2	−1/2					$\frac{5}{12}h^3f''(\xi)$
3	23/12	−16/12	5/12				$\frac{9}{24}h^4f^{(3)}(\xi)$
4	55/24	−59/24	37/24	−9/24			$\frac{251}{720}h^5f^{(4)}(\xi)$
5	1901/720	−2774/720	2616/720	−1274/720	251/720		$\frac{475}{1440}h^6f^{(5)}(\xi)$
6	4277/720	−7923/720	9982/720	−7298/720	2877/720	−475/720	$\frac{19{,}087}{60{,}480}h^7f^{(6)}(\xi)$

Order	β_0	β_1	β_2	β_3	β_4	β_5	Local Truncation Error
2	1/2	1/2					$-\frac{1}{12}h^3f''(\xi)$
3	5/12	8/12	−1/12				$-\frac{1}{24}h^4f^{(3)}(\xi)$
4	9/24	19/24	−5/24	1/24			$-\frac{19}{720}h^5f^{(4)}(\xi)$
5	251/720	646/720	−264/720	106/720	−19/720		$-\frac{27}{1440}h^6f^{(5)}(\xi)$
6	475/1440	1427/1440	−798/1440	482/1440	−173/1440	27/1440	$-\frac{863}{60{,}480}h^7f^{(6)}(\xi)$

How Many Function Evaluations are Done?

Number of function evaluations is the Computational Speed or Efficiency

How many evaluations per step?

No need to repeat the evaluation of function f at previous points

Only one new function evaluation in the predictor

One function evaluation per correction step

# of function evaluations = 1+ number of corrections

Example

Solve

$$\frac{dy}{dx} = 2x + y^2 x \qquad y(0) = 2$$

$h = 0.1,\ \ Use\ \ 2 - step$ Predictor corrector Method

compute y(0.4)

We need two initial conditions to use the

$2 - step$ Predictor corrector Method

We will first use RK2 to estimate y(0.1)

Example

We need two initial conditions

Use $RK2$ *to compute* $y(0.1)$ *then we can use*

the Predictor corrector Method

$$\frac{dy}{dx} = 2x + y^2 x \qquad y(0) = 2, \quad h = 0.1,$$

$$K1 = 0.1(0) = 0$$

$$K2 = 0.1(0.2 + 0.4) = 0.06$$

$$y(0.1) = 2 + 0.5(0.06) = 2.03$$

Example

$$\frac{dy}{dx} = 2x + y^2 x \qquad y_{i\text{-}1} = y(0) = 2, \quad y_i = y(0.1) = 2.03, \qquad h = 0.1$$

$$\text{Predictor}: y_{i+1}^0 = y_i + h\left(\frac{3}{2} f(x_i, y_i) - \frac{1}{2} f(x_{i-1}, y_{i-1})\right)$$

$$= 2.03 + 0.1\left(\frac{3}{2}\left(2(0.1) + 2.03^2(0.1)\right) - \frac{1}{2}(0+0)\right) = 2.1218$$

$$\text{Corrector}: \; y(0.2) = y_{i+1}^1 = y_i + h\left(\frac{1}{2} f(x_{i+1}, y_{i+1}^0) + \frac{1}{2} f(x_i, y_i)\right)$$

$$= 2.03 + 0.1\left(\frac{1}{2}\left(2(0.2) + 2.1218^2(0.2)\right) + \frac{1}{2}\left(2(0.1) + 2.03^2(0.1)\right)\right) = 2.1256$$

Multi-Step Methods

- Single Step Methods
  - Euler and Runge-Kutta are single step methods.
  - Information about *y(x)* is used to estimate *y(x+h)*.
- Multistep Methods
  - Adam-Moulton method is a multi-step method.
  - To estimate *y(x+h),* information about *y(x),* *y(x-h), y(x-2h)…* are used.

Number of Steps

- At each iteration, one prediction step is done and as many correction steps as needed.
- Usually few corrections are done (1 to 3).
- It is usually better (in terms of accuracy) to use smaller step size than corrections.

Boundary-Value and Initial Value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

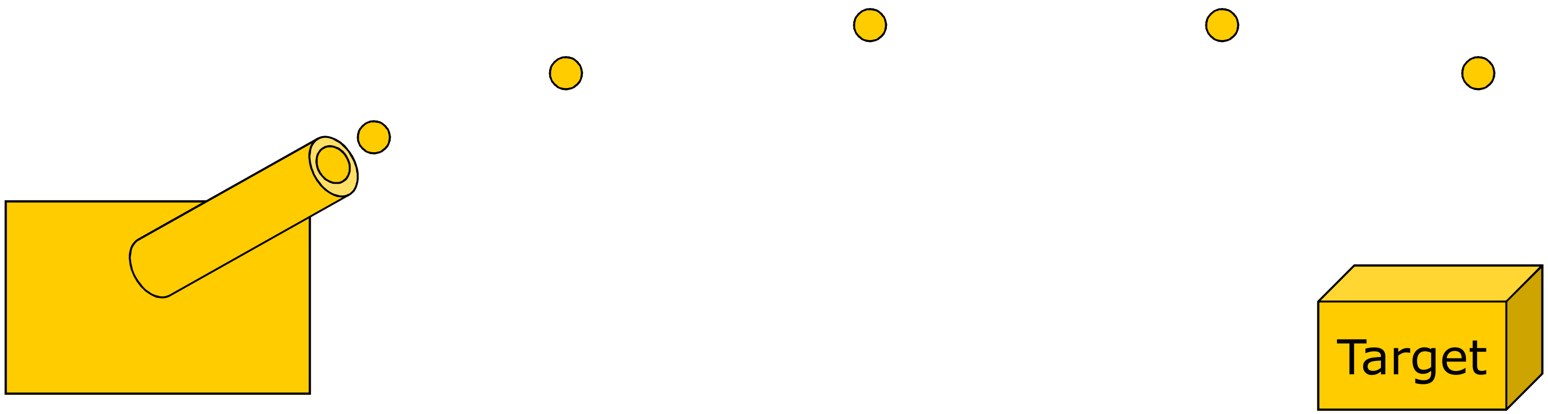
- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problem

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

The Shooting Method



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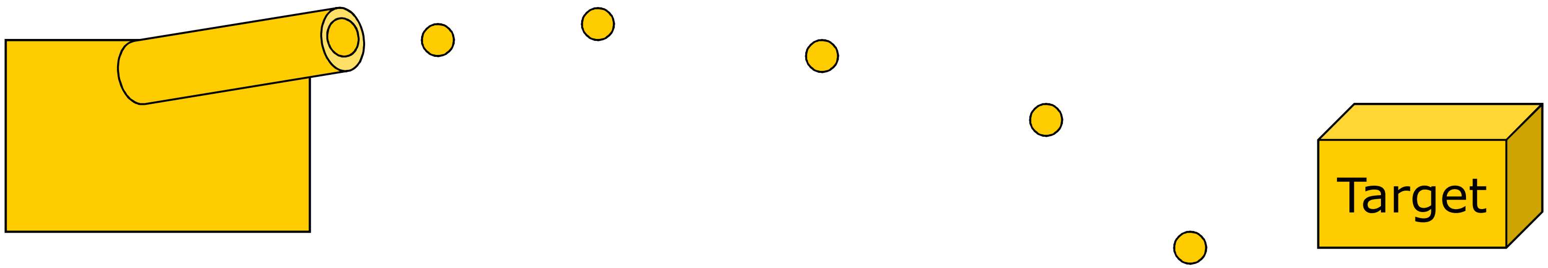

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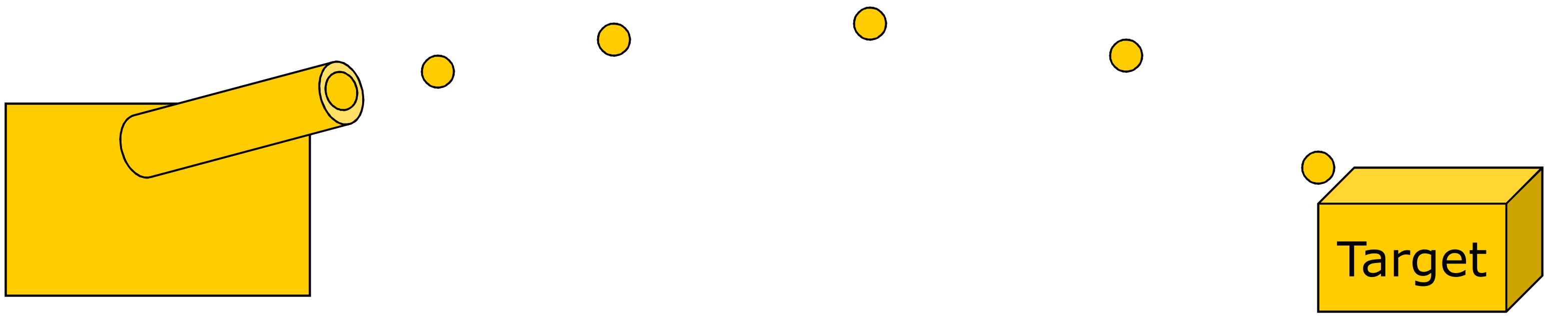


Solution of Boundary-Value Problems

Shooting Method for Boundary-Value Problems

1. Guess a value for the auxiliary conditions at one point of time.
2. Solve the initial value problem using Euler, Runge-Kutta, …
3. Check if the boundary conditions are satisfied, otherwise modify the guess and resolve the problem.

- Use interpolation in updating the guess.
- It is an iterative procedure and can be efficient in solving the BVP.

Solution of Boundary-Value Problems

Shooting Method

Boundary-Value Problem → convert → **Initial-value Problem**

$$Find\ y(x)\ to\ solve\ BVP$$

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

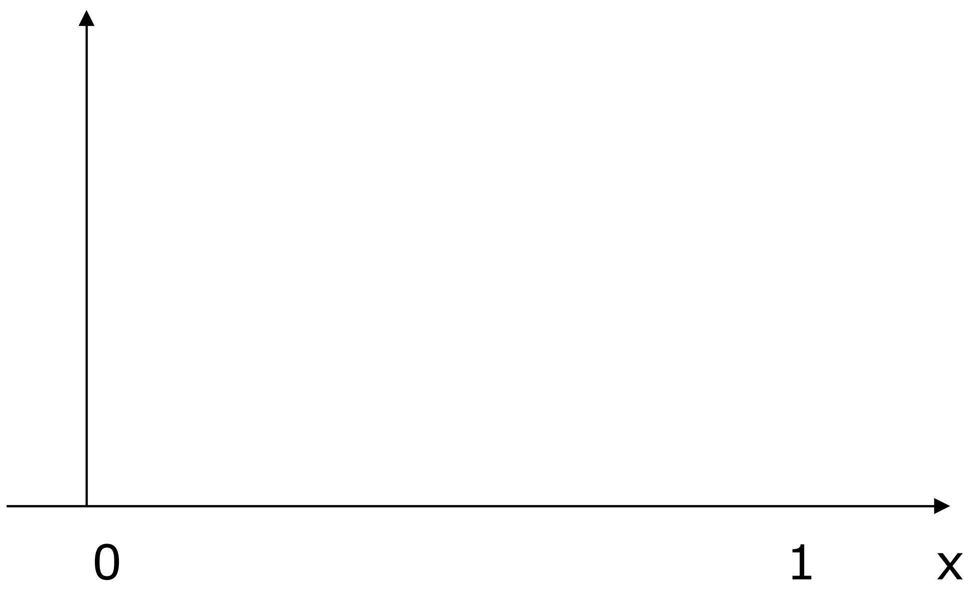
1. **Convert the ODE to a system of first order ODEs.**
2. **Guess the initial conditions that are not available.**
3. **Solve the Initial-value problem.**
4. **Check if the known boundary conditions are satisfied.**
5. **If needed modify the guess and resolve the problem again.**

Example 1

Original BVP

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$



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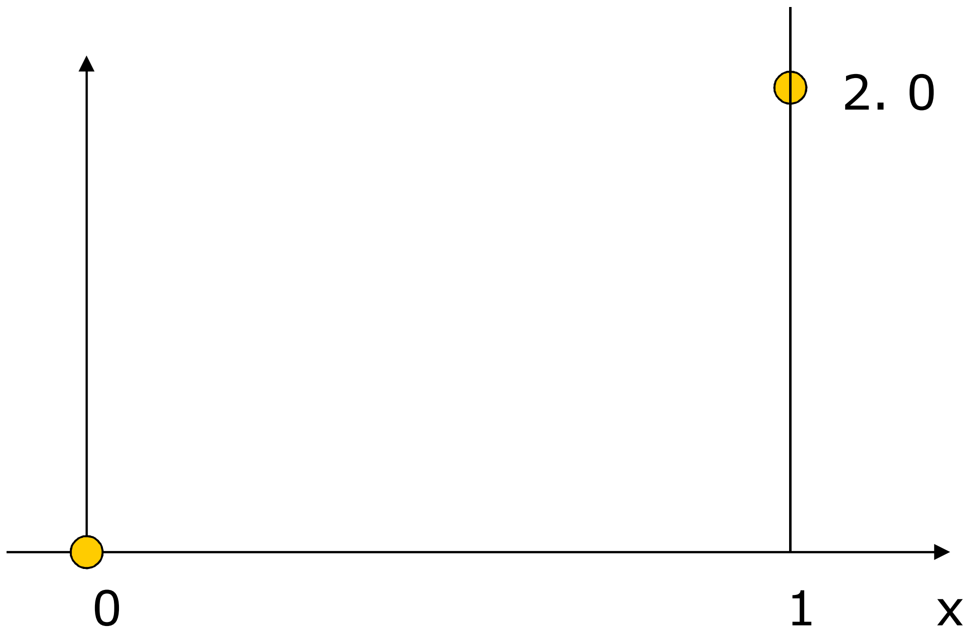

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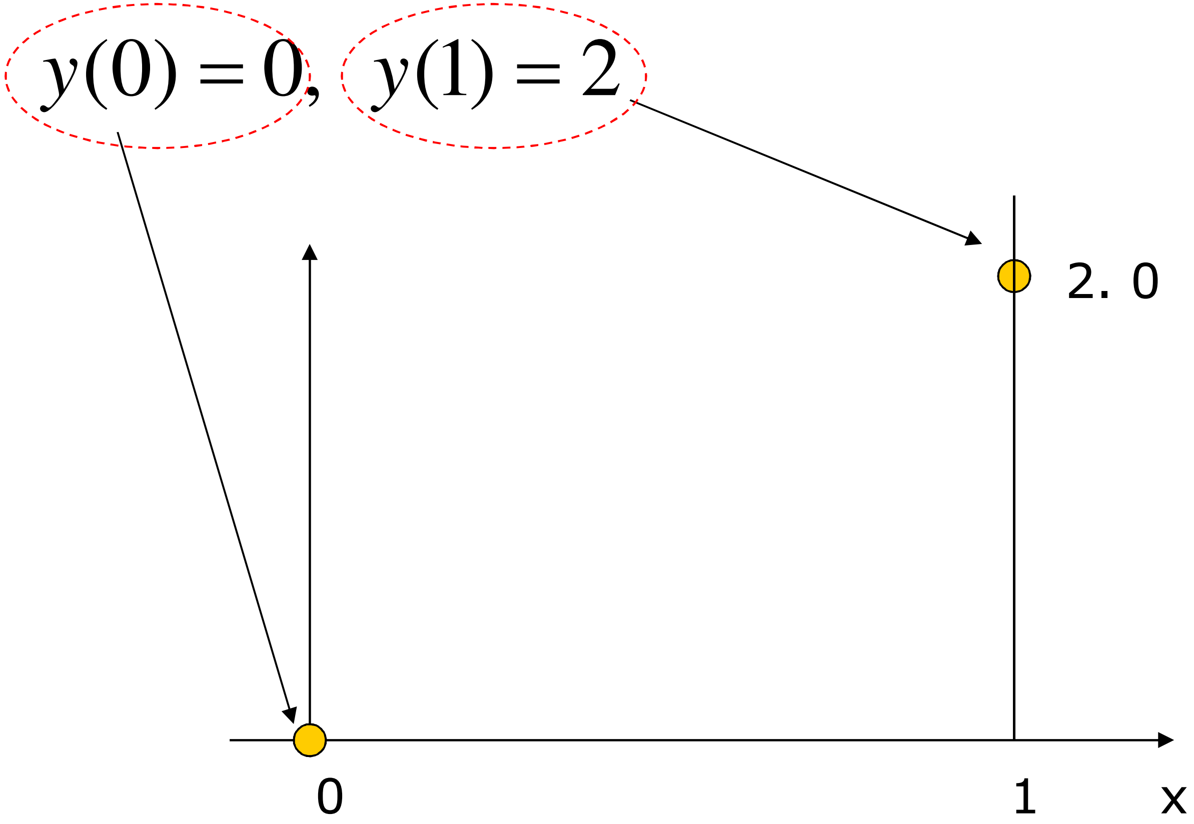

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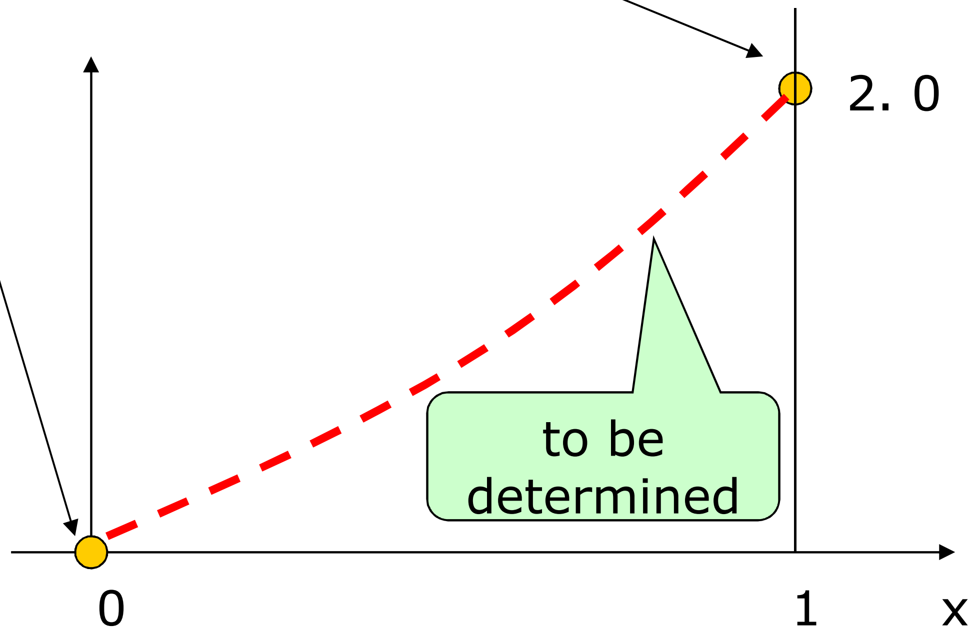
$$y(0) = 0, \quad y(1) = 2$$



Example 1

Step1: Convert to a System of First Order ODEs

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

Convert to a system of first order Equations

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ 4(y_1 - x) \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ ? \end{bmatrix}$$

The problem will be solved using RK2 with $h = 0.01$

for different values of $y_2(0)$ until we have $y(1) = 2$

Example 1

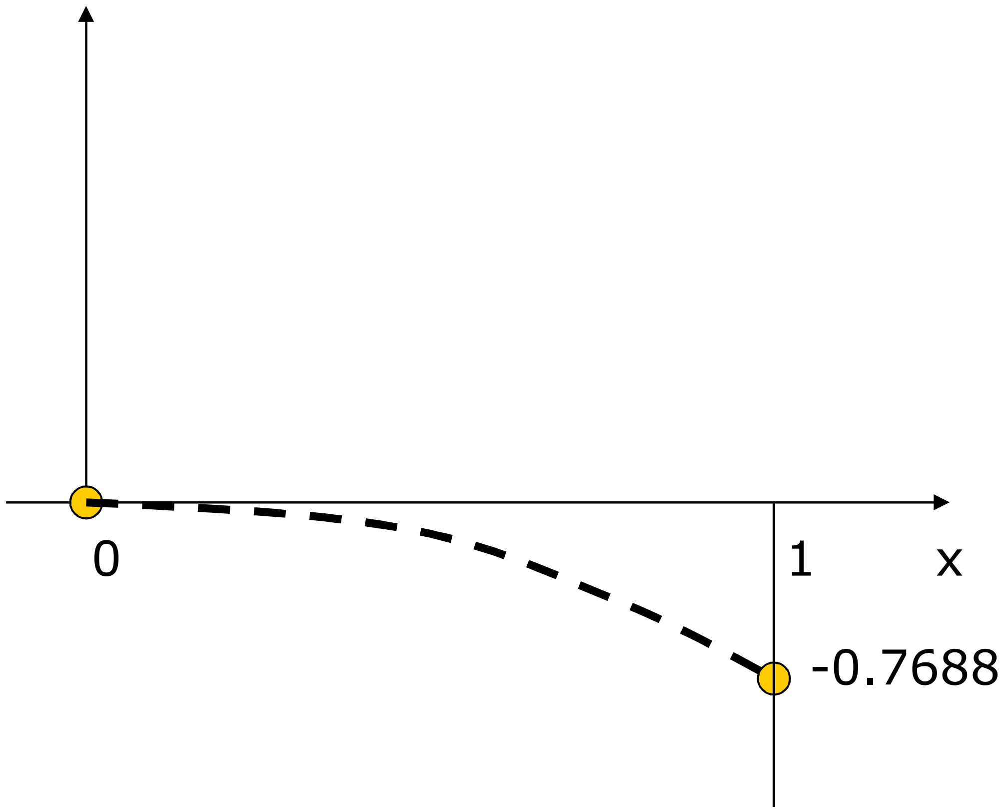
Guess # 1

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

$$Guess\#1$$

$$\dot{y}(0) = 0$$



Example 1

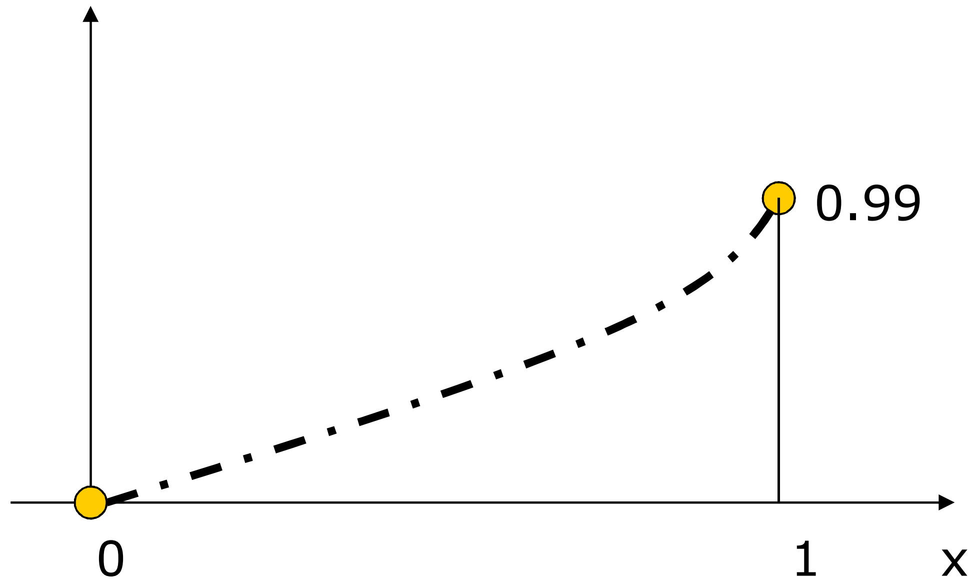
Guess # 2

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

$$Guess\#2$$

$$\dot{y}(0) = 1$$



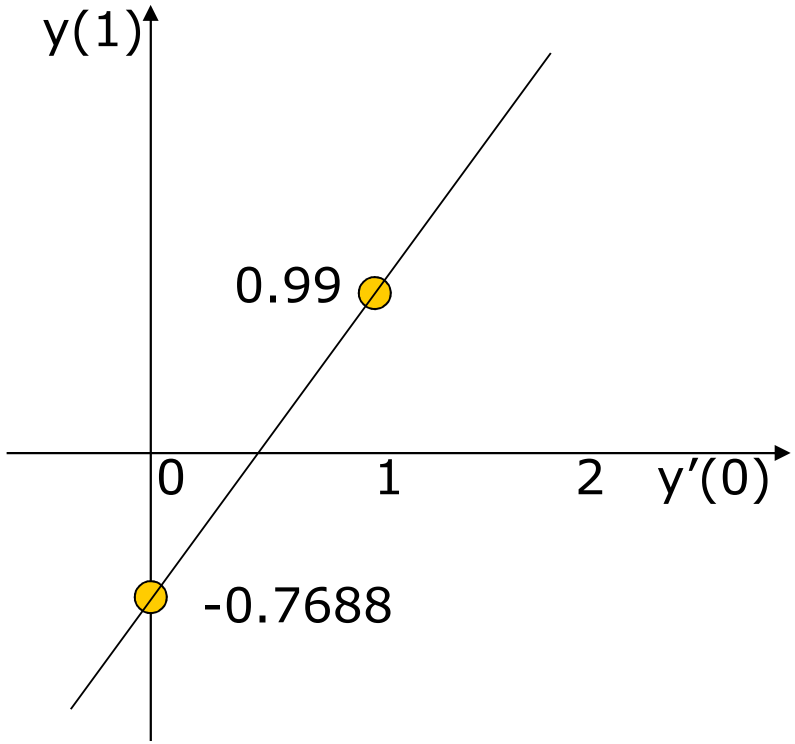
Example 1

Interpolation for Guess # 3

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

Guess	$\dot{y}(0)$	y(1)
1	0	-0.7688
2	1	0.9900



Example 1

Interpolation for Guess # 3

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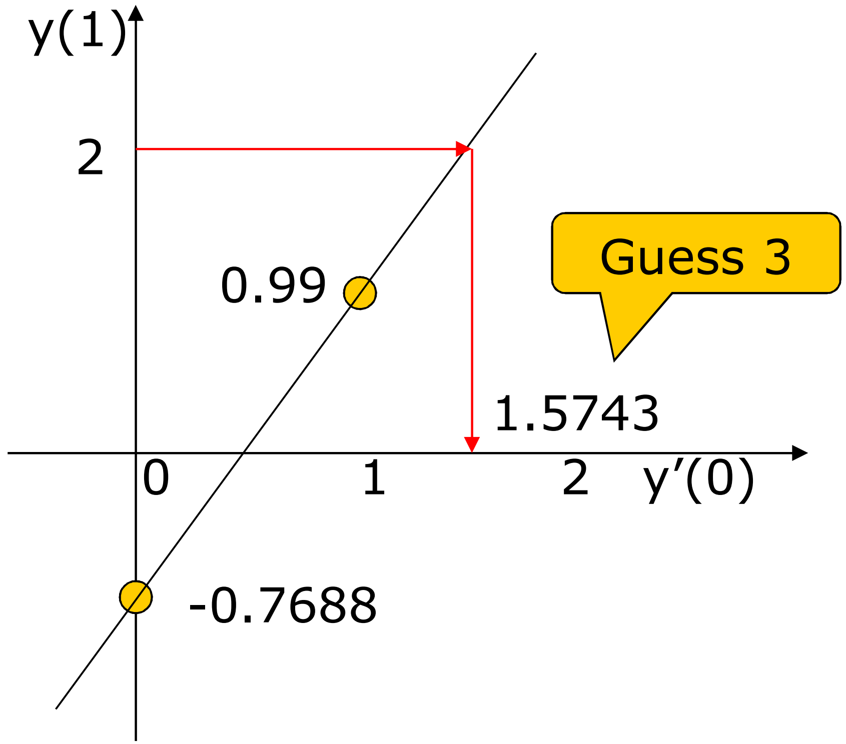


Example 1

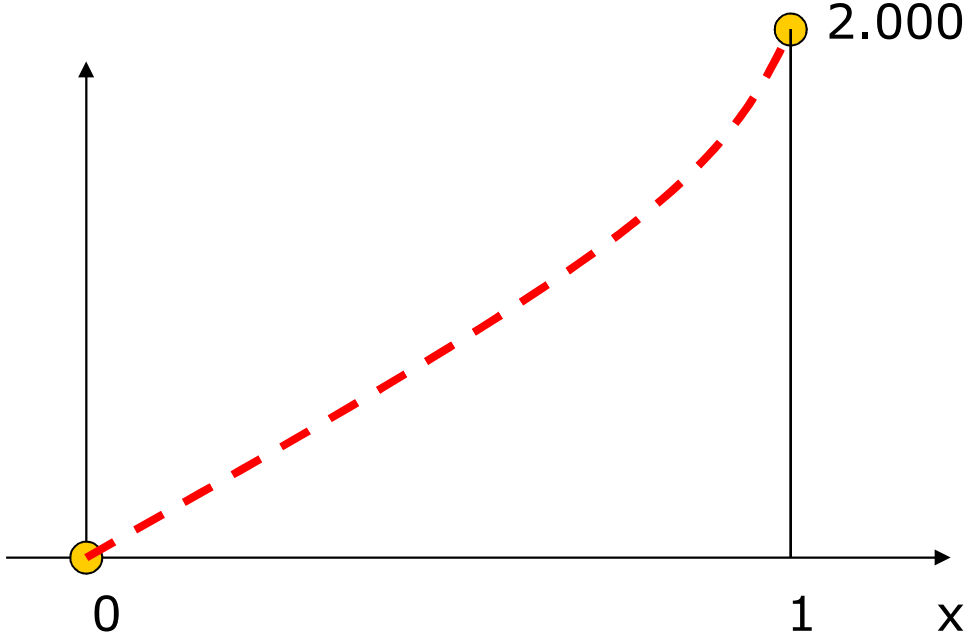
Guess # 3

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

$$Guess\#3$$

$$\dot{y}(0) = 1.5743$$



y(1)=2.000 → This is the solution to the boundary value problem.

Summary of the Shooting Method

1. Guess the unavailable values for the auxiliary conditions at one point of the independent variable.
2. Solve the initial value problem.
3. Check if the boundary conditions are satisfied, otherwise modify the guess and resolve the problem.
4. Repeat (3) until the boundary conditions are satisfied.

Properties of the Shooting Method

1. Using interpolation to update the guess often results in few iterations before reaching the solution.

2. The method can be cumbersome for high order BVP because of the need to guess the initial condition for more than one variable.

Solution of Boundary-Value Problems

Discretization method : Finite Difference Method

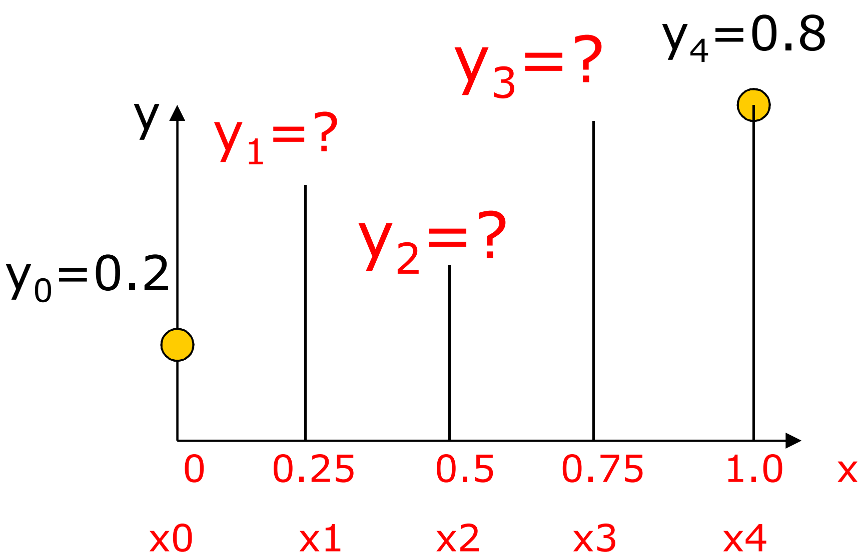
Boundary-Value Problems → convert → **Algebraic Equations**

Find the unknowns y_1, y_2, y_3

Find y(x) to solve BVP

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$



Solution of Boundary-Value Problems

Finite Difference Method

- Divide the interval into *n* sub-intervals.
- The solution of the BVP is converted to the problem of determining the value of function at the base points.
- Use finite approximations to replace the derivatives.
- This approximation results in a set of algebraic equations.
- Solve the equations to obtain the solution of the BVP.

Finite Difference Method

Example

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

Divide the interval [0,1] into n = 4 intervals

Base points are

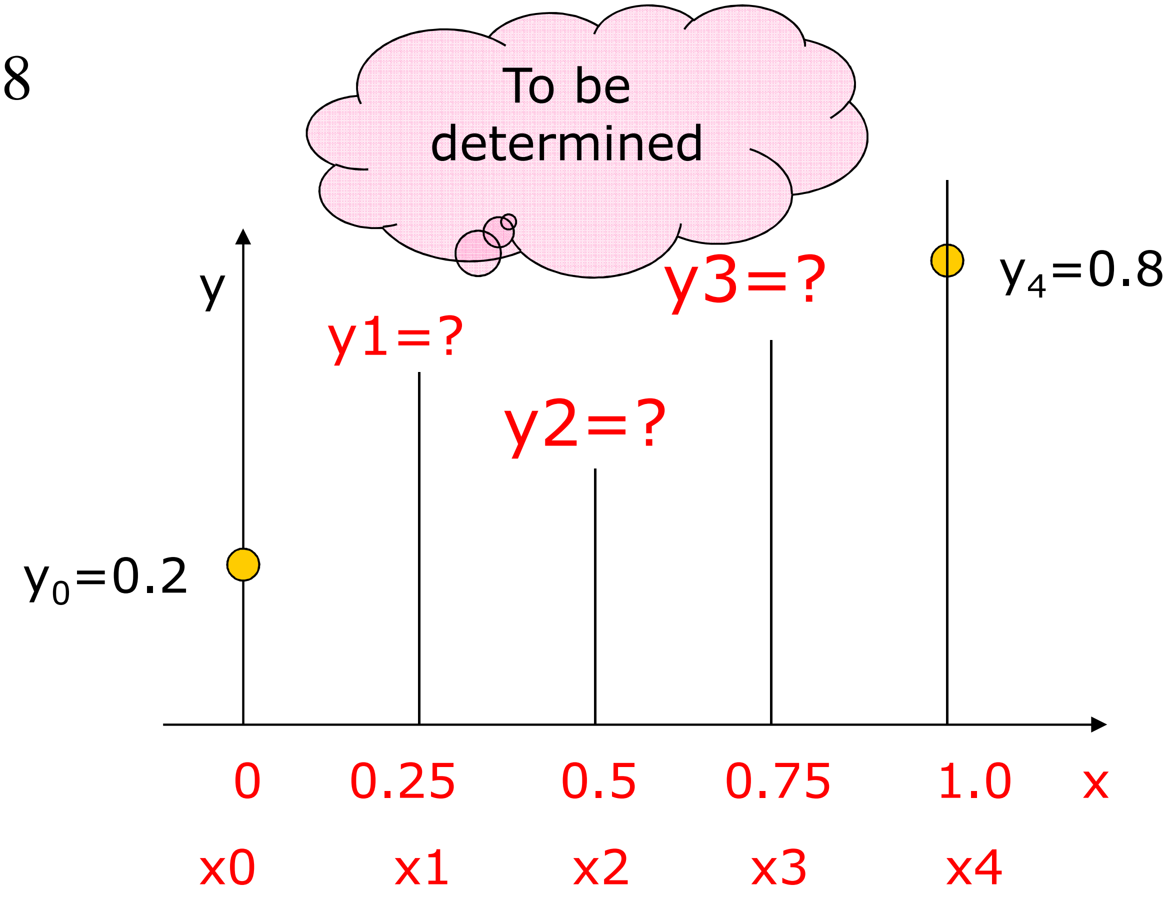
x0=0

x1=0.25

x2=.5

x3=0.75

x4=1.0



Finite Difference Method
Example

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

Divide the interval [0,1] into n = 4 intervals

Base points are

x0=0

x1=0.25

x2=.5

x3=0.75

x4=1.0

Replace

$$\ddot{y} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad \textit{central difference formula}$$

$$\dot{y} = \frac{y_{i+1} - y_{i-1}}{2h} \qquad \textit{central difference formula}$$

$$\ddot{y} + 2\dot{y} + y = x^2$$

Becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_{i-1}}{2h} + y_i = {x_i}^2$$

Second Order BVP

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 \quad with \;\; y(0) = 0.2, \qquad y(1) = 0.8$$

$$Let \quad h = 0.25$$

Base Points

$$x_0 = 0,\; x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$$

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h} = \frac{y_{i+1} - y_i}{h}$$

$$\frac{d^2y}{dx^2} \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Second Order BVP

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_i}{h} + y_i = x_i^2 \quad i = 1,2,3$$

$$x_0 = 0,\ x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$$

$$y_0 = 0.2,\ y_1 = ?, y_2 = ?, y_3 = ?,\quad y_4 = 0.8$$

$$16(y_{i+1} - 2y_i + y_{i-1}) + 8(y_{i+1} - y_i) + y_i = x_i^2$$

$$24y_{i+1} - 39y_i + 16y_{i-1} = x_i^2$$

Second Order BVP

$$24y_{i+1} - 39y_i + 16y_{i-1} = x_i^{\ 2}$$

$$i = 1 \quad 24y_2 - 39y_1 + 16y_0 = x_1^{\ 2}$$

$$i = 2 \quad 24y_3 - 39y_2 + 16y_1 = x_2^{\ 2}$$

$$i = 3 \quad 24y_4 - 39y_3 + 16y_2 = x_3^{\ 2}$$

$$\begin{bmatrix} -39 & 24 & 0 \\ 16 & -39 & 24 \\ 0 & 16 & -39 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.25^2 - 16(0.2) \\ 0.5^2 \\ 0.75^2 - 24(0.8) \end{bmatrix}$$

$$Solution \quad y_1 = 0.4791,\ y_2 = 0.6477,\ y_3 = 0.7436$$

Second Order BVP

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$$

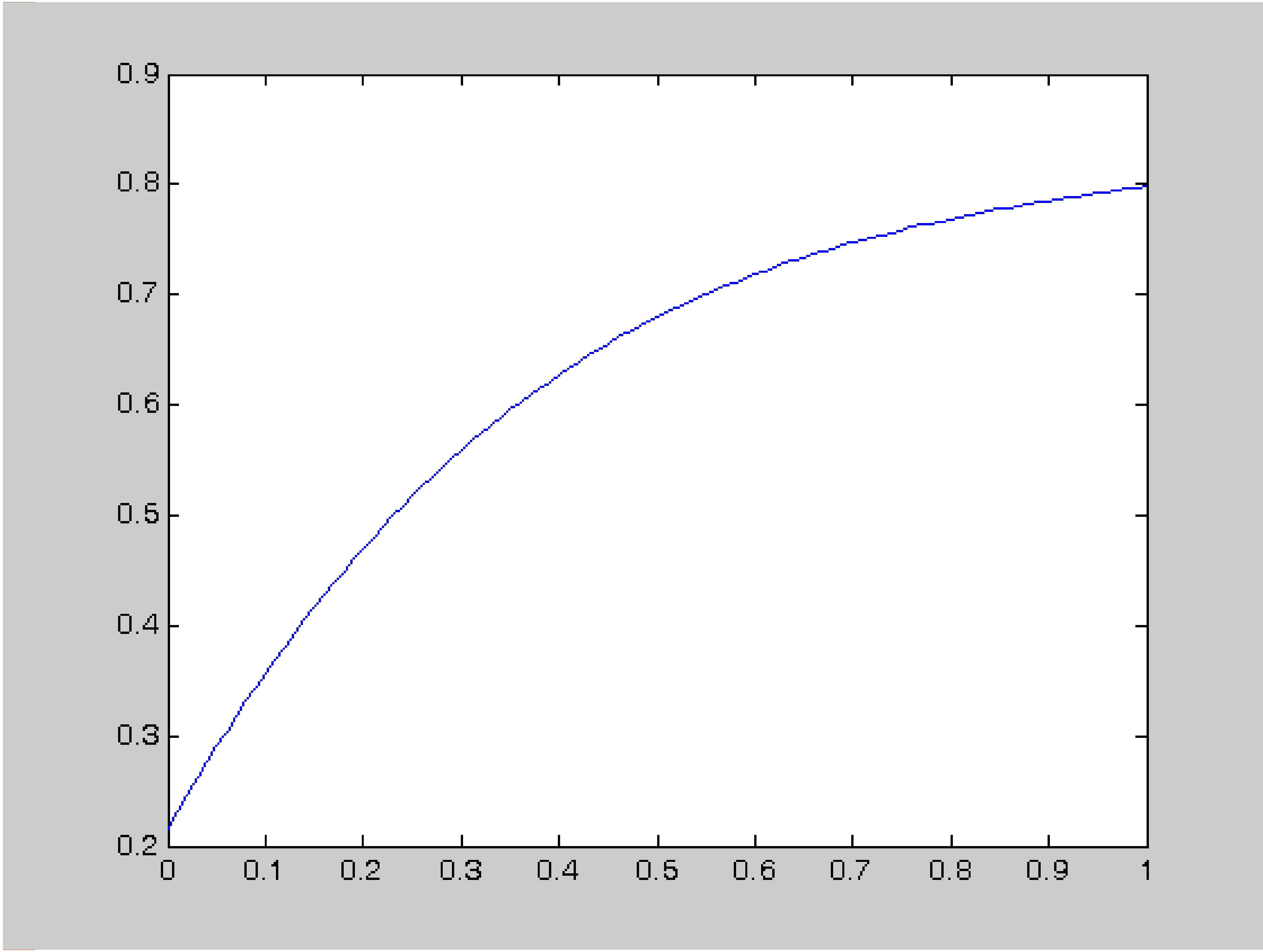
$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_i}{h} + y_i = x_i^{\,2} \quad i = 1,2,\ldots,100$$

$$x_0 = 0,\ x_1 = 0.01, x_2 = 0.02 \quad \ldots \quad x_{99} = 0.99, x_{100} = 1$$

$$y_0 = 0.2,\ y_1 = ?, y_2 = ?, \ \ldots \quad y_{99} = ?, \quad y_{100} = 0.8$$

$$10000\left(y_{i+1} - 2y_i + y_{i-1}\right) + 200\left(y_{i+1} - y_i\right) + y_i = x_i^{\,2}$$

$$10200\,y_{i+1} - 20199\,y_i + 10000\,y_{i-1} = x_i^{\,2}$$


0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Summary of the Discretization Methods

- Select the base points.
- Divide the interval into *n* sub-intervals.
- Use finite approximations to replace the derivatives.
- This approximation results in a set of algebraic equations.
- Solve the equations to obtain the solution of the BVP.

Remarks

<u>Finite Difference Method :</u>

- Different formulas can be used for approximating the derivatives.
- Different formulas lead to different solutions. All of them are approximate solutions.
- For linear second order cases, this reduces to tri-diagonal system.