Lecture 11
Partial Differential Equations

- Partial Differential Equations (PDEs).
- What is a PDE?
- Examples of Important PDEs.
- Classification of PDEs.

Partial Differential Equations

A partial differential equation (**PDE**) is an equation that involves an unknown function and its partial derivatives.

Example :

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

PDE involves two or more independent variables

(in the example x and t are independent variables)

Notation

$$u_{xx} = \frac{\partial^2 \ u(x,t)}{\partial x^2}$$

$$u_{xt} = \frac{\partial^2 \ u(x,t)}{\partial x \ \partial t}$$

Order of the PDE = order of the highest order derivative.

Linear PDE

Classification

A PDE is linear if it is linear in the unknown function and its derivatives

Example of linear PDE :

$$2\,u_{xx} + 1\,u_{xt} + 3\,u_{tt} + 4\,u_x + \cos(2t) = 0$$

$$2\,u_{xx} - 3\,u_t + 4\,u_x = 0$$

Examples of Nonlinear PDE

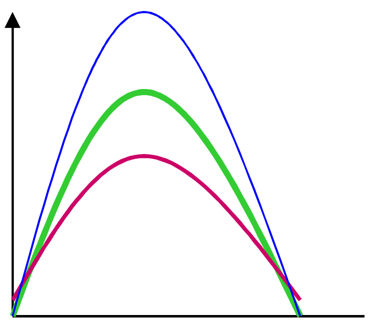
$$2\,u_{xx} + \underline{(u_{xt})^2} + 3\,u_{tt} = 0$$

$$\underline{\sqrt{u_{xx}}} + 2\,u_{xt} + 3u_t = 0$$

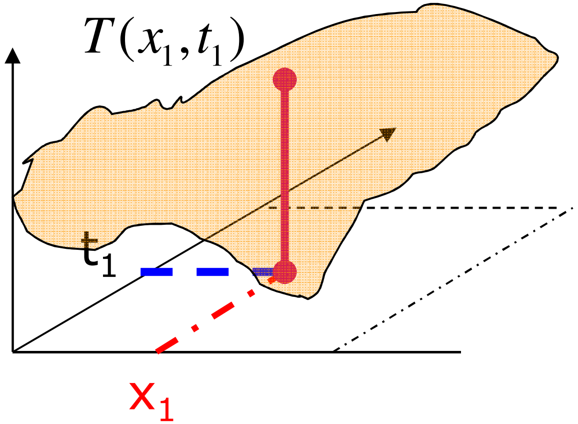
$$2\,u_{xx} + \underline{2\,u_{xt}\,u_t} + 3u_t = 0$$

Representing the Solution of a PDE (Two Independent Variables)

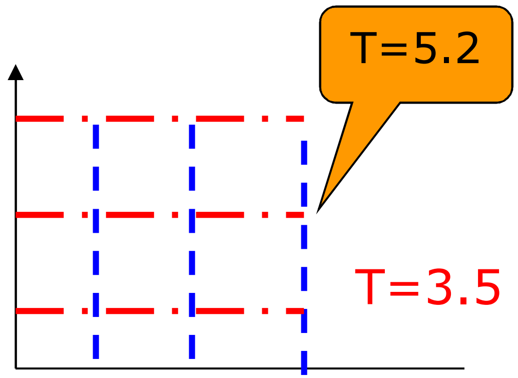
- Three main ways to represent the solution



Different curves are used for different values of one of the independent variable



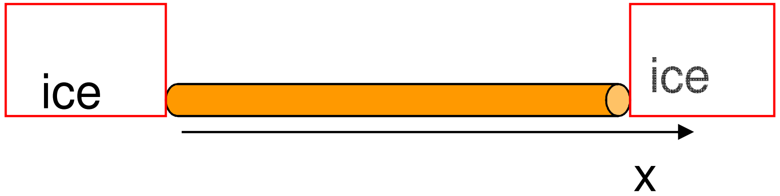
Three dimensional plot of the function T(x,t)



The axis represent the independent variables. The value of the function is displayed at grid points

Heat Equation

Different curve is used for each value of t

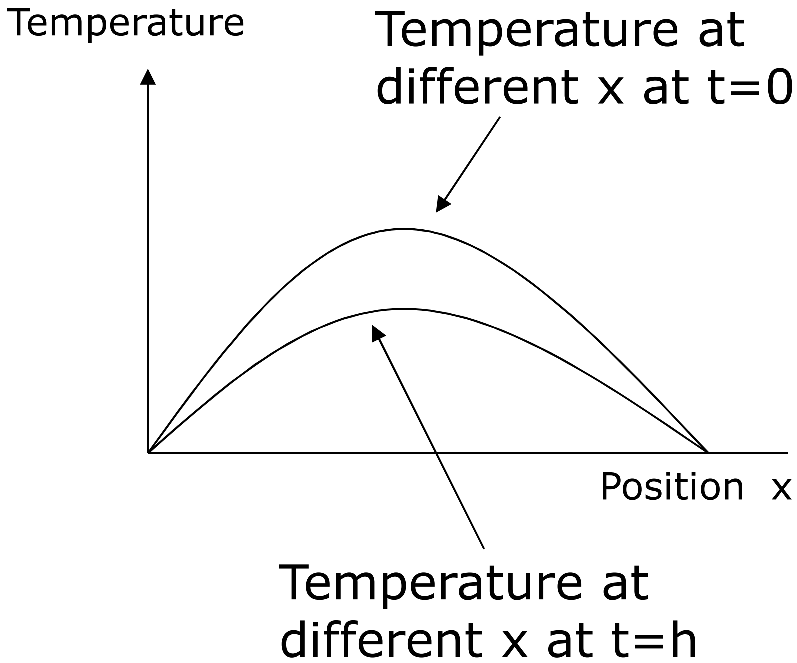


Thin metal rod insulated everywhere except at the edges. At t =0 the rod is placed in ice

$$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0$$

$$T(0,t) = T(1,t) = 0$$

$$T(x,0) = \sin(\pi x)$$



Examples of PDEs

PDEs are used to model many systems in many different fields of science and engineering.

Important Examples:

- Laplace Equation
- Heat Equation
- Wave Equation

Laplace Equation

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

Used to describe the steady state distribution of heat in a body.

Also used to describe the steady state distribution of electric charge (or electric potential) in a body.

Heat Equation (Diffusion Equation)

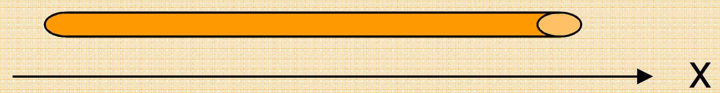
$$\frac{\partial u(x,y,z,t)}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

The function u(x,y,z,t) is used to represent the temperature at time *t* in a physical body at a point with coordinates (x,y,z)

α is the thermal diffusivity. It is sufficient to consider the case α **= 1**.

Simpler Heat Equation

$$\frac{\partial\, T(x,t)}{\partial t} = \frac{\partial^2\, T(x,t)}{\partial x^2}$$



T(x,t) is used to represent the temperature at time *t* at the point *x* of the thin rod.

Wave Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

The function *u(x,y,z,t)* is used to represent the displacement at time t of a particle whose position at rest is *(x,y,z)* .

The constant *c* represents the propagation speed of the wave.

Classification of PDEs

Linear Second order PDEs are important sets of equations that are used to model many systems in many different fields of science and engineering.

Classification is important because:

- Each category relates to specific engineering problems.
- Different approaches are used to solve these categories.

Review of Conic Sections
Classification of Quadratic Curve

A quadratic curve, generally given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

(with A, B, C, D, E, F are real)

is classified based on $(B^2 - 4AC)$ as follows:

$B^2 - 4AC < 0$ Elliptic ex: $x^2 + xy + y^2 - 1 = 0$

$B^2 - 4AC = 0$ Parabolic ex: $3x^2 - 6xy + 3y^2 + 2x - 5 = 0$

$B^2 - 4AC > 0$ Hyperbolic ex: $xy - y^2 - 5y + 1 = 0$

Linear Second Order PDEs
Classification

A second order linear PDE (2-independent variables)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0,$$

A, B, and C are functions of x and y

D is a function of x, y, u, u_x, and u_y

is classified based on $(B^2 - 4AC)$ as follows:

$B^2 - 4AC < 0$	Elliptic
$B^2 - 4AC = 0$	Parabolic
$B^2 - 4AC > 0$	Hyperbolic

Linear Second Order PDE

Examples (Classification)

Laplace Equation $$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$$A = 1, B = 0, C = 1 \Rightarrow B^2 - 4AC < 0$$

$\Rightarrow$ Laplace Equation *is Elliptic*

One possible solution: $u(x,y) = e^x \sin y$

$$u_x = e^x \sin y, \quad u_{xx} = e^x \sin y$$

$$u_y = e^x \cos y, \quad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = 0$$

Linear Second Order PDE

Examples (Classification)

Heat Equation $\alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$

$A = \alpha,\ B = 0,\ C = 0 \Rightarrow B^2 - 4AC = 0$

$\Rightarrow$ Heat Equation *is Parabolic*

Wave Equation $c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = 0$

$A = c^2 > 0,\ B = 0,\ C = -1 \Rightarrow B^2 - 4AC > 0$

$\Rightarrow$ Wave Equation *is Hyperbolic*

Boundary Conditions for PDEs

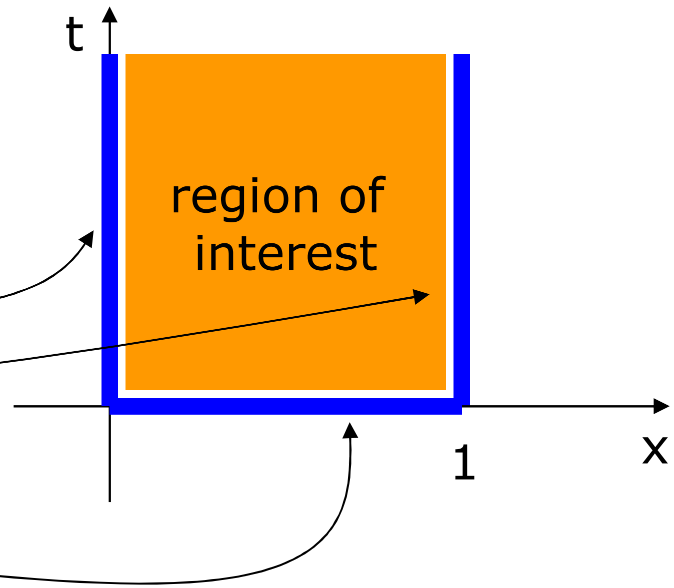
- To uniquely specify a solution to the PDE, a set of boundary conditions are needed.
- Both regular and irregular boundaries are possible.

$$\text{Heat Equation}: \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$



The Solution Methods for PDEs

- Analytic solutions are possible for simple and special (idealized) cases only.
- To make use of the nature of the equations, different methods are used to solve different classes of PDEs.
- The methods discussed here are based on the finite difference technique.

Parabolic Equations

A second order linear PDE (2 - independent variables x , y)

$$A\ u_{xx} + B\ u_{xy} + C\ u_{yy} + D = 0,$$

A, B, and C are functions of x and y

D is a function of x, y, u, u_x, and u_y

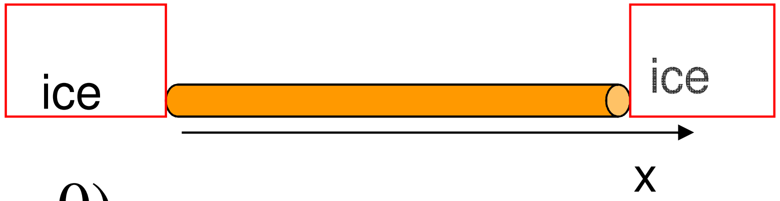
is parabolic if $B^2 - 4AC = 0$

Parabolic Problems

Heat Equation : $$\frac{\partial\, T(x,t)}{\partial t} = \frac{\partial^2\, T(x,t)}{\partial x^2}$$

$$T(0,t) = T(1,t) = 0$$

$$T(x,0) = \sin(\pi\, x)$$



* Parabolic problem $(B^2 - 4AC = 0)$

* Boundary conditions are needed to uniquely specify a solution.

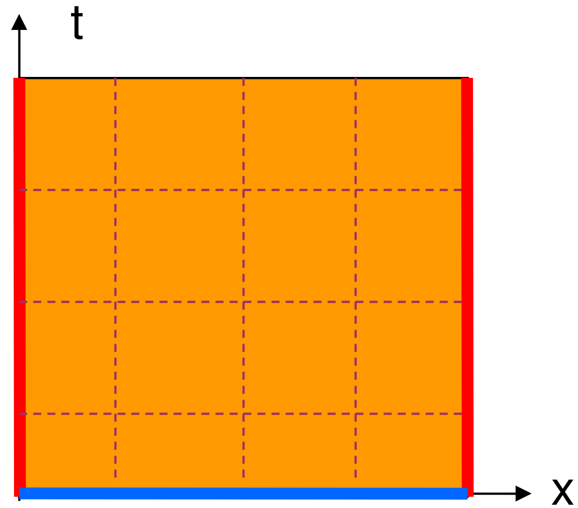
Minority Carrier Diffusion Equation : D_n,D_p denote electron, hole diffusion coefficients

$$\frac{\partial \Delta n_p}{\partial t} = D_n \frac{\partial^2 \Delta n_p}{\partial x^2} - \frac{\Delta n_p}{\tau_n} + G_L$$

$$\frac{\partial \Delta p_n}{\partial t} = D_p \frac{\partial^2 \Delta p_n}{\partial x^2} - \frac{\Delta p_n}{\tau_p} + G_L$$

Finite Difference Methods

- Divide the interval *x* into sub-intervals, each of width *h*
- Divide the interval *t* into sub-intervals, each of width *k*
- A grid of points is used for the finite difference solution
- $T_{i,j}$ represents $T(x_i, t_j)$
- Replace the derivatives by finite-difference formulas



Finite Difference Methods

Replace the derivatives by finite difference formulas

Central Difference Formula for $\dfrac{\partial^2 T}{\partial x^2}$:

$$\frac{\partial^2 T(x,t)}{\partial x^2} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2} = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h^2}$$

Forward Difference Formula for $\dfrac{\partial T}{\partial t}$:

$$\frac{\partial T(x,t)}{\partial t} \approx \frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \frac{T_{i,j+1} - T_{i,j}}{k}$$

Solution of the Heat Equation

- Two solutions to the Parabolic Equation (Heat Equation) will be presented:

1. Explicit Method:

   Simple, Stability Problems.

2. Crank-Nicolson Method:

   Involves the solution of a Tridiagonal system of equations, Stable.

Explicit Method

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

$$\frac{T(x,t+k) - T(x,t)}{k} = \frac{T(x-h,t) - 2T(x,t) + T(x+h,t)}{h^2}$$

$$T(x,t+k) - T(x,t) = \frac{k}{h^2}\left(T(x-h,t) - 2T(x,t) + T(x+h,t)\right)$$

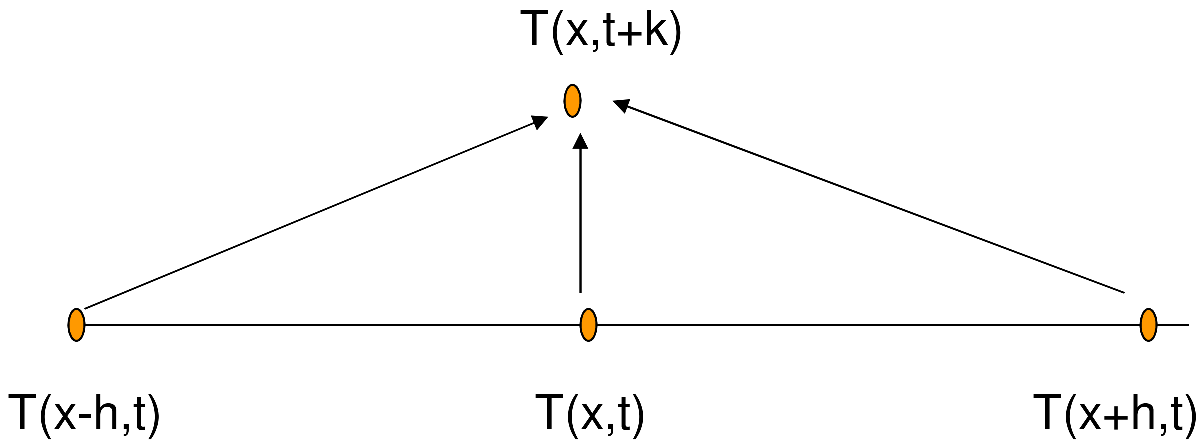
$$Define \quad \lambda = \frac{k}{h^2}$$

$$T(x,t+k) = \lambda\, T(x-h,t) + (1-2\lambda)\, T(x,t) + \lambda\, T(x+h,t)$$

Explicit Method
How Do We Compute?

$$T(x,t+k) = \lambda\, T(x-h,t) + (1-2\lambda)\, T(x,t) + \lambda\, T(x+h,t)$$

means



Convergence and Stability

$T(x, t+k)$ can be computed directly using:

$$T(x, t+k) = \lambda\, T(x-h, t) + (1-2\lambda)\, T(x, t) + \lambda\, T(x+h, t)$$

Can be unstable (errors are magnified)

To guarantee stability, $(1-2\lambda) \geq 0 \Rightarrow \lambda \leq \frac{1}{2} \Rightarrow k \leq \frac{h^2}{2}$

This means that k is much smaller than h

$\Rightarrow$ short time step $\Rightarrow$ "slow" speed.

Convergence and Stability of the Solution

- Convergence

  The solutions converge means that the solution obtained using the finite difference method approaches the true solution as the steps Δx and Δt approach zero.

- Stability:

  An algorithm is stable if the errors at each stage of the computation are not magnified as the computation progresses.

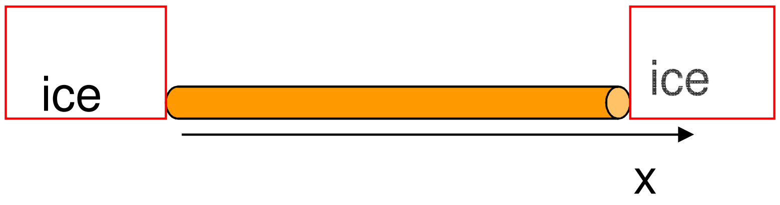
Example 1: Heat Equation

Solve the PDE :

$$\frac{\partial^2 u(x,t)}{\partial^2 x} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$



$Use \quad h = 0.25, \quad k = 0.25$ to find $u(x,t)$ for $x \in [0,1], t \in [0,1]$

$$\lambda = \frac{k}{h^2} = 4$$

Example 1

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

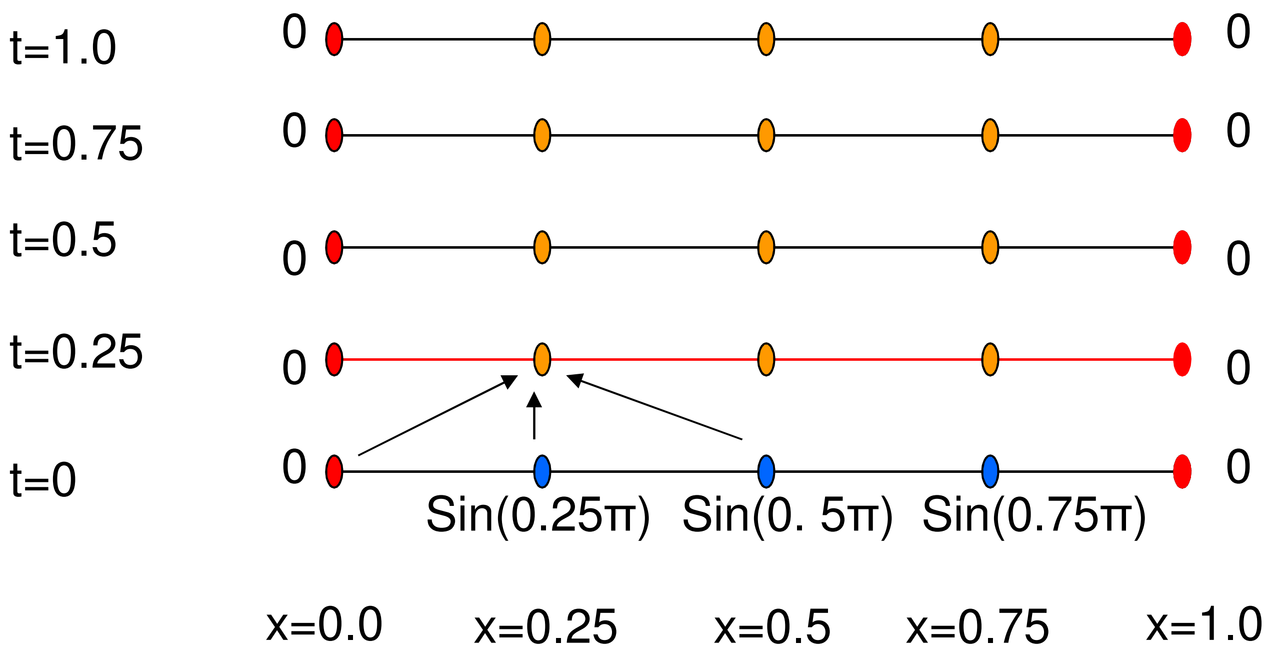
$$\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} - \frac{u(x,t+k) - u(x,t)}{k} = 0$$

$$16\big(u(x-h,t) - 2u(x,t) + u(x+h,t)\big) - 4\big(u(x,t+k) - u(x,t)\big) = 0$$

$$u(x,t+k) = 4\ u(x-h,t) - 7\ u(x,t) + 4\ u(x+h,t)$$

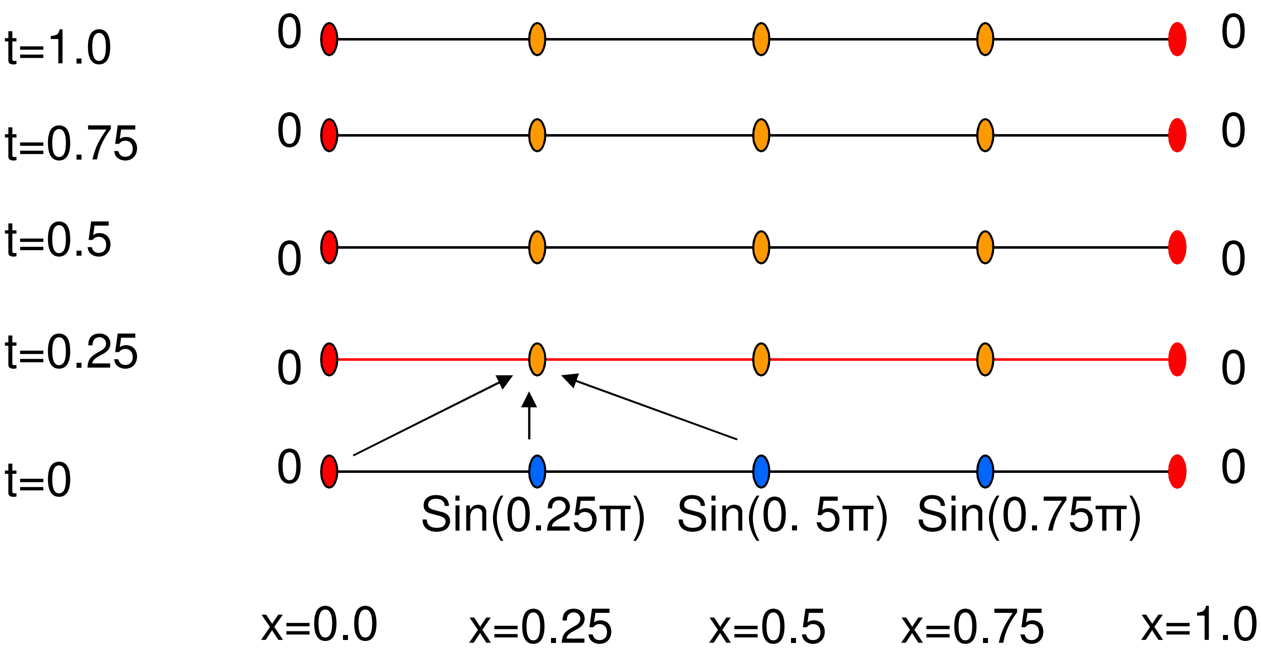
Example 1

$$u(x,t+k) = 4\ u(x-h,t) - 7\ u(x,t) + 4\ u(x+h,t)$$



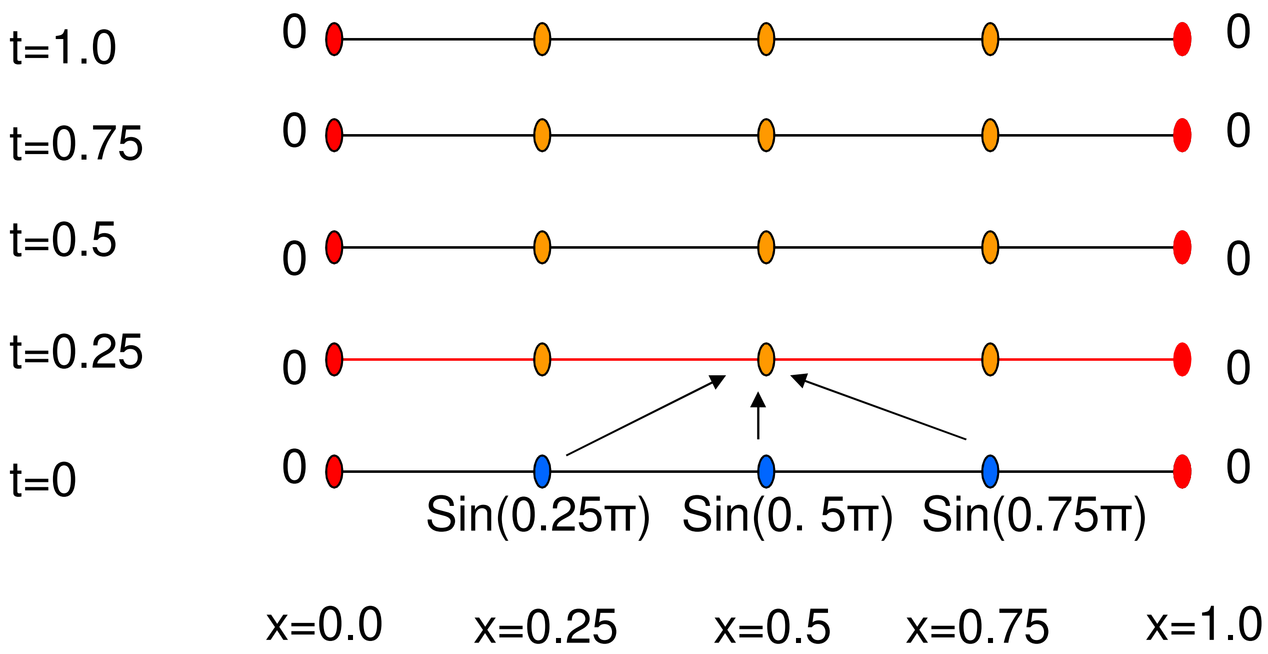
Example 1

$$u(0.25,0.25) = 4\ u(0,0) - 7\ u(0.25,0) + 4\ u(0.5,0)$$
$$= 0 - 7\sin(\pi/4) + 4\sin(\pi/2) = -0.9497$$



Example 1

$$u(0.5,0.25) = 4\ u(0.25,0) - 7\ u(0.5,0) + 4\ u(0.75,0)$$
$$= 4\sin(\pi/4) - 7\sin(\pi/2) + 4\sin(3\pi/4) = -0.1716$$



Remarks on Example 1

The obtained results are probably not accurate

because: $1 - 2\lambda = -7$

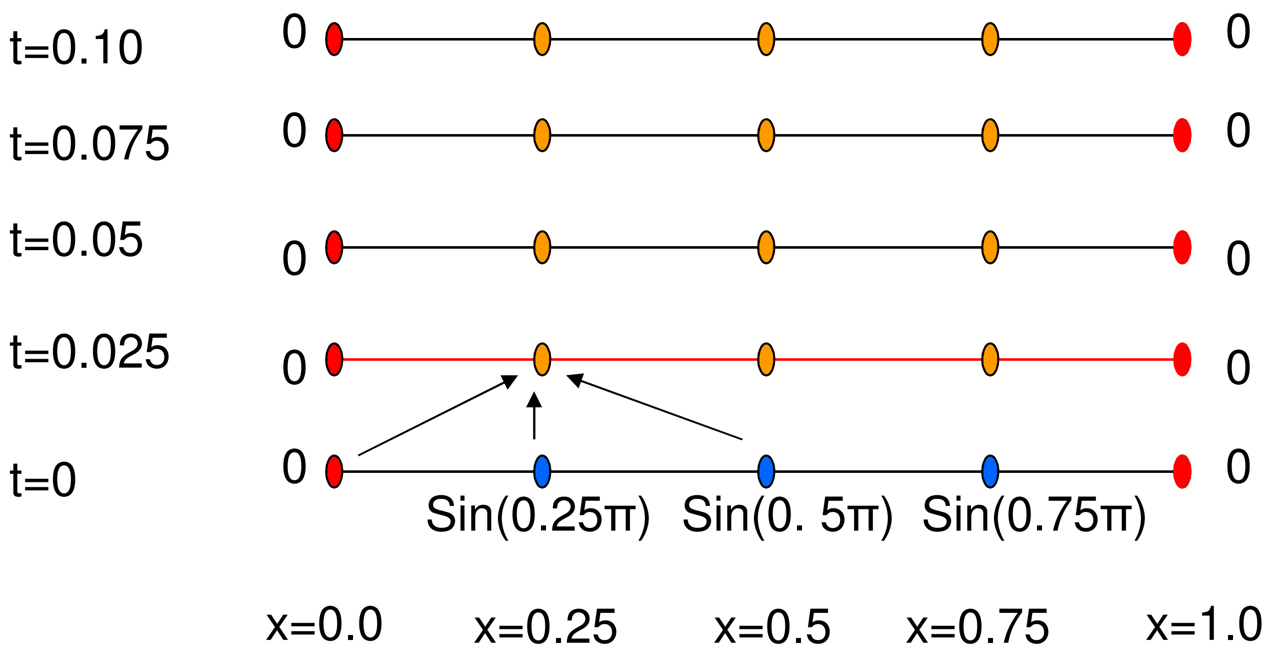
For accurate results: $1 - 2\lambda \geq 0$

One needs to select $k \leq \dfrac{h^2}{2} = \dfrac{(0.25)^2}{2} = 0.03125$

For example, choose $k = 0.025$, then $\lambda = \dfrac{k}{h^2} = 0.4$

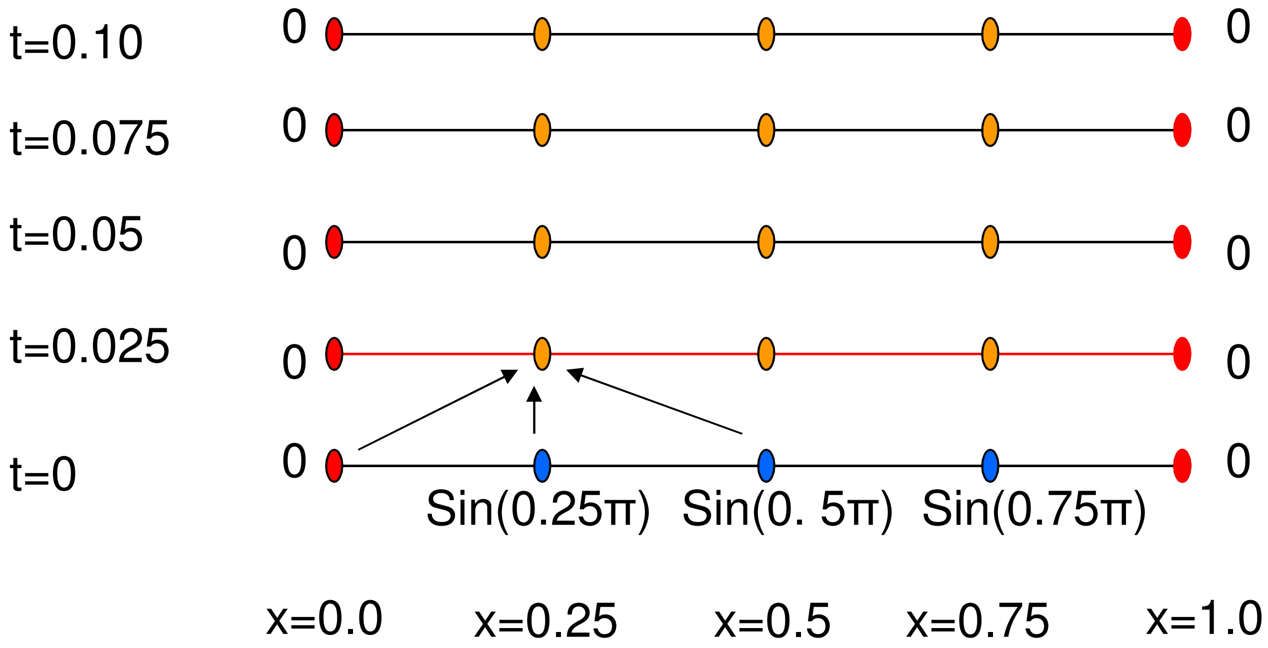
Example 1 – cont'd

$$u(x,t+k)=0.4\ u(x-h,t)+0.2\ u(x,t)+0.4\ u(x+h,t)$$



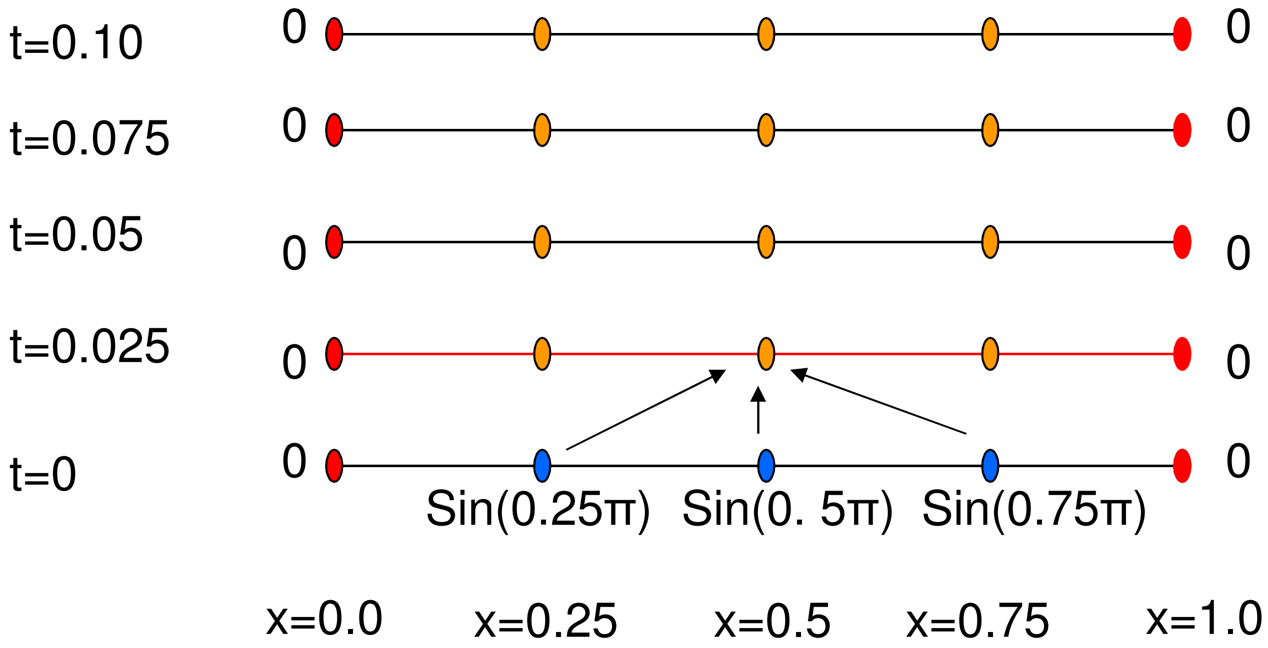
Example 1 – cont'd

$$u(0.25, 0.025) = 0.4\ u(0,0) + 0.2\ u(0.25,0) + 0.4\ u(0.5,0)$$
$$= 0 + 0.2\sin(\pi/4) + 0.4\sin(\pi/2) = 0.5414$$



Example 1 – cont'd

$$u(0.5,0.025) = 0.4\ u(0.25,0) + 0.2\ u(0.5,0) + 0.4\ u(0.75,0)$$
$$= 0.4\sin(\pi/4) + 0.2\sin(\pi/2) + 0.4\sin(3\pi/4) = 0.7657$$



Crank-Nicolson Method

The method involves solving a Tridiagonal system of linear equations.

The method is stable (No magnification of error).

$\rightarrow$ We can use larger h, k (compared to the Explicit Method).

Based on the finite difference method

1. Divide the interval x into subintervals of width h

2. Divide the interval t into subintervals of width k

3. Replace the first and second partial derivatives with their *backward* and *central difference* formulas respectively :

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t) - u(x,t-k)}{k}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2}$$

Crank-Nicolson Method

Heat Equation : $\dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t}$ becomes

$$\frac{u(x-h,t)-2u(x,t)+u(x+h,t)}{h^2} = \frac{u(x,t)-u(x,t-k)}{k}$$

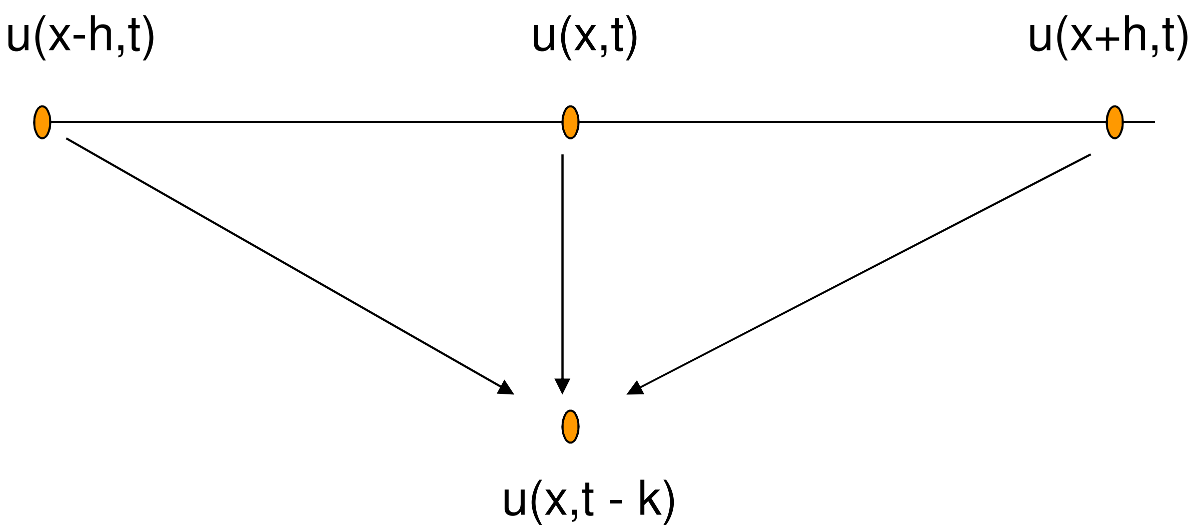
$$\frac{k}{h^2}\big(u(x-h,t)-2u(x,t)+u(x+h,t)\big) = u(x,t)-u(x,t-k)$$

$$-\frac{k}{h^2}u(x-h,t)+(1+2\frac{k}{h^2})\,u(x,t)-\frac{k}{h^2}u(x+h,t) = u(x,t-k)$$

Crank-Nicolson Method

Define $\lambda = \dfrac{k}{h^2}$ then Heat equation becomes :

$$-\lambda\, u(x-h,t) + (1+2\lambda)\, u(x,t) - \lambda\, u(x+h,t) = u(x,t-k)$$



Crank-Nicolson Method

The equation:

$$-\lambda\, u(x-h,t)+(1+2\lambda)\, u(x,t)-\lambda\, u(x+h,t)=u(x,t-k)$$

can be rewritten as:

$$-\lambda\, u_{i-1,j}+(1+2\lambda)\, u_{i,j}-\lambda\, u_{i+1,j}=u_{i,j-1}$$

and can be expanded as a system of equations (fix $j=1$):

$$-\lambda\, u_{0,1}+(1+2\lambda)\, u_{1,1}-\lambda\, u_{2,1}=u_{1,0}$$

$$-\lambda\, u_{1,1}+(1+2\lambda)\, u_{2,1}-\lambda\, u_{3,1}=u_{2,0}$$

$$-\lambda\, u_{2,1}+(1+2\lambda)\, u_{3,1}-\lambda\, u_{4,1}=u_{3,0}$$

$$-\lambda\, u_{3,1}+(1+2\lambda)\, u_{4,1}-\lambda\, u_{5,1}=u_{4,0}$$

Crank-Nicolson Method

$$-\lambda\, u(x-h,t)+(1+2\lambda)\, u(x,t)-\lambda\, u(x+h,t)=u(x,t-k)$$

can be expressed as a Tridiagonal system of equations:

$$\begin{bmatrix} 1+2\lambda & -\lambda & & \\ -\lambda & 1+2\lambda & -\lambda & \\ & -\lambda & 1+2\lambda & -\lambda \\ & & -\lambda & 1+2\lambda \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{bmatrix} = \begin{bmatrix} u_{1,0}+\lambda\, u_{0,1} \\ u_{2,0} \\ u_{3,0} \\ u_{4,0}+\lambda\, u_{5,1} \end{bmatrix}$$

where $u_{1,0}, u_{2,0}, u_{3,0}$, and $u_{4,0}$ are the initial temperature values

at $x = x_0 + h,\ x_0 + 2h,\ x_0 + 3h,$ and $x_0 + 4h$

$u_{0,1}$ and $u_{5,1}$ are the boundary values at $x = x_0$ and $x_0 + 5h$

Crank-Nicolson Method

The solution of the tridiagonal system produces:

The temperature values $u_{1,1}, u_{2,1}, u_{3,1}$, and $u_{4,1}$ at $t = t_0 + k$

To compute the temperature values at $t = t_0 + 2k$

Solve a second tridiagonal system of equations ($j = 2$)

$$\begin{bmatrix} 1+2\lambda & -\lambda & & \\ -\lambda & 1+2\lambda & -\lambda & \\ & -\lambda & 1+2\lambda & -\lambda \\ & & -\lambda & 1+2\lambda \end{bmatrix} \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} = \begin{bmatrix} u_{1,1} + \lambda\, u_{0,2} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} + \lambda\, u_{5,2} \end{bmatrix}$$

To compute $u_{1,2}, u_{2,2}, u_{3,2}$, and $u_{4,2}$

Repeat the above step to compute temperature values at $t_0 + 3k$, etc.

Example 2

Solve the PDE :

$$\frac{\partial^2 u(x,t)}{\partial^2 x} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi\, x)$$

Solve using Crank - Nicolson method

$Use \quad h = 0.25, \quad k = 0.25$ to find $u(x,t)$ for $x \in [0,1], t \in [0,1]$

Example 2

Crank-Nicolson Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} = \frac{u(x,t) - u(x,t-k)}{k}$$

$$16\big(u(x-h,t) - 2u(x,t) + u(x+h,t)\big) - 4\big(u(x,t) - u(x,t-k)\big) = 0$$

$$Define \quad \lambda = \frac{k}{h^2} = 4$$

$$-4\,u(x-h,t) + 9\,u(x,t) - 4\,u(x+h,t) = u(x,t-k)$$

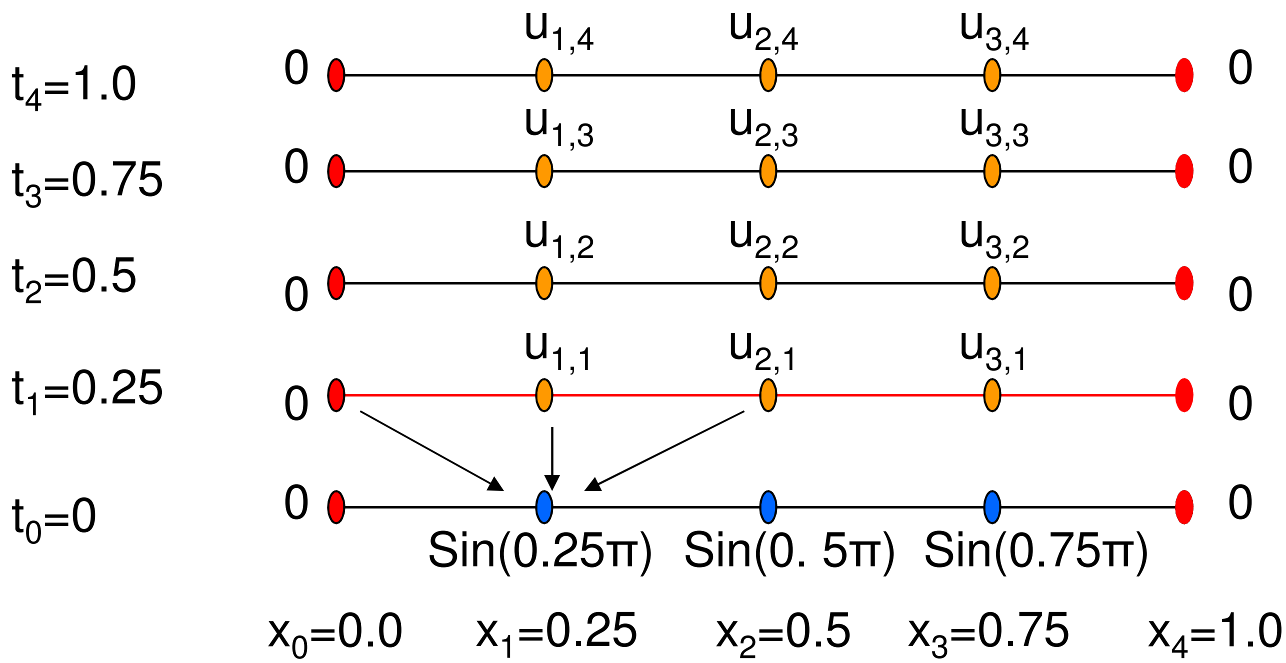
$$-4\,u_{i-1,j} + 9\,u_{i,j} - 4\,u_{i+1,j} = u_{i,j-1}$$

Example 2

$$-4u_{0,1}+9u_{1,1}-4u_{2,1}=u_{1,0} \Rightarrow \quad 9u_{1,1}-4u_{2,1} = \sin(\pi/4)$$

$$-4u_{1,1}+9u_{2,1}-4u_{3,1}=u_{2,0} \Rightarrow -4u_{1,1}+9u_{2,1}-4u_{3,1} = \sin(\pi/2)$$

$$-4u_{2,1}+9u_{3,1}-4u_{4,1}=u_{3,0} \Rightarrow \quad -4u_{2,1}+9u_{3,1} = \sin(3\pi/4)$$



Example 2

Solution of Row 1 at t1=0.25 sec

The Solution of the PDE at $t_1 = 0.25$ sec is the solution of the following tridiagonal system of equations:

$$\begin{bmatrix} 9 & -4 & \\ -4 & 9 & -4 \\ & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} \sin(0.25\pi) \\ \sin(0.5\pi) \\ \sin(0.75\pi) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 0.21151 \\ 0.29912 \\ 0.21151 \end{bmatrix}$$

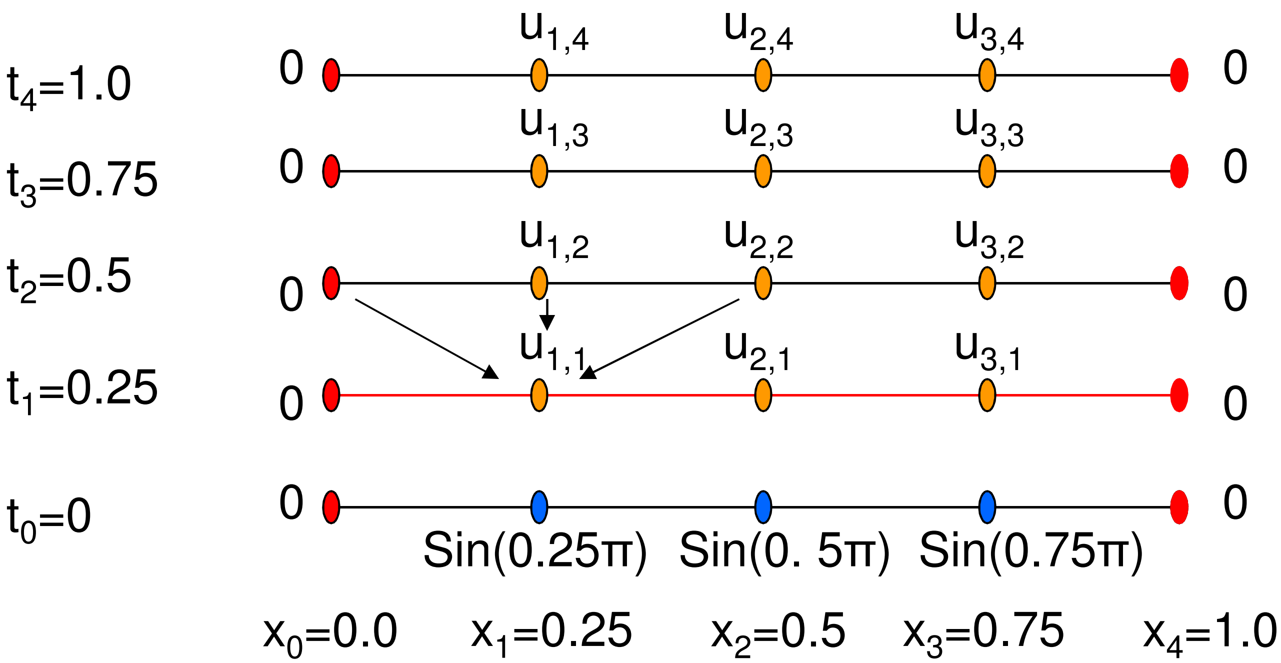
Example 2:

Second Row at t2=0.5 sec

$$-4u_{0,2}+9u_{1,2}-4u_{2,2}=u_{1,1}\Rightarrow\quad 9u_{1,2}-4u_{2,2}=0.21151$$

$$-4u_{1,2}+9u_{2,2}-4u_{3,2}=u_{2,1}\Rightarrow -4u_{1,2}+9u_{2,2}-4u_{3,2}=0.29912$$

$$-4u_{2,2}+9u_{3,2}-4u_{4,2}=u_{3,1}\Rightarrow\quad -4u_{2,2}+9u_{3,2}=0.21151$$



Example 2

Solution of Row 2 at t2=0.5 sec

The Solution of the PDE at $t_2 = 0.5$ sec is the solution of the following tridiagonal system of equations:

$$\begin{bmatrix} 9 & -4 & \\ -4 & 9 & -4 \\ & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 0.21151 \\ 0.29912 \\ 0.21151 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} 0.063267 \\ 0.089473 \\ 0.063267 \end{bmatrix}$$

Example 2

Solution of Row 3 at t3=0.75 sec

The Solution of the PDE at $t_3 = 0.75$ sec is the solution of the following tridiagonal system of equations :

$$\begin{bmatrix} 9 & -4 & \\ -4 & 9 & -4 \\ & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} 0.063267 \\ 0.089473 \\ 0.063267 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} 0.018924 \\ 0.026763 \\ 0.018924 \end{bmatrix}$$

Example 2

Solution of Row 4 at t4=1 sec

The Solution of the PDE at $t_4 = 1$ sec is the solution of the following tridiagonal system of equations:

$$\begin{bmatrix} 9 & -4 & \\ -4 & 9 & -4 \\ & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{1,4} \\ u_{2,4} \\ u_{3,4} \end{bmatrix} = \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} 0.018924 \\ 0.026763 \\ 0.018924 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{1,4} \\ u_{2,4} \\ u_{3,4} \end{bmatrix} = \begin{bmatrix} 0.0056606 \\ 0.0080053 \\ 0.0056606 \end{bmatrix}$$

Remarks

The Explicit Method:

•One needs to select small k to ensure **stability.**

•Computation per point is very simple but many points are needed.

Cranks Nicolson:

- Requires the solution of a **Tridiagonal** system.
- Stable (Larger k can be used).

Elliptic Equations

A second order linear PDE (2 - independent variables x , y)

$$A\, u_{xx} + B\, u_{xy} + C\, u_{yy} + D = 0,$$

A, B, and C are functions of x and y

D is a function of x, y, u, u_x, and u_y

is Elliptic if $B^2 - 4AC < 0$

Laplace Equation

Laplace equation appears in several engineering problems such as:

- Studying the steady state distribution of heat in a body.
- Studying the steady state distribution of electrical charge in a body.

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = f(x,y)$$

T : steady state temperature at point (x, y)

$f(x,y)$: heat source (or heat sink)

Laplace Equation

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = f(x,y)$$

$$A = 1, B = 0, C = 1$$

$$B^2 - 4AC = -4 \quad < 0 \quad Elliptic$$

- Temperature is a function of the position (x and y)
- When no heat source is available → *f(x,y)=0*
- In Electrostatics: Poisson's & Laplace's EQ

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \nabla^2 V = \Delta V = -\frac{\rho_v}{\varepsilon}$$

Laplacian Operator

Solution Technique

- A grid is used to divide the region of interest.
- Since the PDE is satisfied at each point in the area, it must be satisfied at each point of the grid.
- A finite difference approximation is obtained at each grid point.

$$\frac{\partial^2 T(x,y)}{\partial x^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}, \quad \frac{\partial^2 T(x,y)}{\partial y^2} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

Solution Technique

$$\frac{\partial^2 \; T(x,y)}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2},$$

$$\frac{\partial^2 \; T(x,y)}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

$$\Rightarrow \frac{\partial^2 \; T(x,y)}{\partial x^2} + \frac{\partial^2 \; T(x,y)}{\partial y^2} = 0$$

is approximated by:

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

Solution Technique

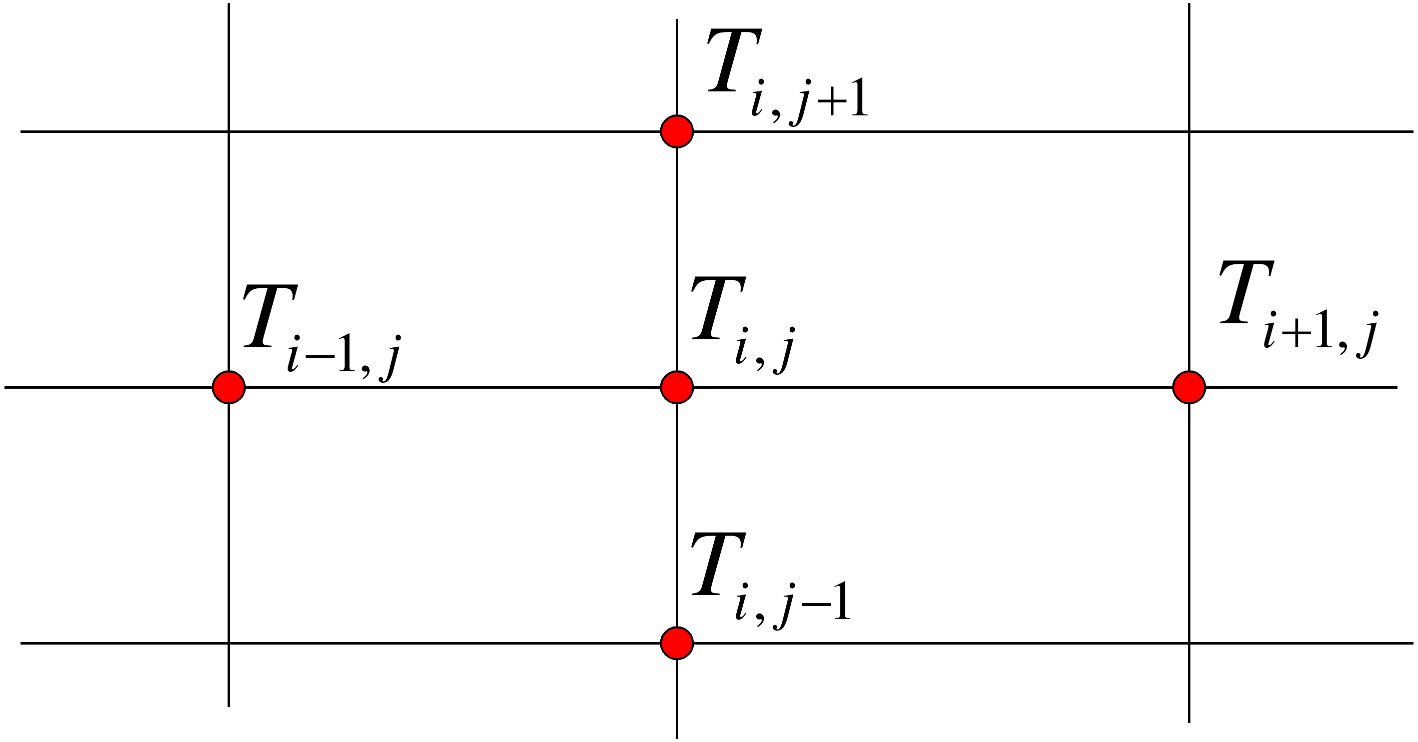
$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

$$(Laplacian\ Difference\ Equation)$$

$$Assume: \Delta x = \Delta y = h$$

$$\Rightarrow T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

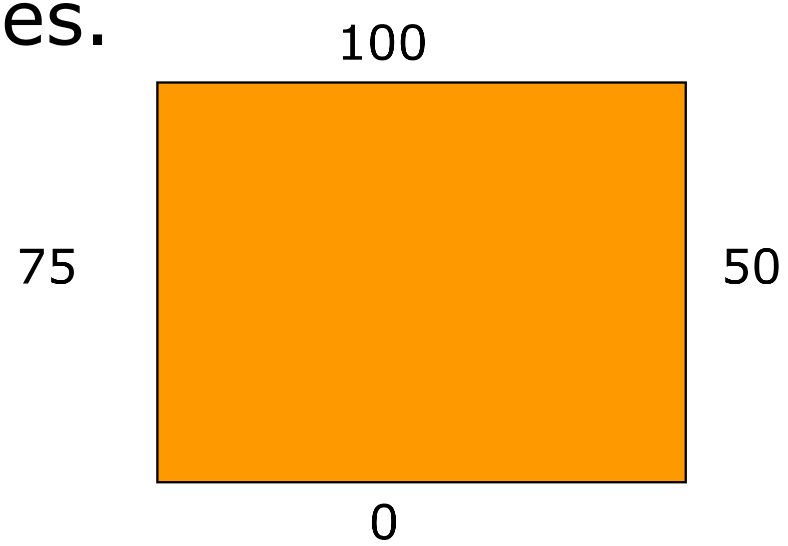
Solution Technique



$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

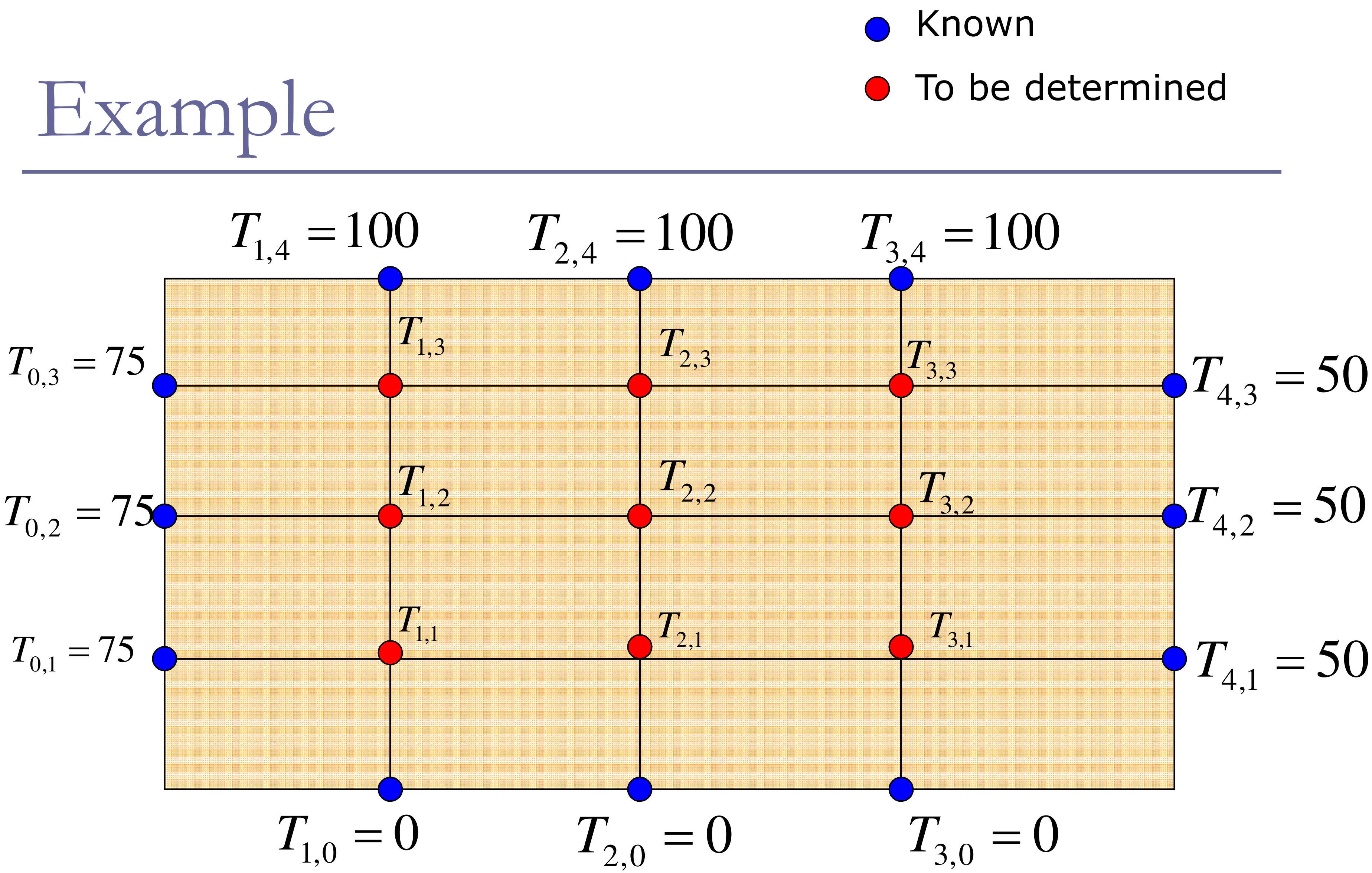
Example

It is required to determine the steady state temperature at all points of a heated sheet of metal. The edges of the sheet are kept at a constant temperature: 100, 50, 0, and 75 degrees.



The sheet is divided to 5X5 grids.

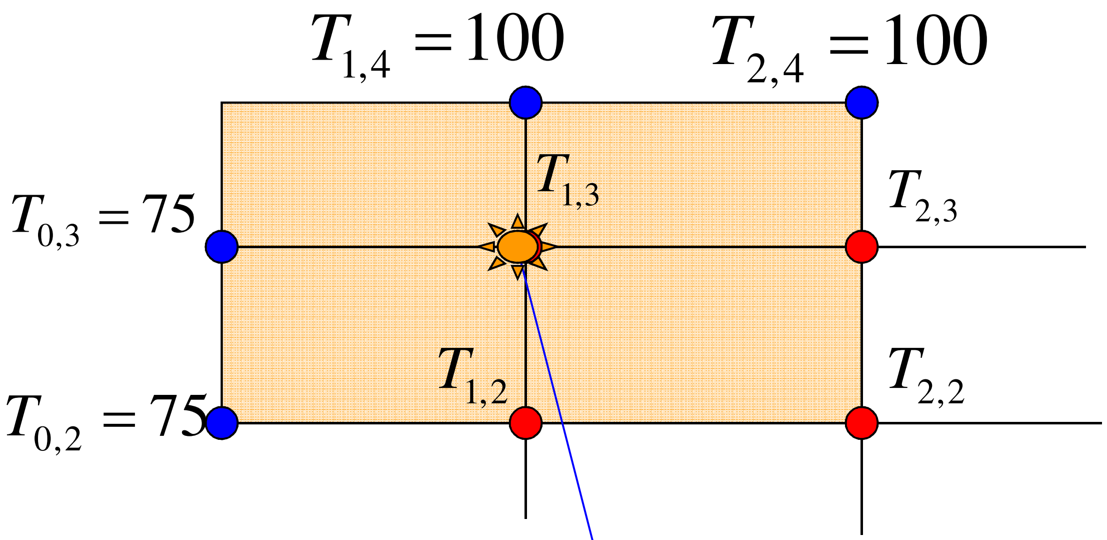
Example



Known
To be determined
$T_{1,4} = 100$
$T_{2,4} = 100$
$T_{3,4} = 100$
$T_{0,3} = 75$
$T_{0,2} = 75$
$T_{0,1} = 75$
$T_{1,3}$
$T_{2,3}$
$T_{3,3}$
$T_{1,2}$
$T_{2,2}$
$T_{3,2}$
$T_{1,1}$
$T_{2,1}$
$T_{3,1}$
$T_{4,3} = 50$
$T_{4,2} = 50$
$T_{4,1} = 50$
$T_{1,0} = 0$
$T_{2,0} = 0$
$T_{3,0} = 0$

First Equation

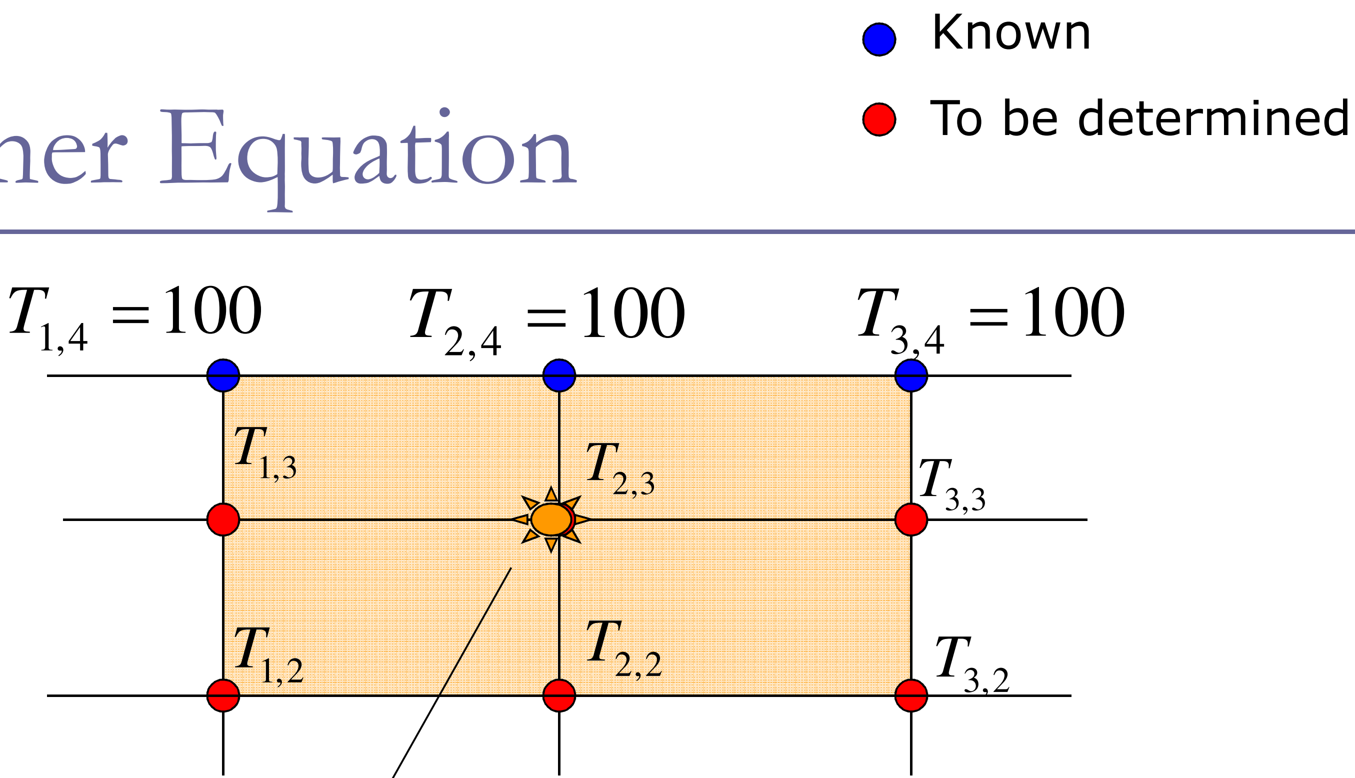
- Known
- To be determined



$$T_{0,3} + T_{1,4} + T_{1,2} + T_{2,3} - 4T_{1,3} = 0$$

$$75 + 100 + T_{1,2} + T_{2,3} - 4T_{1,3} = 0$$

Another Equation



$$T_{1,3} + T_{2,4} + T_{3,3} + T_{2,2} - 4T_{2,3} = 0$$

$$T_{1,3} + 100 + T_{3,3} + T_{2,2} - 4T_{2,3} = 0$$

Solution

The Rest of the Equations

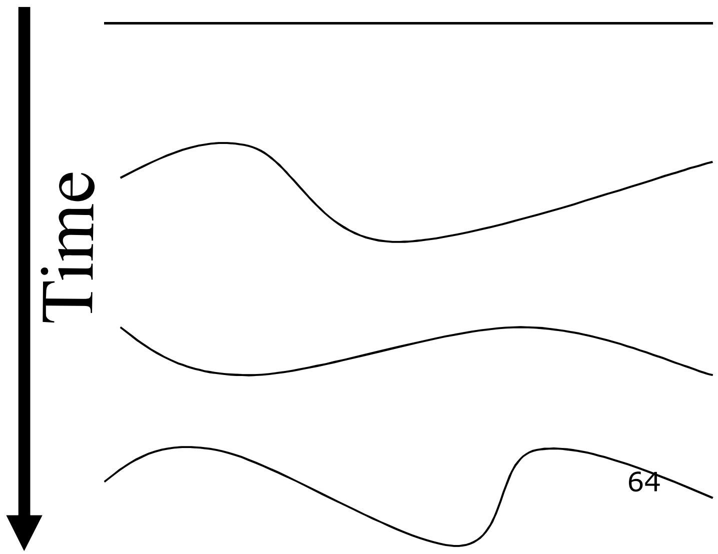
$$
\begin{pmatrix}
4 & -1 & 0 & -1 & & & & & \\
-1 & 4 & -1 & 0 & -1 & & & & \\
0 & -1 & 4 & 0 & 0 & -1 & & & \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & & \\
 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & \\
 & & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
 & & & -1 & 0 & 0 & 4 & -1 & 0 \\
 & & & & -1 & 0 & -1 & 4 & -1 \\
 & & & & & -1 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix}
T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{1,2} \\ T_{2,2} \\ T_{3,2} \\ T_{1,3} \\ T_{2,3} \\ T_{3,3}
\end{pmatrix}
=
\begin{pmatrix}
75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150
\end{pmatrix}
$$

Hyperbolic PDE

- A continuously-vibrating violin or guitar string.
- Acoustic waves inside pipe or horn.
- Electromagnetic wave in space.
- Voltage across transmission Line.

- Example – wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$



Hyperbolic Problems

Wave Equation : $$\frac{\partial u^2(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} ;0 \le x \le 1, t \ge 0$$

$u(0,t) = u(1,t) = 0$ (Boundary conditions)

$u(x,0) = f(x)$ (Given initial displacement)

$u_t(x,0) = g(x)$ (Given initial velocity)

* Hyperbolic problem $(B^2 - 4AC < 0)$

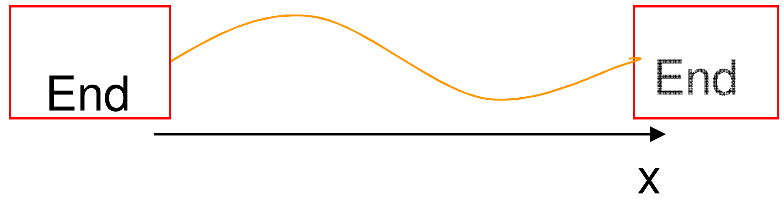
* Boundary conditions are needed to uniquely specify a solution.

* Note that $u_{tt} = c^2 u_{xx}$ can be reduced to $u_{tt} = u_{xx}$ by a linear transformation of x and t.

Hyperbolic Problems
Vibrating string

Wave Equation : $$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

$0 \le x \le 1, t \ge 0$



$u(x,0) = f(x)$

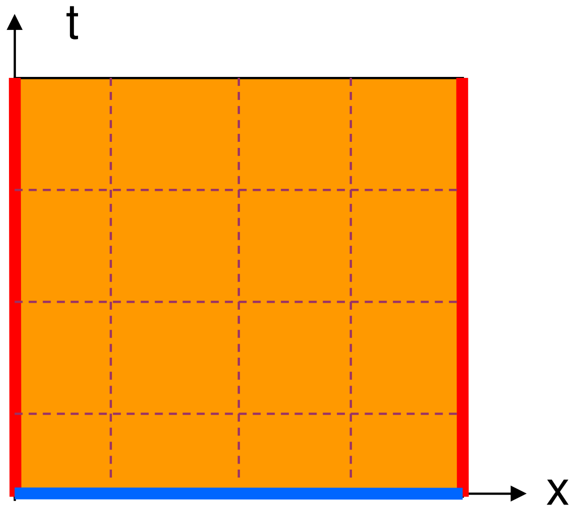
$u_t(x,0) = g(x)$

$u(0,t) = u(1,t) = 0$

* Hyperbolic problem $(B^2 - 4AC > 0)$

Finite Difference Methods (as before)

- Divide the interval *x* into sub-intervals, each of width *h*
- Divide the interval *t* into sub-intervals, each of width *k*
- A grid of points is used for the finite difference solution
- $u_{i,j}$ represents $u(x_i, t_j)$
- Replace the derivatives by finite-difference formulas



Finite Difference Methods

Replace the derivatives by finite difference formulas

Central Difference Formula for $\dfrac{\partial^2 u}{\partial x^2}$:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Central Difference Formula for $\dfrac{\partial^2 u}{\partial t^2}$:

$$\frac{\partial^2 u(x,t)}{\partial t^2} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta t)^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Finite Difference Methods (Cont.)

Thus

$$\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{k^2}=\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}$$

Choosing $r^*=k^2/h^2=1$ yields

$$u_{i,j+1}=u_{i-1,j}+u_{i+1,j}-u_{i,j-1}$$

Furthermore, from $u_t(x,0)=g(x)$

$$\frac{1}{2k}\left(u_{i,1}-u_{i,-1}\right)=g_i \Rightarrow u_{i,-1}=u_{i,1}-2kg_i$$

where $g_i=g(ih)$. Since $u_{i,1}=u_{i-1,0}+u_{i+1,0}-u_{i,-1}$,

$$u_{i,1}=\frac{1}{2}\left(u_{i-1,0}+u_{i+1,0}\right)+kg_i$$

Example: $f(x)=\sin \pi x$, $g(x)=0$, $h=k=0.2$

Initial Condition : $u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right)$, then

$u_{11} = \left(u_{00} + u_{20}\right)/2 = 0.951057/2 = 0.475528$

$u_{21} = \left(u_{10} + u_{30}\right)/2 = 1.538842/2 = 0.769421$

$u_{31} = u_{21}, u_{41} = u_{11}$ by symmetry.

Using $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

along with $u_{0,j} = 0$ and $j = 1$ yields

$u_{12} = u_{01} + u_{21} - u_{10} = 0.769421 - 0.587785 = 0.181636$

$u_{22} = u_{11} + u_{31} - u_{20} = 0.475528 + 0.769421 - 0.951057 = 0.293892$

and $u_{32} = u_{22}, u_{42} = u_{12}$

Solution

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
0.0	0	0.588	0.951	0.951	0.588	0
0.2	0	0.476	0.769	0.769	0.476	0
0.4	0	0.182	0.294	0.294	0.182	0
0.6	0	−0.182	−0.294	−0.294	−0.182	0
0.8	0	−0.476	−0.769	−0.769	−0.476	0
1.0	0	−0.588	−0.951	−0.951	−0.588	0

Exact solution : $u(x,t) = \sin \pi x \cos \pi t$

Helmholtz equation

Wave equation

$$\nabla^2 u(x,y,z,t) = \frac{1}{c^2}\frac{\partial^2 u(x,y,z,t)}{\partial^2 t}, \quad x,y,z \in D$$

Time domain

Fourier transformation

$$(\nabla^2 + k^2)u(x,y,z) = 0, \quad x \in D$$

Frequency domain

Helmholtz equation

Solution to Helmholtz Equation

- Finite Difference Method (Similar to Laplace’s equation)
- Finite Element Method (FEM)
- Boundary Element Method (BEM)