Function Minimization Algorithms

- Quine McClusky Method (Q-M Method, Tabular Method)
  - Use adjacency property, e.g., *ab*+*ab'*=a.
- Iterated Consensus
  - Use consensus operation and absorption property, e.g., *ab ¢ a'c*=*bc, a*+*ab*=a.
- Prime Implicant Table
- Multiple-output Problems
- Cubical Technique

General Strategy

1. Create the prime implicant (PI) table (or chart) using either Quine-McCluskey method or iterated consensus method.

2. (a) Find all essential PI and then “minimum” SOP solution.

   (b) Apply Petrick’s method to find a “minimum” SOP solution.

Recall that in K-map, first find all prime implicants, then essential PI’s, finally determine the minimum SOP solution.

Quine-McCluskey Method for Generating Prime Implicants

1. **<u>Group</u>** minterms by the number of 1's

2. Apply adjacency (a b' + a b = a) to each pair of terms, forming a second list. Check those terms in the first list that are covered by the new terms. Note that only terms in adjacent groups (that differ by one 1) need be paired.
   (Find **<u>common</u>** terms)

3. Repeat process with second list (and again if multiple terms are formed on a third list).
   (Make prime implicant table)

Q-M Method Example

- Consider K-map example 2:

$$f(w,x,y,z) = \sum m(0,4,5,7,8,11,12,15)$$

1. Group by # of 1's:

A 0 0 0 0
B 0 1 0 0
C 1 0 0 0
D 0 1 0 1
E 1 1 0 0
F 0 1 1 1
G 1 0 1 1
H 1 1 1 1

2. Use adjacency property:

A + B = J = 0 – 0 0
A + C = K = - 0 0 0
B + D = L = 0 1 0 –
B + E = M = - 1 0 0
C + D = none
C + E = N = 1 – 0 0
D + F = O = 0 1 - 1
F + H = P = - 1 1 1
G + H = Q = 1 – 1 1

J + N = R = - - 0 0 = K + M

Table 4.1 Quine-McCluskey prime implicant computation.

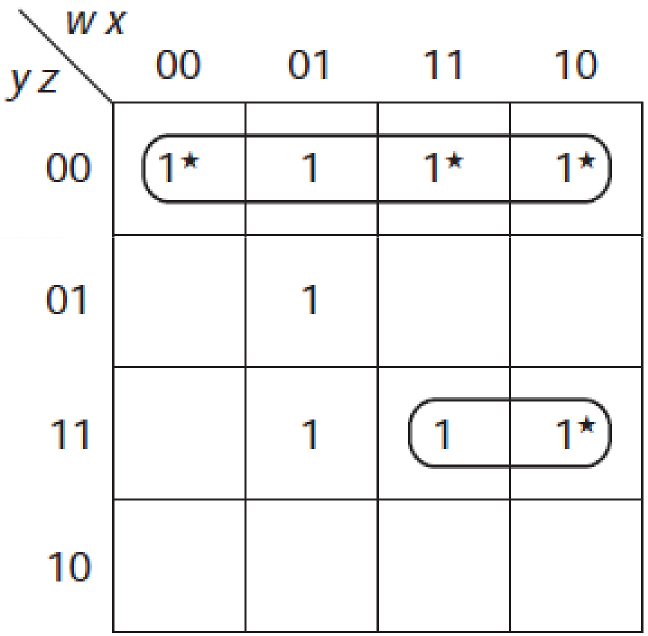
A 0 0 0 0 √	J 0 – 0 0 √	R – – 0 0
-------------	K – 0 0 0 √	
B 0 1 0 0 √	-------------	
C 1 0 0 0 √	L 0 1 0 –	
-------------	M – 1 0 0 √	
D 0 1 0 1 √	N 1 – 0 0 √	
E 1 1 0 0 √	-------------	
-------------	O 0 1 – 1	
F 0 1 1 1 √	-------------	
G 1 0 1 1 √	P – 1 1 1	
-------------	Q 1 – 1 1	
H 1 1 1 1 √		

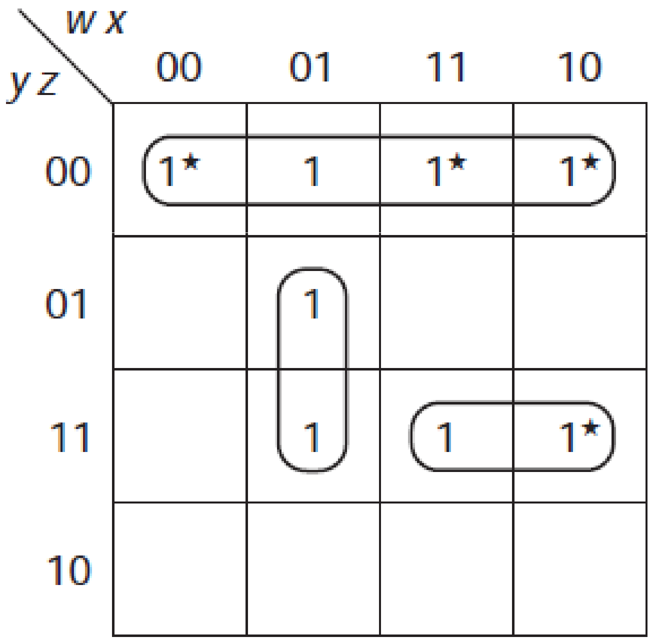
Prime Implicants :

R, L, O, P, Q

K-map Example 2

$$f(w,x,y,z)=\sum m(0,4,5,7,8,11,12,15)$$





$$f = y'z' + wyz + w'xz$$

Note: *w'xy', xyz are redundant*

From Marcovitz's Introduction to Logic Design

The iterated consensus algorithm for single functions is as follows:

1. Find a list of product terms (implicants) that cover the function. Make sure that no term is equal to or included in any other term on the list. (These terms could be prime implicants or minterms or any other set of implicants. However, the rest of the algorithm proceeds more quickly if we start with prime implicants.)
2. For each pair of terms, t_i and t_j (including terms added to the list in step 3), compute $t_i ¢ t_j$.
3. If the consensus is defined, and the consensus term is not equal to or included in a term already on the list, add it to the list.
4. Delete all terms that are included in the new term added to the list.
5. The process ends when all possible consensus operations have been performed. The terms remaining on the list are ALL of the prime implicants.

$$f(w, x, y, z) = \sum m(0,4,5,7,8,11,12,15)$$

Table 4.2 Computing the prime implicants.

~~A~~	~~$w'x'y'z'$~~		
B	$w'xy'$		
~~C~~	~~$wy'z'$~~		
D	xyz		
E	wyz		
~~F~~	~~$w'y'z'$~~	B ¢ $A \geq A$ (remove A)	
~~G~~	~~$xy'z'$~~	C ¢ B	
		D ¢ C	undefined
H	$w'xz$	D ¢ B	
		E ¢ D	undefined
		E ¢ C	undefined
		E ¢ B	undefined
		F ¢ E	undefined
		F ¢ D	undefined
J	$y'z'$	F ¢ $C \geq G, F, C$ (remove G, F, C)	
		H ¢ $E = D$ (do not add)	
		H ¢ D	undefined
		H ¢ B	undefined
		J ¢ $H = B$ (do not add)	
		J ¢ E	undefined
		J ¢ D	undefined
		J ¢ B	undefined

Table 4.3 Numeric computation of prime implicants.

~~A~~	~~0~~	~~0~~	~~0~~	~~0~~	
B	0	1	0	–	
~~C~~	~~1~~	~~–~~	~~0~~	~~0~~	
D	–	1	1	1	
E	1	–	1	1	
~~F~~	~~0~~	~~–~~	~~0~~	~~0~~	B ¢ $A \geq A$
~~G~~	~~–~~	~~1~~	~~0~~	~~0~~	C ¢ B
H	0	1	–	1	D ¢ B (D ¢ C undefined)
					(E ¢ D, E ¢ C, E ¢ B, F ¢ E, F ¢ D undefined)
J	–	–	0	0	F ¢ $C \geq G, F, C$
					(H ¢ $E = D$; H ¢ D, H ¢ B undefined; J ¢ $H = B$; J ¢ E, J ¢ D, J ¢ B undefined)

Prime Implicants :

B, D, E, H, J

Table 4.4 A prime implicant (PI) table.

PI	Numeric	$	Label	0	4	5	7	8	11	12	15
$w'xy'$	0 1 0 –	4	*A*		X	X					
xyz	– 1 1 1	4	*B*				X				X
wyz	1 – 1 1	4	*C*						X		X
$w'xz$	0 1 – 1	4	*D*			X	X				
$y'z'$	– – 0 0	3	*E*	X	X			X		X	

Gate Inputs

Table 4.5 Finding essential prime implicants.

PI	Numeric	$	Label	0	4	5	7	8	11	12	15
				√	√			√	√	√	√
$w'xy'$	0 1 0 –	4	*A*		X	X					
xyz	– 1 1 1	4	*B*				X				X
wyz*	1 – 1 1	4	*C*						X		X
$w'xz$	0 1 – 1	4	*D*			X	X				
$y'z'$*	– – 0 0	3	*E*	X	X			X		X	

Table 4.6 The reduced table.

$	Label	5	7
4	*A*	X	
4	*B*		X
4	*D*	X	X

Final Solution : *E*, *C*, *D* or $f = y'z' + wyz + w'xz$

$$f(a,b,c,d)=\sum m(1,3,4,6,7,9,11,12,13,15)$$

		$		√ 1	√ 3	4	6	7	√ 9	√ 11	12	13	15
*b′d**	– 0 – 1	3	*A*	X	X				X	X			
cd	– – 1 1	3	*B*		X			X		X			X
ad	1 – – 1	3	*C*						X	X		X	X
abc′	1 1 0 –	4	*D*								X	X	
bc′d′	– 1 0 0	4	*E*			X					X		
a′bd′	0 1 – 0	4	*F*			X	X						
a′bc	0 1 1 –	4	*G*				X	X					

$		4	6	7	12	13	15
3	*B*			X			X
3	*C*					X	X
4	*D*				X	X	
4	*E*	X			X		
4	*F*	X	X				
4	*G*		X	X			

$		6	7	13	15
3	*B*		X		X
3	*C*			X	X
4	*D*			X	
4	*F*	X			
4	*G*	X	X		

Choose E-> A + C + E + G

$		7	12	13	15
3	*B*	X			X
3	*C*			X	X
4	*D*		X	X	
4	*E*		X		
4	*G*	X			

Choose F -> A + F + B + D

Minimum Cover Solutions :
A, *C*, *E*, *G* or
A, *B*, *D*, *F*

Petrick's Method

1. Reduce the prime implicant chart by eliminating the essential prime implicant rows and the corresponding columns.
2. Label the rows of the reduced prime implicant chart P_1, P_2, P_3, etc.
3. Form a logical function which is true when all the columns are covered. P consists of a product of sums where each sum term has the form (P_{i0}, P_{i1}, …, P_{iN}) , where each P_{ij} represents a row covering column j.
4. Reduce P to a minimum sum of products by multiplying out and applying $X + XY = X$.
5. Each term in the result represents a solution, that is, a set of rows which covers all of the minterms in the table. To determine the minimum solutions, first find those terms which contain a minimum number of prime implicants.
6. Next, for each of the terms found in step five, count the number of literals in each prime implicant and find the total number of literals.
7. Choose the term or terms composed of the minimum total number of literals, and write out the corresponding sums of prime implicants.

Petrick's Method

From second table:

$$(E + F)(F + G)(B + G)(D + E)(C + D)(B + C)$$

$$= (F + EG)(B + CG)(D + CE)$$

$$= (BF + BEG + CFG + CEG)(D + CE)$$

$$= \underline{BDF} + BDEG + CDFG + CDEG +$$

$$BCEF + BCEG + CEFG + \underline{CEG}$$

Minimum Cover Solutions : A, B, D, F or A, C, E, G

Additional Example

$$f(a,b,c,d)=\sum m(0,2,5,6,7,8,9,13)+\sum d(1,12,15)$$

List 1

0	0 0 0 0	✓
1	0 0 0 1	✓
2	0 0 1 0	✓
8	1 0 0 0	✓
5	0 1 0 1	✓
6	0 1 1 0	✓
9	1 0 0 1	✓
12	1 1 0 0	✓
7	0 1 1 1	✓
13	1 1 0 1	✓
15	1 1 1 1	✓

List 2

0,1	0 0 0 x	✓
0,2	0 0 x 0	
0,8	x 0 0 0	✓
1,5	0 x 0 1	✓
2,6	0 x 1 0	
1,9	x 0 0 1	✓
8,9	1 0 0 x	✓
8,12	1 x 0 0	✓
5,7	0 1 x 1	✓
6,7	0 1 1 x	
5,13	x 1 0 1	✓
9,13	1 x 0 1	✓
12,13	1 1 0 x	✓
7,15	x 1 1 1	✓
13,15	1 1 x 1	✓

List 3

0,1,8,9	x 0 0 x
1,5,9,13	x x 0 1
8,9,12,13	1 x 0 x
5,7,13,15	x 1 x 1

From Brown's Fundamentals of digital logic

Additional Example (2)

Prime implicant	Minterm 0	2	5	6	7	8	9	13
p_1 = 0 0 x 0	√	√						
p_2 = 0 x 1 0		√		√				
p_3 = 0 1 1 x				√	√			
p_4 = x 0 0 x	√					√	√	
p_5 = x x 0 1			√				√	√
p_6 = 1 x 0 x						√	√	√
p_7 = x 1 x 1			√		√			√

Prime Implicant Table

After removing dominating columns (9,13)

Prime implicant	Minterm 0	2	5	6	7	8
p_1 = 0 0 x 0	√	√				
p_2 = 0 x 1 0		√		√		
p_3 = 0 1 1 x				√	√	
p_4 = x 0 0 x	√					√
p_5 = x x 0 1			√			
p_6 = 1 x 0 x						√
p_7 = x 1 x 1			√		√	

Additional Example (3)

Prime implicant	0	2	5	6	7	8
p_1	✓	✓				
p_2		✓		✓		
p_3				✓	✓	
p_4	✓					✓
p_7			✓		✓	

After removing dominated rows (p_5,p_6)

Prime implicant	2	6
p_1	✓	
p_2	✓	✓
p_3		✓

After removing rows p_4,p_7

Final solution : $f = b'c' + bd + a'cd'$

Multiple-output Problems

$$f(a,b,c) = \sum m(2,3,7); g(a,b,c) = \sum m(4,5,7)$$

A	0 1 0	– 0	(Included in *f* not in *g*)
B	1 0 0	0 –	(Included in *g* not in *f*)
C	0 1 1	– 0	
D	1 0 1	0 –	
E	1 1 1	– –	(Included in both *f* and *g*)

$A + C = F = 0\,1 - \quad -0$

$B + D = G = 1\,0 - \quad 0 -$

$C + E = H = -1\,1 \quad -0$

$D + E = J = 1 - 1 \quad 0 -$

Table 4.7 Multiple output Quine-McCluskey method.

A	0 1 0	– 0 √	*F*	0 1 –	– 0
B	0 1 1	– 0 √	*G*	1 0 –	0 –
C	1 0 0	0 – √	*H*	– 1 1	– 0
D	1 0 1	0 – √	*J*	1 – 1	0 –
E	1 1 1	– –			

a'	b	c'	g'	0	1	0	–	0
a'	b	c	g'	0	1	1	–	0
a	b'	c'	f'	1	0	0	0	–
a	b'	c	f'	1	0	1	0	–
a	b	c		1	1	1	–	–

Table 4.8 Iterated consensus for multiple output functions.

~~*A*~~	~~0 1 0~~	~~– 0~~				
~~*B*~~	~~0 1 1~~	~~– 0~~				
~~*C*~~	~~1 0 0~~	~~0 –~~				
~~*D*~~	~~1 0 1~~	~~0 –~~				
E	1 1 1	– –				
F	0 1 –	– 0	$B ¢ A \geq B, A$			
G	1 0 –	0 –	$D ¢ C \geq D, C$			
H	– 1 1	– 0	$F ¢ E$			
J	1 – 1	0 –	$G ¢ E$	($G ¢ F$ undefined)		

$H ¢ G$ zero tag; $H ¢ F$, $H ¢ E$ undefined

$J ¢ H$, $J ¢ F$ zero tag; $J ¢ G$, $J ¢ E$ undefined

Zero tag means a grouping made of a 1 from one function with a 1 from the other.

Table 4.9 A multiple output prime implicant table.

			f			g		
			√	√		√	√	
	$		2	3	7	4	5	7
1 1 1	4	*A*			X			X
0 1 –*	3	*B*	X	X				
1 0 –*	3	*C*				X	X	
– 1 1	3	*D*		X	X			
1 – 1	3	*E*					X	X

Table 4.10 A reduced prime implicant table.

			f	g
	$		7	7
1 1 1	4	*A*	X	X
– 1 1	3	*D*	X	
1 – 1	3	*E*		X

$$f = a'b + abc$$

$$g = ab' + abc$$

$$f(a, b, c, d) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$$

$$g(a, b, c, d) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$$

A	0 0 0 0	– – √

B	0 0 0 1	– – √
C	0 0 1 0	– – √
D	0 1 0 0	– 0 √

E	0 0 1 1	– 0 √
F	0 1 1 0	– – √
G	1 0 0 1	– 0 √
H	1 0 1 0	0 – √
I	1 1 0 0	– – √

J	1 0 1 1	– – √
K	1 1 1 0	– – √

L	1 1 1 1	– – √

AA	0 0 0 –	– –
AB	0 0 – 0	– –
AC	0 – 0 0	– 0 √

AD	0 0 – 1	– 0 √
AE	– 0 0 1	– 0 √
AF	0 0 1 –	– 0 √
AG	0 – 1 0	– –
AH	– 0 1 0	0 – √
AI	0 1 – 0	– 0 √
AJ	– 1 0 0	– 0 √

AK	– 0 1 1	– 0 √
AL	– 1 1 0	– –
AM	1 0 – 1	– 0 √
AN	1 0 1 –	0 – √
AO	1 – 1 0	0 – √
AP	1 1 – 0	– –

AQ	1 – 1 1	– –
AR	1 1 1 –	– –

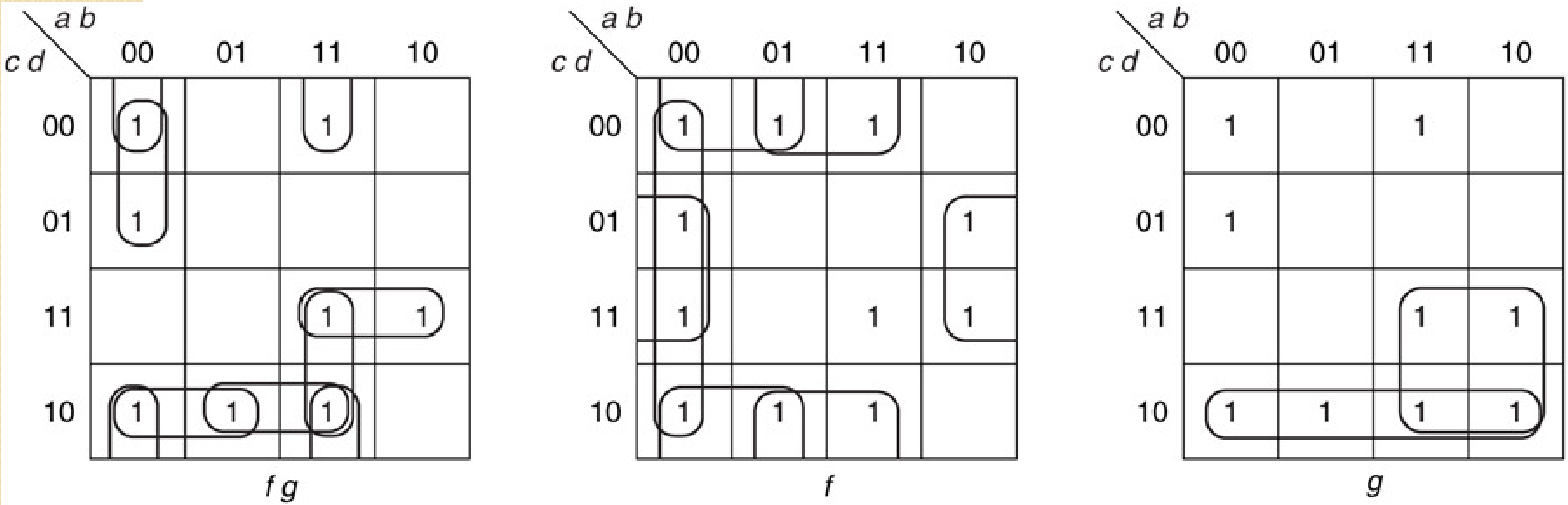
BA	0 0 – –	– 0
BB	0 – – 0	– 0

BC	– 0 – 1	– 0
BD	– – 1 0	0 –
BE	– 1 – 0	– 0

BF	1 – 1 –	0 –

$$f(a, b, c, d) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$$

$$g(a, b, c, d) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$$



0 0 0 –	– –	0 0 – –	– 0	– – 1 0	0 –
0 0 – 0	– –	0 – – 0	– 0	1 – 1 –	0 –
0 – 1 0	– –	– 1 – 0	– 0		
– 1 1 0	– –	– 0 – 1	– 0		
1 1 1 –	– –				
1 1 – 0	– –				
1 – 1 1	– –				

			f							g				
				√			√	√						√
			2	**3**	**4**	**6**	**9**	**11**	**12**	**2**	**6**	**10**	**11**	**12**
0 0 0 –	*A*	4												
0 0 – 0	*B*	4	X							X				
0 – 1 0	*C*	4	X			X				X	X			
– 1 1 0	*D*	4				X					X			
1 – 1 1	*E*	4						X					X	
1 1 1 –	*F*	4												
1 1 – 0*	*G*	4							X					X
– 1 – 0	*H*	3			X	X			X					
0 – – 0	*J*	3	X		X	X								
0 0 – –	*K*	3	X	X										
– 0 – 1*	*L*	3		X			X	X						
– – 1 0	*M*	3								X	X	X		
1 – 1 –	*N*	3										X	X	

			f				g			
									√	√
			2	4	6	12	2	6	10	11
0 0 – 0	*B*	4	X				X			
0 – 1 0	*C*	4	X		X		X	X		
– 1 1 0	*D*	4			X			X		
1 – 1 1	*E*	4								X
1 1 – 0*	*G*	1				X				
– 1 – 0	*H*	3		X	X	X				
0 – – 0	*J*	3	X	X	X					
0 0 – –	*K*	3	X							
– – 1 0	*M*	3					X	X	X	
1 – 1 –	*N*	3							X	X

			f				g	
			2	4	6	12	2	6
0 0 – 0	*B*	4	X				X	
0 – 1 0	*C*	4	X		X		X	X
– 1 1 0	*D*	4			X			X
1 1 – 0	*G*	1				X		
– 1 – 0	*H*	3		X	X	X		
0 – – 0	*J*	3	X	X	X			
0 0 – –	*K*	3	X					
– – 1 0	*M*	3					X	X

Choose C for f, g, and H for f :

$f = b'd + a'cd' + bd'$
$g = ac + a'cd' + abd'$

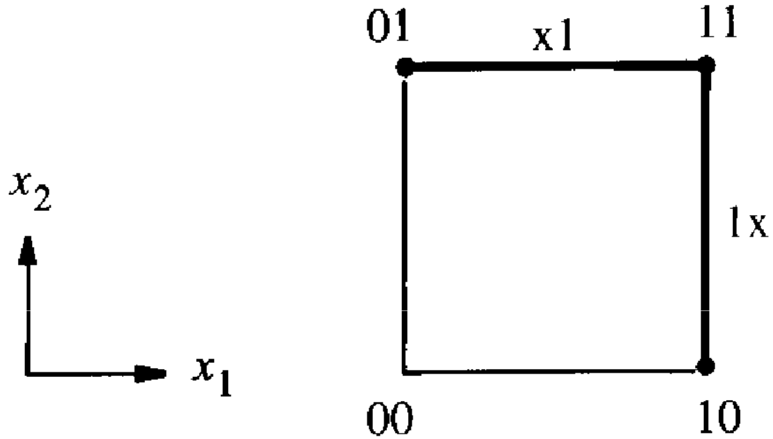
✓*Choose M for g, and J, G for f :*

$f = b'd + abd' + a'd'$
$g = ac + abd' + cd'$

Cubical Representation

Two-Dimensional Cube

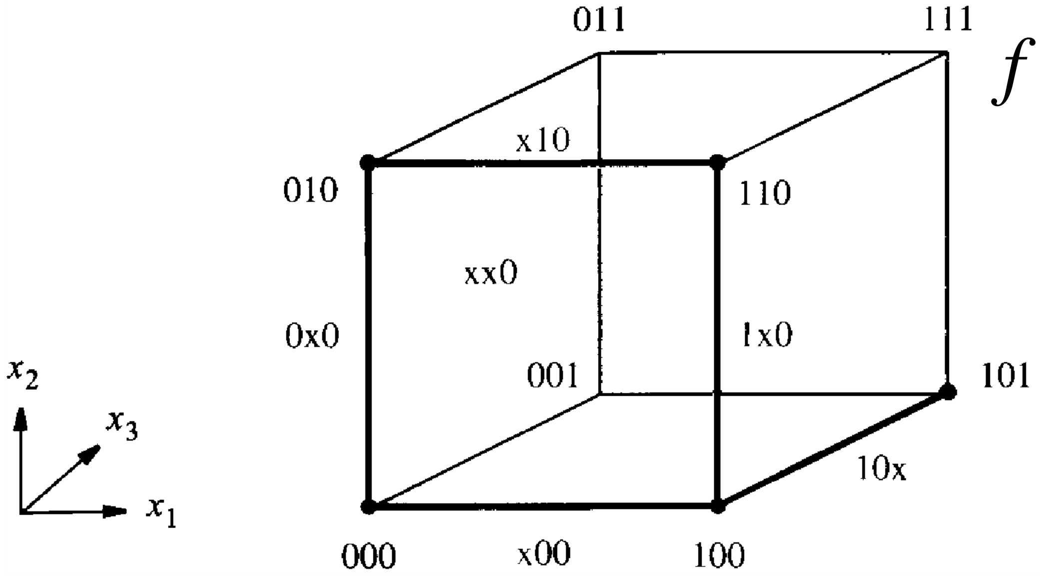
$$f(x_1, x_2) = \sum m(1,2,3) = x_1 + x_2$$



x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

$$f(x_1, x_2, x_3) = \sum m(0,2,4,5,6)$$

$$= \overline{x}_3 + x_1\overline{x}_2$$



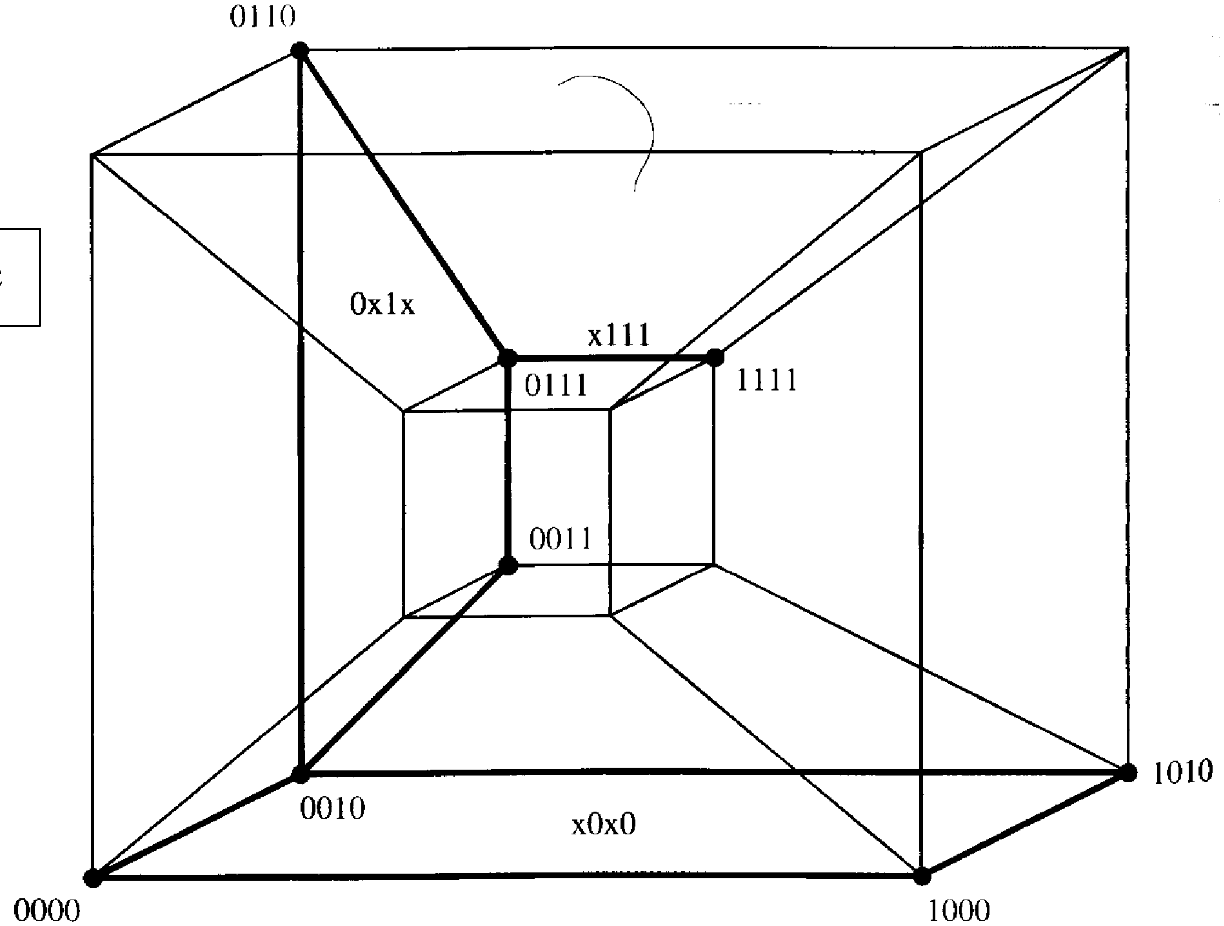
Three-Dimensional Cube

Cubical Representation (2)

$$f(x_1, x_2, x_3, x_4) = \sum m(0,2,3,6,7,8,10,15)$$

$$= \bar{x}_2\bar{x}_4 + \bar{x}_1 x_3 + x_2 x_3 x_4$$

Four-Dimensional Cube



Cubical Technique for Minimization

1. **Use *-operation to find prime implicants.**

2. **Use #-operation to determine "essential" prime implicants.**

3. **Choose minimum cover.**

-product operation (-operation)

*-operation is a simple way to combine two cubes.

Let A, B be two cubes, then C = A*B such that

1. $C=\phi$ if $A_i*B_i=\phi$ for more than one i.
2. $C_i=A_i*B_i$ when $A_i*B_i\neq\phi$ and $C_i=x$ for the coordinate where $A_i*B_i=\phi$.

Examples:

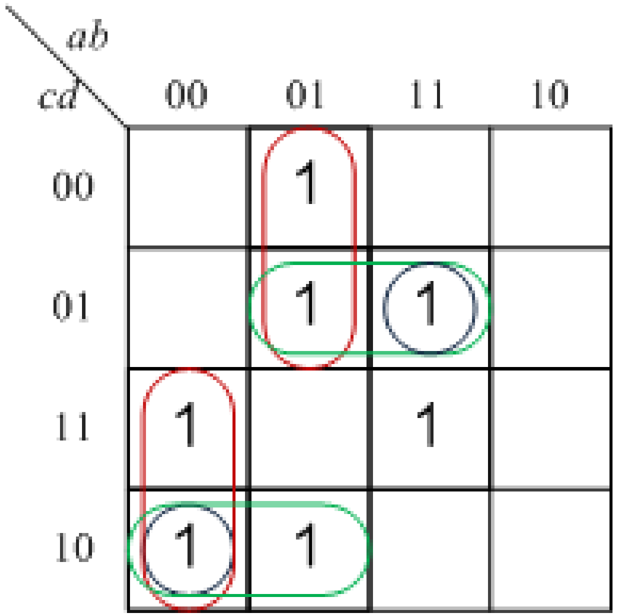
0101*1111=ϕ

0101*0100=010x

010x*1101=x101

0x10*001x=0010

00xx*10xx=x0xx



A_i \ B_i	0	1	x
0	0	ϕ	0
1	ϕ	1	1
x	0	1	x

A_i*B_i

Use *-operation to find PI

Initial Input: $C^0=\{c^1,c^2,\ldots,c^N\}$: set of implicants.
Repeat the calculation of
$G^{k+1} = c^m * c^n$ for all $c^m * c^n \in C^k$, $k=0,1,\ldots$
$C^{k+1}=C^k \cup G^{k+1}$ - redundant cubes
Until $C^{k+1}=C^k$.
Example: $f(x_1, x_2, x_3) = \sum m(0,1,2,3,7)$
C^0={000,001,010,011,111};
G^1={00x,0x0,0x1,01x,x11}, $C^1=G^1$;
G^2={000,001,0xx,0x1,010,01x,011}, C^2={x11,0xx}
G^3={011}, C^3={x11,0xx}=C^2
PI : x11, 0xx (x_2x_3, x_1')

#-operation (sharp operation)

The result of #-operation, A#B, gives "that part of A that is not covered by B".

Let A, B be two cubes, then C = A#B such that

1. C=A if $A_i \# B_i = \phi$ for some i.
2. C=ϕ if $A_i \# B_i = \varepsilon$ for all i.
3. Otherwise, $C = \bigcup_i (A_1, A_2, \ldots, \overline{B}_i, \ldots A_n)$ for all i, A_i=x AND $B_i \neq$x.

$A_i \backslash B_i$	0	1	x
0	ε	∅	ε
1	∅	ε	ε
x	1	0	ε

$A_i \# B_i$

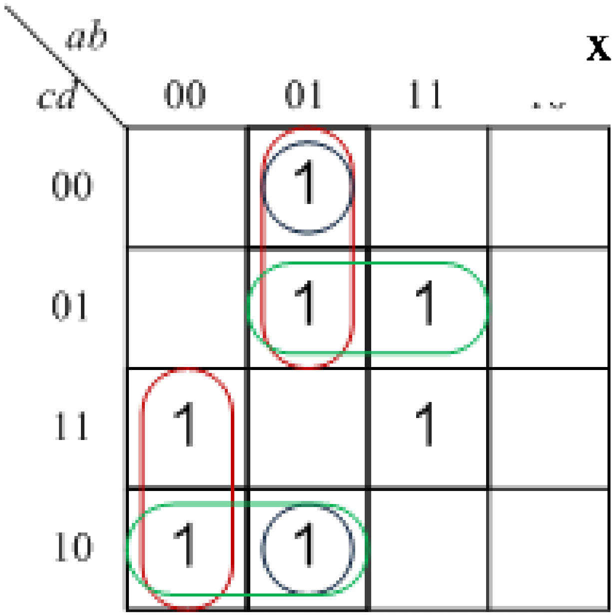
Examples:

0101#1111=0101

0101#010x=ϕ

010x#x101=0100

001x#0x10=0110



Use #-operation to find essential PI

Initial Input: $P=\{p^1,p^2,\ldots,p^N\}$: set of PI's,
DC=$\{d^1,d^2,\ldots,d^M\}$: don't care set.
p^i is "essential" $\leftrightarrow$ $p^i \#(P-p^i)\# DC \neq \phi$
Example if $P=\{p^1,p^2,p^3,p^4\}$, $DC=\{d^1,d^2\}$, then
p^3 is "essential" if $((((p^3 \# p^1)\# p^2)\# p^4)\# d^1)\# d^2 \neq \phi$
Example: P = {x11, 0xx} (x_2x_3, x_1')
$P^1\#p^2$={111}; $P^2\#p^1$={00x,0x0}.
Thus, both PI's are "essential".

Cubic Technique Example

*-operation

$$f(w,x,y,z)=\sum m(0,4,57,8,11,12,15)$$

$$C^0=\{0000,0100,0101,0111,1000,1011,1100,1111\}$$

$$G^1=\{0x00,x000,010x,x100,01x1,x111,1x00,1x11\}$$

$$C^1=C^0\bigcup G^1=G^1;\ G^2=\{xx00,\ldots\}$$

$$C^2=\{01x1,x111,1x00,1x11,xx00\}=C^3$$

$$P=\{01x1,x111,1x00,1x11,xx00\}=\{p^1,p^2,p^3,p^4,p^5\}$$

#-operation

$$p^1\#p^2=\{0101\};0101\#p^3=0101\#p^4=0101\#p^5=\{0101\}\neq\phi$$

$$p^2\#p^1=p^2;p^2\#p^3=\{1111\};1111\#p^4=\phi$$

$$p^3\#p^1=p^3\#p^2=p^3\#p^4=p^3;p^3\#p^5=\phi$$

$$p^4\#p^1=p^4;p^4\#p^2=\{1011\};1011\#p^3=1011\#p^5\neq\phi$$

$$p^5\#p^1=p^5\#p^2=p^5;p^5\#p^3=p^5\#p^4=\{0x00\}\neq\phi$$

Essential PI : p^1, p^4, p^5

Minimum Cover : $f=w'xz+wyz+y'z'$