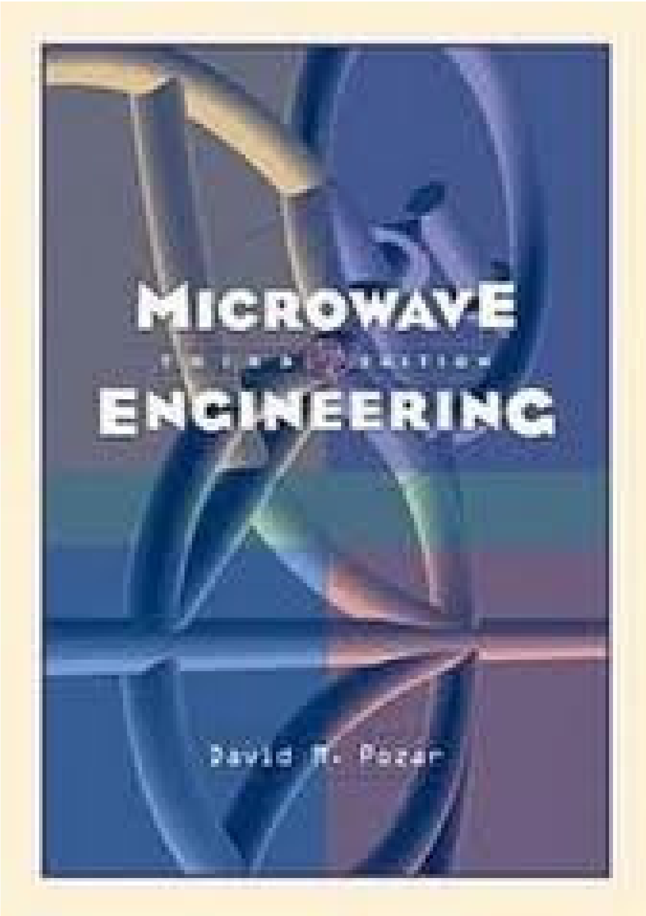
ECE 5317-6351
Microwave Engineering

Fall 2011

Prof. David R. Jackson
Dept. of ECE



Notes 14

Network Analysis
Multiport Networks

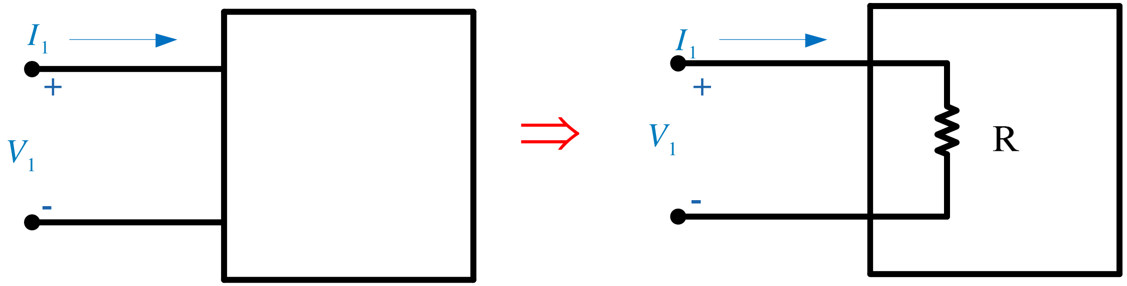
Multiport Networks

A general circuit can be represented by a multi-port network, where the "ports" are defined as access terminals at which we can define voltages and currents.

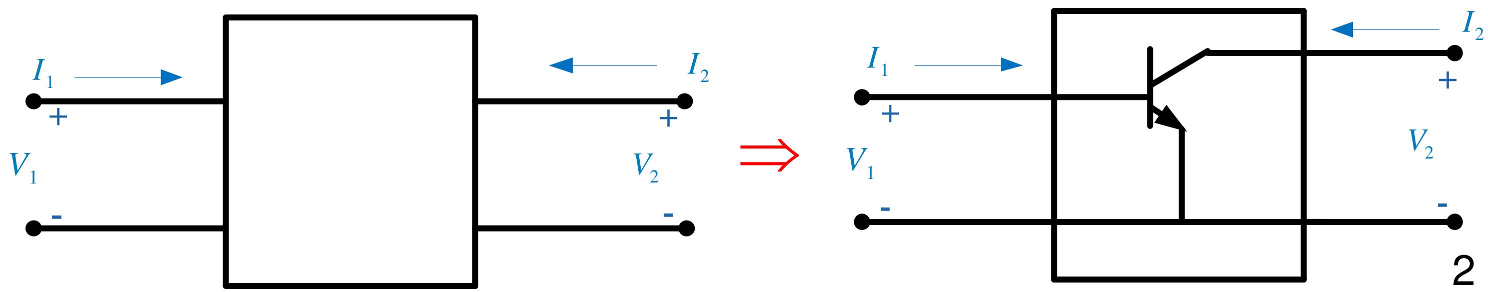
Note: Equal and opposite currents are assumed on the two wires of a port.

Examples:

1) One-port network

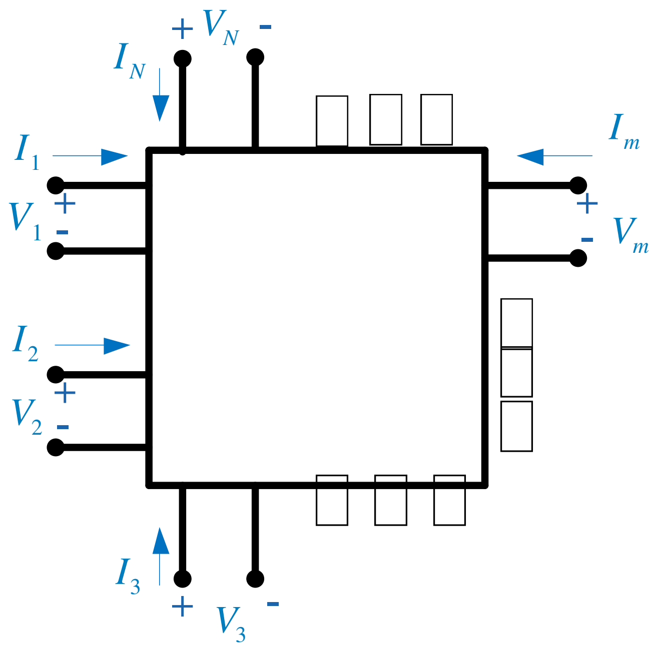


2) Two-port network



Multiport Networks (cont.)

3) N-port Network



To represent multi-port networks we use:

- Z (impedance) parameters
- Y (admittance) parameters
- h (hybrid) parameters
- $ABCD$ parameters

} Not easily measurable at high frequency

- S (scattering) parameters

} Measurable at high frequency

Poynting Theorem (Phasor Domain)

$$\int_V -\frac{1}{2}\left(\underline{E}\cdot\underline{J}^{i*}\right)dV = \oint_S \underline{S}\cdot\underline{\hat{n}}\,dS$$

$$+\int_V\left(\frac{1}{2}\omega\varepsilon_c''\left|\underline{E}\right|^2+\frac{1}{2}\omega\mu''\left|\underline{H}\right|^2\right)dV$$

$$+2j\omega\int_V\left(\frac{1}{4}\mu'\left|\underline{H}\right|^2-\frac{1}{4}\varepsilon_c'\left|\underline{E}\right|^2\right)dV$$

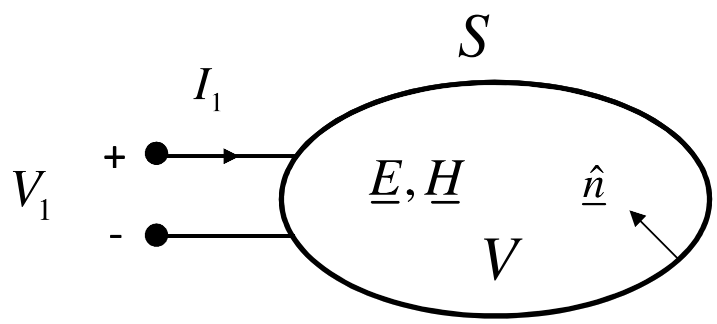
The last term is the VARS consumed by the region.

The notation < > denotes time-average.

$$P_s = P_f + \langle \mathrm{P}_d \rangle + j(2\omega)\left(\langle \mathrm{W}_m \rangle - \langle \mathrm{W}_e \rangle\right)$$

Self Impedance

Consider a general one-port network



Complex power delivered to network:

$$P_{in} = \frac{1}{2} \oint_S \left(\underline{E} \times \underline{H}^* \right) \cdot \hat{\underline{n}} \; ds = P_d + j2\omega \left(W_m - W_e \right)$$

$$= \frac{1}{2} V_1 I_1^*$$

Average power dissipated in [W]

$$P_d = \langle \mathrm{P}_d \rangle$$

Average magnetic energy (in [J]) stored inside V

$$W_m = \langle \mathrm{W}_m \rangle$$

Average electric energy (in [J]) stored inside V

$$W_e = \langle \mathrm{W}_e \rangle$$

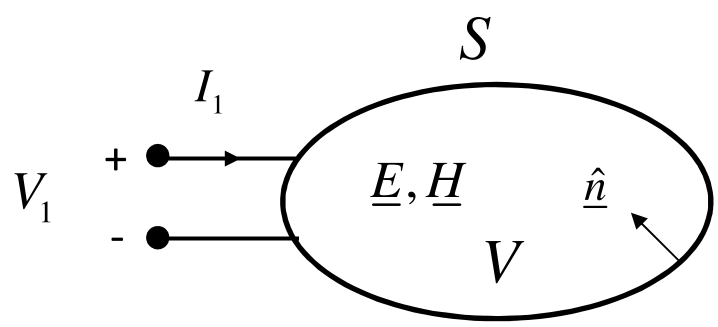
Define Self Impedance (Z_{in})

$$Z_{in} \equiv \frac{V_1}{I_1} = \frac{V_1 I_1^*}{|I_1|^2} = \frac{\frac{1}{2} V_1 I_1^*}{\frac{1}{2}|I_1|^2} = \frac{P_{in}}{\frac{1}{2}|I_1|^2}$$

$$= R_{in} + jX_{in} = \frac{P_d + j2\omega(W_m - W_e)}{\frac{1}{2}|I_1|^2}$$

$$R_{in} = \frac{2P_d}{|I_1|^2}$$

$$X_{in} = \frac{4\omega(W_m - W_e)}{|I_1|^2}$$



Self Impedance (cont.)

We can show that for physically realizable networks the following apply:

Please see the Pozar book for a proof.

$$V_1(-\omega) = V_1^*(\omega)$$

$$\Rightarrow Z_{in}(-\omega) = Z_{in}^*(\omega)$$

$$I_1(-\omega) = I_1^*(\omega)$$

$\Rightarrow$ $R_{in}(\omega)$ is an even function of ω

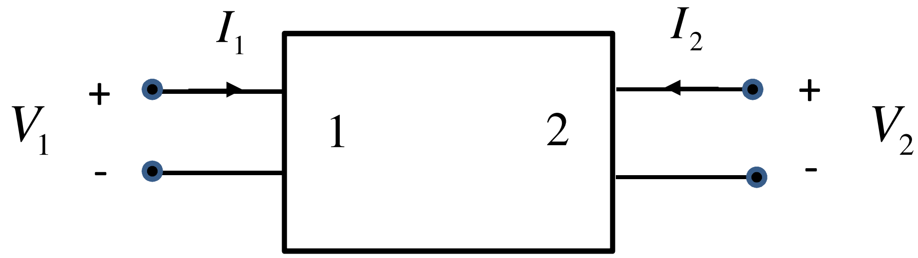
$$Z_{in}(\omega) = R_{in}(\omega) + jX_{in}(\omega)$$

$X_{in}(\omega)$ is an odd function of ω

Note: Frequency is usually defined as a positive quantity. However, we consider the analytic continuation of the functions into the complex frequency plane.

Two-Port Networks

Consider a general 2-port linear network:



In terms of Z-parameters, we have (from superposition)

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

Impedance (Z) matrix

$$\Rightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad \Rightarrow \quad [V] = [Z][I]$$

Elements of Z-Matrix: Z-Parameters

(open-circuit parameters)

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

Port 2 open circuited

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

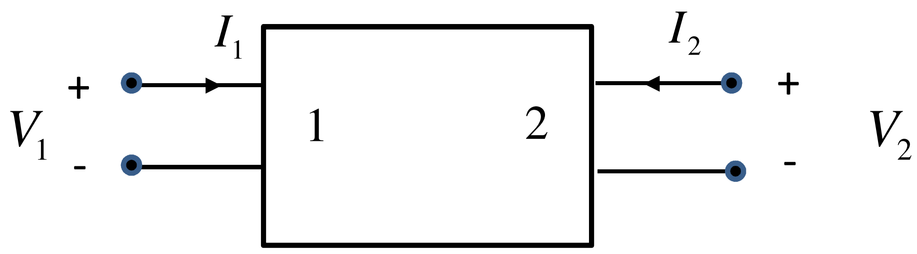
$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$Z_{ij} = \left.\frac{V_i}{I_j}\right|_{I_k=0 \quad k \neq j}$$

Port 1 open circuited

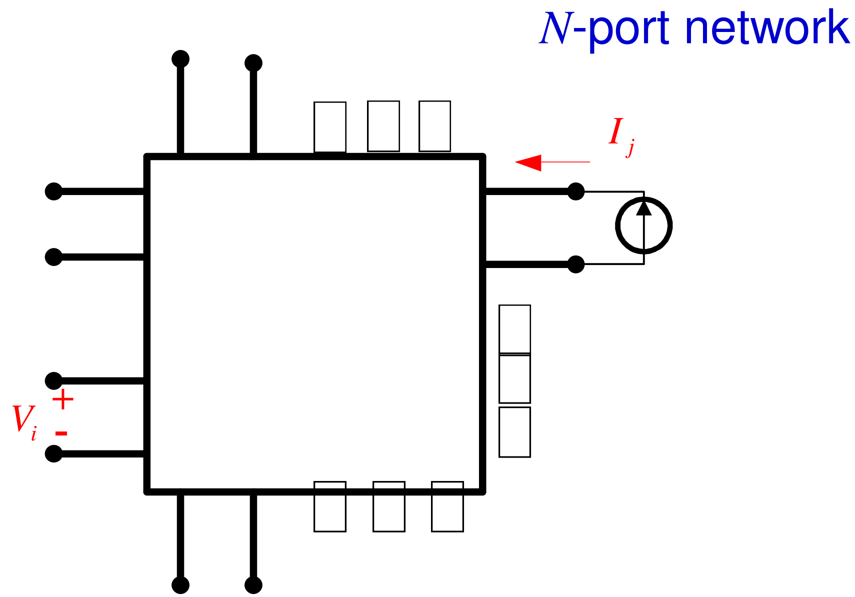
$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$



Z-Parameters (cont.)

$$Z_{ij} = \left. \frac{V_i}{I_j} \right|_{I_k = 0 \quad k \neq j}$$



We inject a current into port j and measure the voltage (with an ideal voltmeter) at port i. All ports are open-circuited except j.

Z-Parameters (cont.)

Z-parameters are convenient for series connected networks.

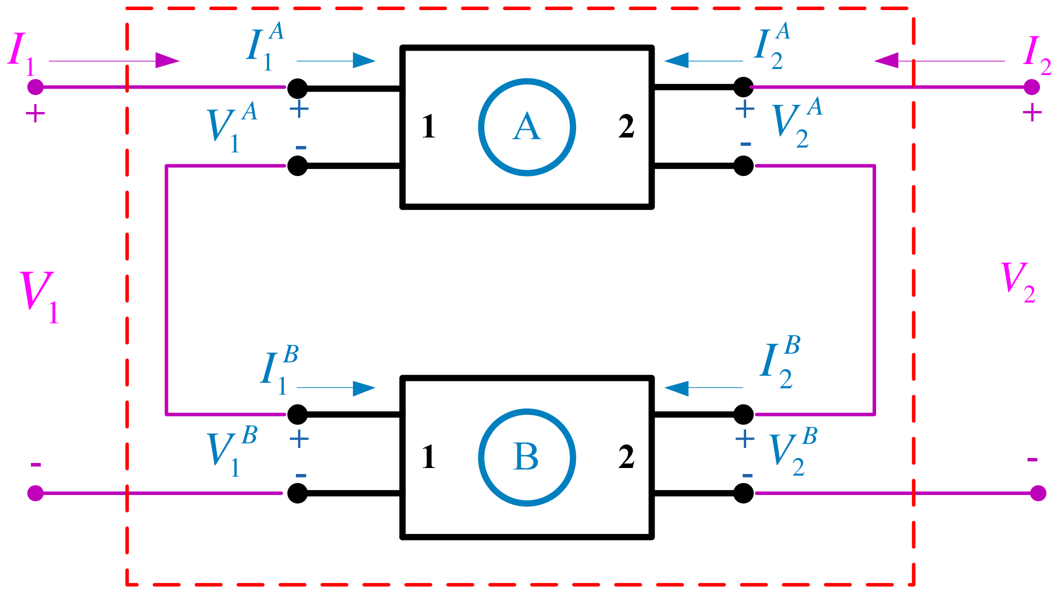
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1^A \\ V_2^A \end{bmatrix} + \begin{bmatrix} V_1^B \\ V_2^B \end{bmatrix}$$

$$= \left[Z^A\right]\left[I^A\right] + \left[Z^B\right]\left[I^B\right]$$

$$= \left(\left[Z^A\right] + \left[Z^B\right]\right)\left[I\right]$$

$$= \left(\left[Z^A\right] + \left[Z^B\right]\right)\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$= \left[Z^A + Z^B\right]\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$



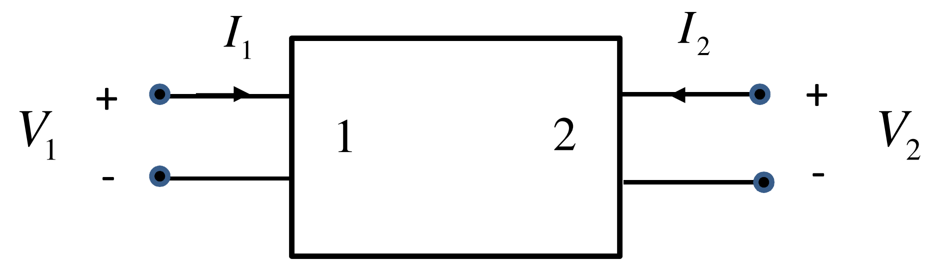
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11}^A + Z_{11}^B & Z_{12}^A + Z_{12}^B \\ Z_{21}^A + Z_{21}^B & Z_{22}^A + Z_{22}^B \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Series $\Rightarrow$ $I_1 = I_1^A = I_1^B$

$I_2 = I_2^A = I_2^B$

Admittance (Y) Parameters

Consider a 2-port network:



$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

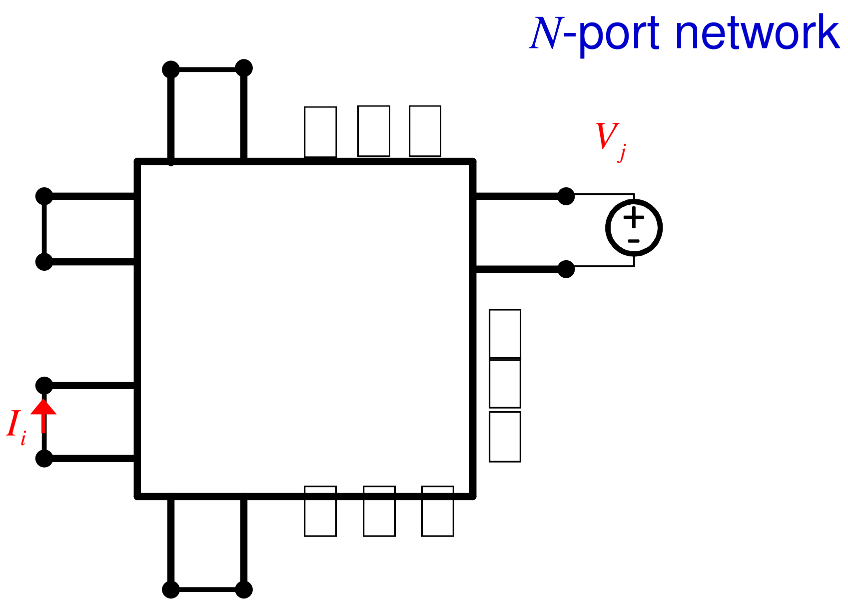
Admittance matrix

or $$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow [I] = [Y][V]$$

$$Y_{ij} = \left. \frac{I_i}{V_j} \right|_{V_k = 0 \quad k \neq j}$$ ← Short-circuit parameters

Y-Parameters (cont.)

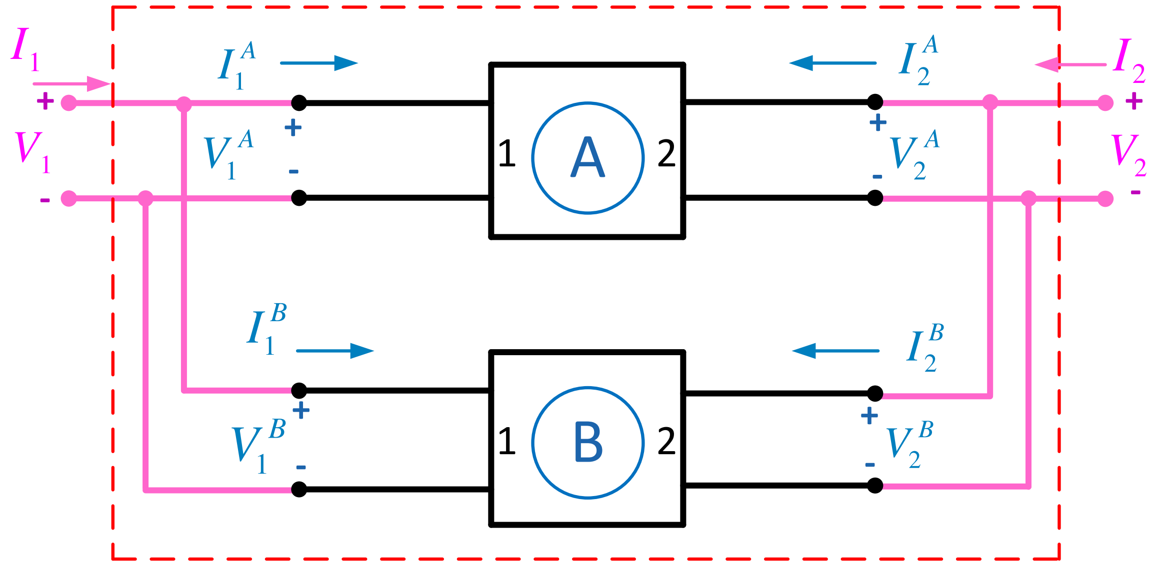
$$Y_{ij} = \left. \frac{I_i}{V_j} \right|_{V_k = 0 \quad k \neq j}$$



We apply a voltage across port j and measure the current (with an ideal current meter) at port i. All ports are short-circuited except j.

Admittance (Y) Parameters

Y-parameters are convenient for parallel connected networks.



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_1^A \\ I_2^A \end{bmatrix} + \begin{bmatrix} I_1^B \\ I_2^B \end{bmatrix}$$

$$= \begin{bmatrix} Y_{11}^A + Y_{11}^B & Y_{12}^A + Y_{12}^B \\ Y_{21}^A + Y_{21}^B & Y_{22}^A + Y_{22}^B \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Parallel $\Rightarrow V_1 = V_1^A = V_1^B$

$V_2 = V_2^A = V_2^B$

Admittance (Y) Parameters

Relation between [Z] and [Y] matrices:

$$[V]=[Z][I]$$

$$[I]=[Y][V]$$

Hence $$[V]=[Z]([Y][V]) = ([Z][Y])[V]$$

$$[Z][Y]=[U]= \text{Identity Matrix}$$

Therefore $$[Y]=[Z]^{-1}$$

Reciprocal Networks

If a network does not contain non-reciprocal devices or materials* (i.e. ferrites, or active devices), then the network is "reciprocal."

$$\Rightarrow Z_{ij} = Z_{ji} \quad \left(Y_{ij} = Y_{ji}\right)$$

Note: The inverse of a symmetric matrix is symmetric.

$$\Rightarrow [Z] \text{ and } [Y] \text{ are symmetric}$$

* A reciprocal material is one that has reciprocal permittivity and permeability tensors.
  A reciprocal device is one that is made from reciprocal materials

Example of a nonreciprocal material: a biased ferrite

(This is very useful for making isolators and circulators.)

Reciprocal Materials

$$\underline{D} = \underline{\underline{\varepsilon}} \cdot \underline{E}$$

$$\underline{B} = \underline{\underline{\mu}} \cdot \underline{H}$$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \qquad \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} \mu_x \\ \mu_y \\ \mu_z \end{bmatrix}$$

Reciprocal: $\varepsilon_{ij} = \varepsilon_{ji}, \;\; \mu_{ij} = \mu_{ji}$

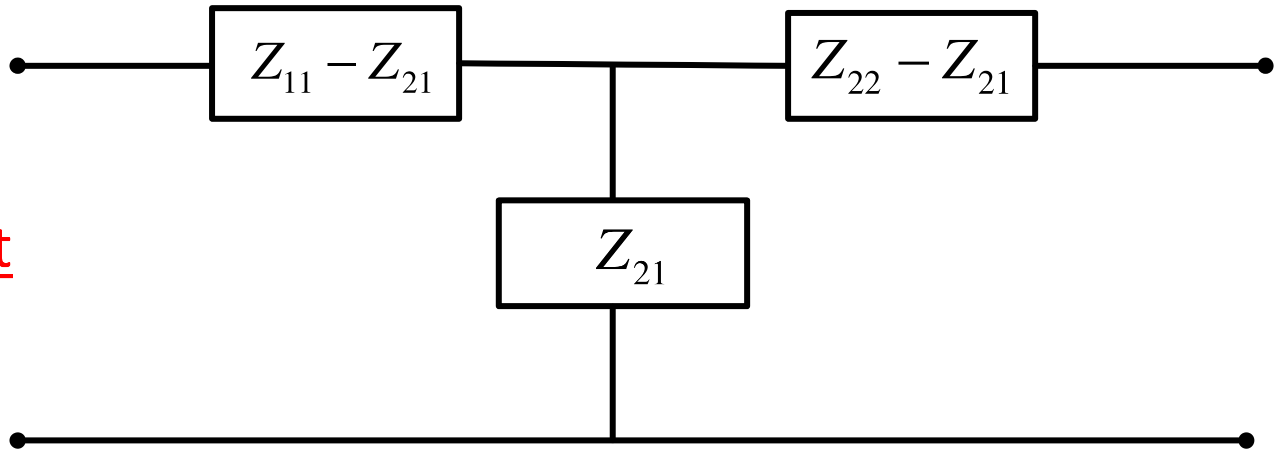
Ferrite: $$\underline{\underline{\mu}} = \mu_0 \begin{bmatrix} \alpha & j\gamma & 0 \\ -j\gamma & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{\underline{\mu}}$ is <u>not</u> symmetric!

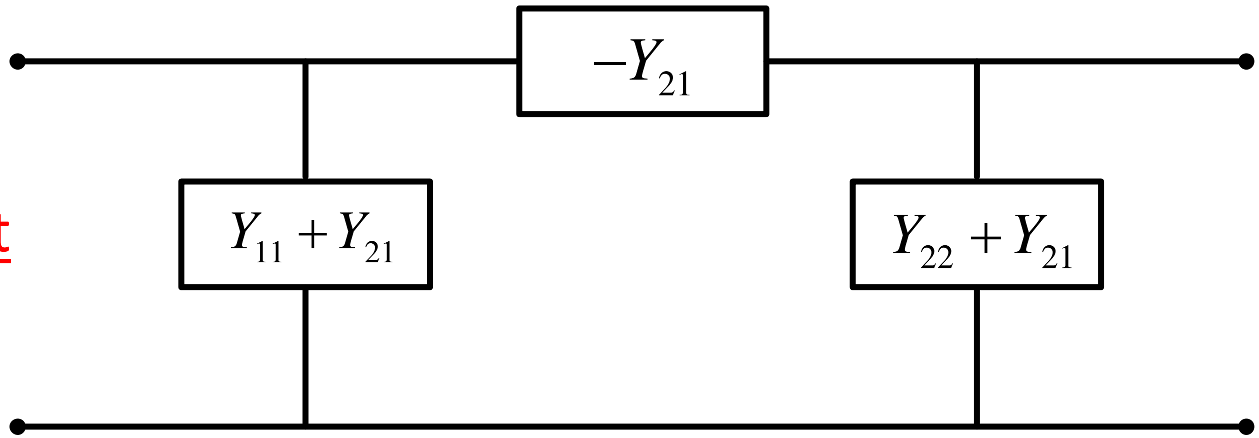
Reciprocal Networks (cont.)

We can show that the equivalent circuits for reciprocal 2-port networks are:

T-equivalent



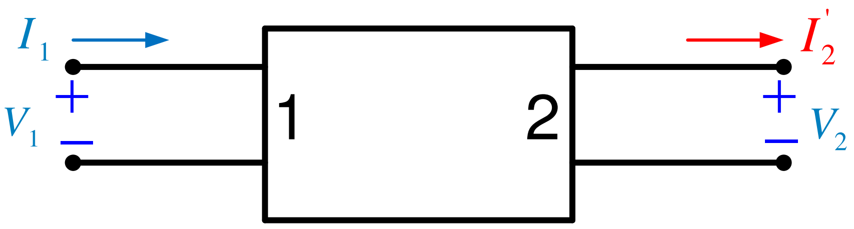
Pi-equivalent



ABCD-Parameters

There are defined only for 2-port networks.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2' \end{bmatrix}$$

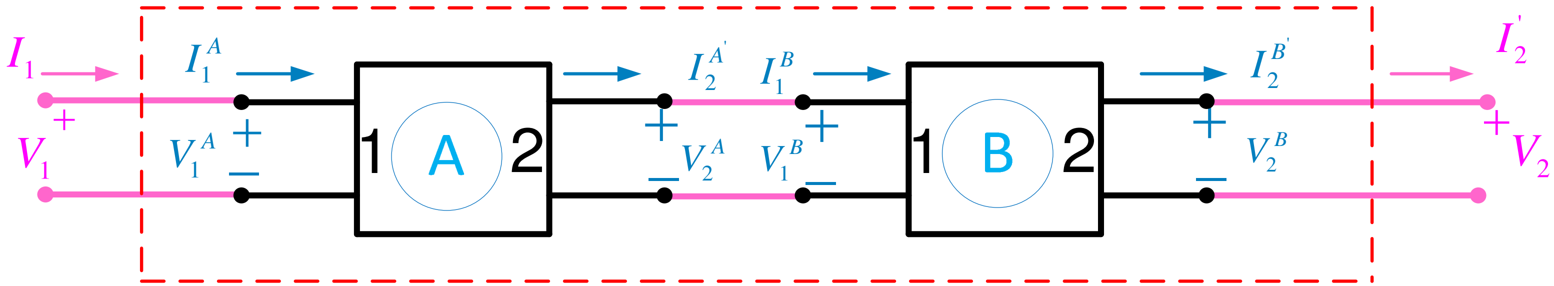


$$I_2' = -I_2$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2'=0} \qquad C = \left.\frac{I_1}{V_2}\right|_{I_2'=0}$$

$$B = \left.\frac{V_1}{I_2'}\right|_{V_2=0} \qquad D = \left.\frac{I_1}{I_2'}\right|_{V_2=0}$$

Cascaded Networks



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_1^A \\ I_1^A \end{bmatrix} = \left[ABCD^A \right] \begin{bmatrix} V_2^A \\ I_2^{A'} \end{bmatrix}$$

$$= \left[ABCD^A \right] \begin{bmatrix} V_1^B \\ I_1^B \end{bmatrix}$$

$$= \underbrace{\left[ABCD^A \right] \left[ABCD^B \right]} \begin{bmatrix} V_2^B \\ I_2^{B'} \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \left[ABCD^{AB} \right] \begin{bmatrix} V_2 \\ I_2^{'} \end{bmatrix}$$

A nice property of the ABCD matrix is that it is easy to use with cascaded networks: you simply multiply the ABCD matrices together.

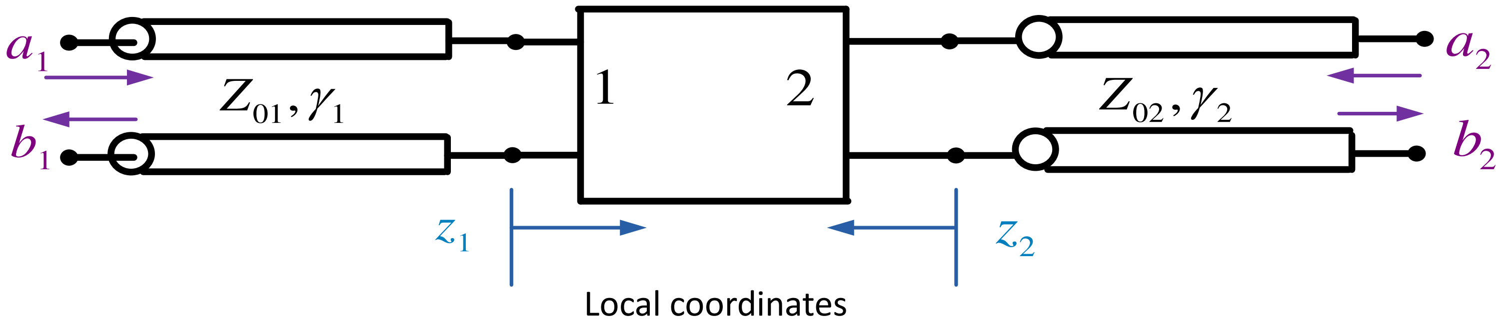
Scattering Parameters

- At high frequencies, Z, Y, h & ABCD parameters are difficult (if not impossible) to measure.
  - V and I are not uniquely defined
  - Even if defined, V and I are extremely difficult to measure (particularly I).
  - Required open and short-circuit conditions are often difficult to achieve.

- Scattering (S) parameters are often the best representation for multi-port networks at high frequency.

Scattering Parameters (cont.)

S-parameters are defined
assuming transmission lines are connected to each port.



On each transmission line:

$$V_i(z_i) = V_{i0}^{+} e^{-\gamma_i z_i} + V_{i0}^{-} e^{+\gamma_i z_i} = V_i^{+}(z_i) + V_i^{-}(z_i)$$

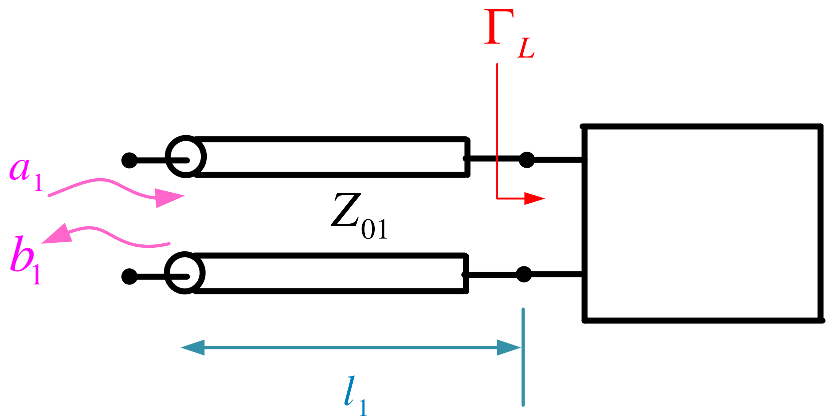
$$I_i(z_i) = \frac{V_i^{+}(z_i)}{Z_{0i}} - \frac{V_i^{-}(z_i)}{Z_{0i}} \qquad i = 1, 2$$

Incoming wave function $\equiv a_i(z_i) \equiv V_i^{+}(z_i) / \sqrt{Z_{0i}}$

Outgoing wave function $\equiv b_i(z_i) \equiv V_i^{-}(z_i) / \sqrt{Z_{0i}}$

For a One-Port Network

$$\Gamma_L = \frac{V_1^-(0)\Big/\sqrt{Z_{01}}}{V_1^+(0)\Big/\sqrt{Z_{01}}}$$



$$= \frac{b_1(0)}{a_1(0)}$$

$$\Rightarrow \quad b_1(0) = \Gamma_L a_1(0)$$

$$= S_{11}\, a_1(0)$$

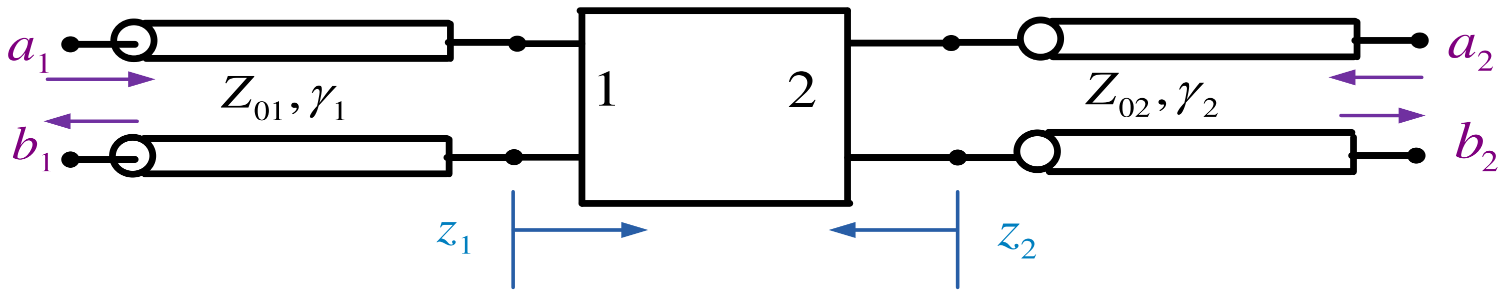
For a one-port network, S_{11} is defined to be the same as Γ_L.

$$= S_{11}$$

Incoming wave function $\equiv a_i(z_i) \equiv V_i^+(z_i)\Big/\sqrt{Z_{0i}}$

Outgoing wave function $\equiv b_i(z_i) \equiv V_i^-(z_i)\Big/\sqrt{Z_{0i}}$

For a Two-Port Network



$$b_1(0) = S_{11}a_1(0) + S_{12}a_2(0)$$
$$b_2(0) = S_{21}a_1(0) + S_{22}a_2(0)$$

Scattering matrix

$$\Rightarrow \begin{bmatrix} b_1(0) \\ b_2(0) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \end{bmatrix} \Rightarrow [b] = [S][a]$$

Scattering Parameters

$$b_1(0) = S_{11}a_1(0) + S_{12}a_2(0)$$
$$b_2(0) = S_{21}a_1(0) + S_{22}a_2(0)$$

$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0}$$ Output is matched ← input reflection coef. w/ output matched

$$S_{12} = \left.\frac{b_1(0)}{a_2(0)}\right|_{a_1=0}$$ Input is matched ← reverse transmission coef. w/ input matched

$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0}$$ Output is matched ← forward transmission coef. w/ output matched

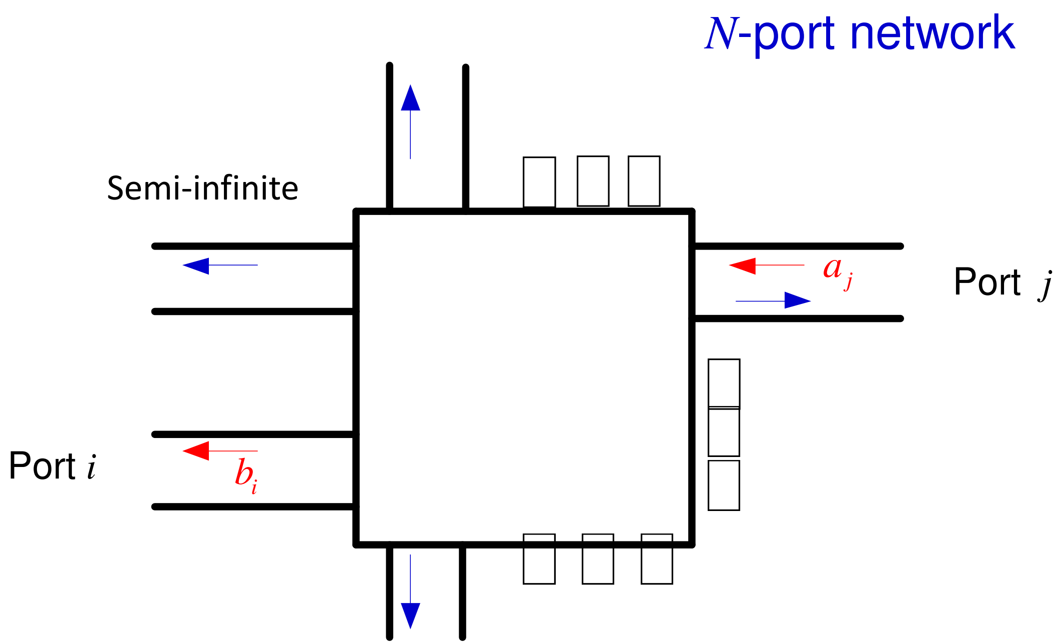
$$S_{22} = \left.\frac{b_2(0)}{a_2(0)}\right|_{a_1=0}$$ Input is matched ← output reflection coef. w/ input matched

Scattering Parameters (cont.)

For a general multiport network:

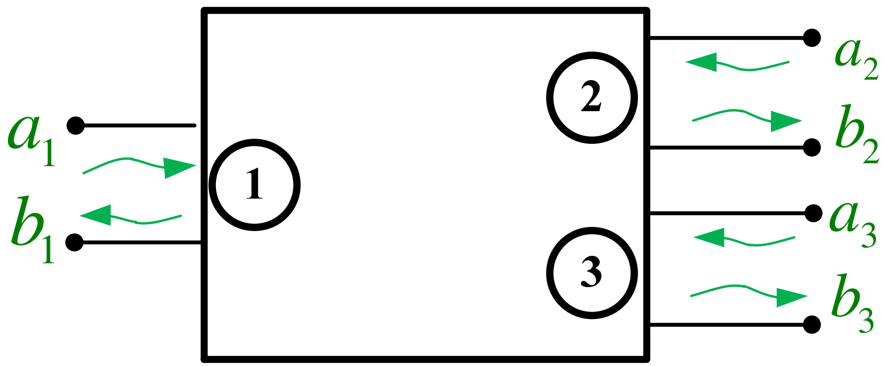
$$S_{ij} = \left. \frac{b_i(0)}{a_j(0)} \right|_{a_k = 0 \;\; k \neq j}$$

All ports except j are semi-infinite (or matched)



Scattering Parameters (cont.)

Illustration of a three-port network



Scattering Parameters (cont.)

For reciprocal networks, the S-matrix is symmetric.

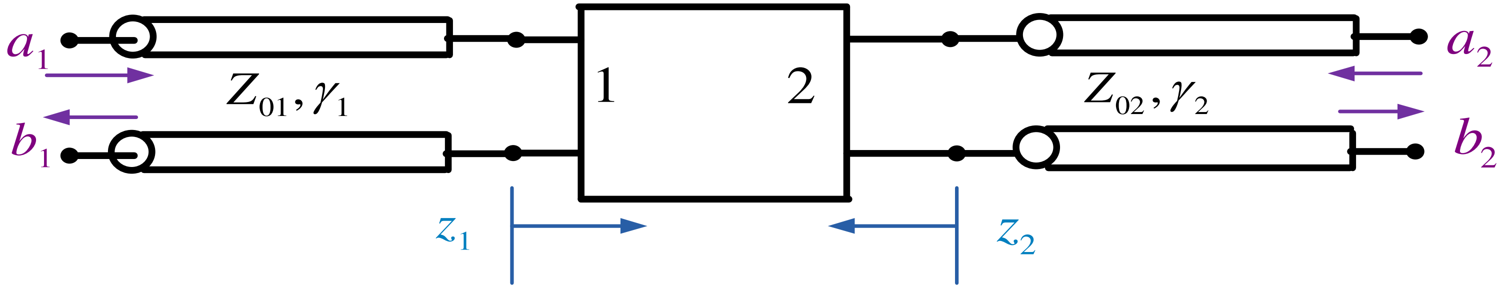
$$\Rightarrow S_{ij} = S_{ji} \qquad i \neq j$$

Note: If all lines entering the network have the same characteristic impedance, then

$$S_{ij} = \left. \frac{V_i^-(0)}{V_j^+(0)} \right|_{V_k^+=0 \;\; k \neq j}$$

Scattering Parameters (cont.)

Why are the wave functions (a and b) defined as they are?



$$P_i^+(0) = \frac{1}{2}\operatorname{Re}\left[V_i^+(0)\, I_i^{+*}(0)\right] = \frac{1}{2}\frac{\left|V_i^+(0)\right|^2}{Z_{0i}}$$

(assuming lossless lines)

Note:

$$a_i(0) = V_i^+(0)\big/\sqrt{Z_{0i}}$$

$$\Rightarrow P_i^+(0) = \frac{1}{2}\left|a_i(0)\right|^2$$

Scattering Parameters (cont.)

Similarly,

$$P_i^-(0) = \frac{1}{2}\frac{\left|V_i^-(0)\right|^2}{Z_{0i}} = \frac{1}{2}\left|b_i(0)\right|^2$$

Also,

$$V_i^+(-l_i) = V_i^+(0)\, e^{+\gamma_i l_i}$$

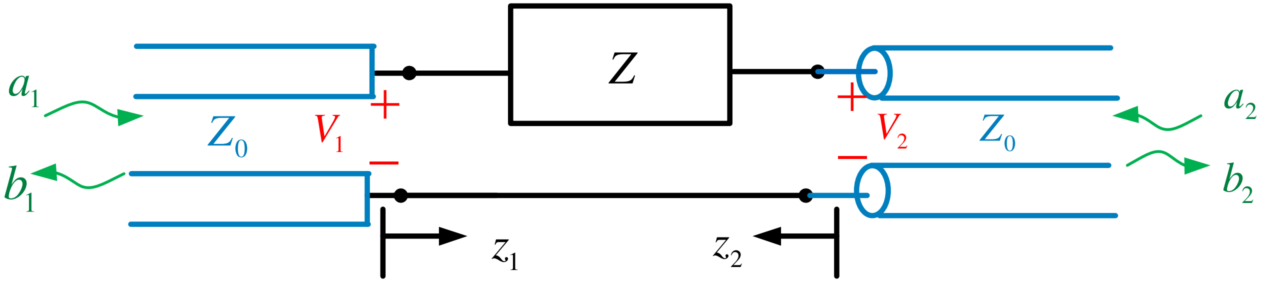
$$V_i^-(-l_i) = V_i^-(0)\, e^{-\gamma_i l_i}$$

$$\Rightarrow\; P_i^+(-l_i) = \frac{1}{2}\left|a_i(-l_i)\right|^2 = \frac{1}{2}\left|a_i(0)\right|^2 e^{+2\alpha_i l_i}$$

$$P_i^-(-l_i) = \frac{1}{2}\left|b_i(-l_i)\right|^2 = \frac{1}{2}\left|b_i(0)\right|^2 e^{-2\alpha_i l_i}$$

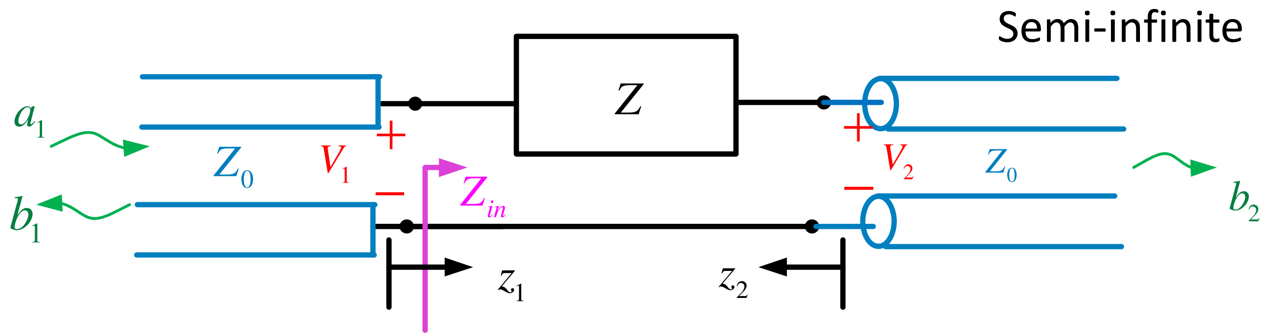
Example

Find the S parameters for a series impedance Z.



Note that two different coordinate systems are being used here!

Example (cont.)



S_{11} Calculation:

$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0} = \frac{V_1^-(0)}{V_1^+(0)} = \left.\frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right|_{a_2=0} = \frac{(Z + Z_0) - Z_0}{(Z + Z_0) + Z_0}$$

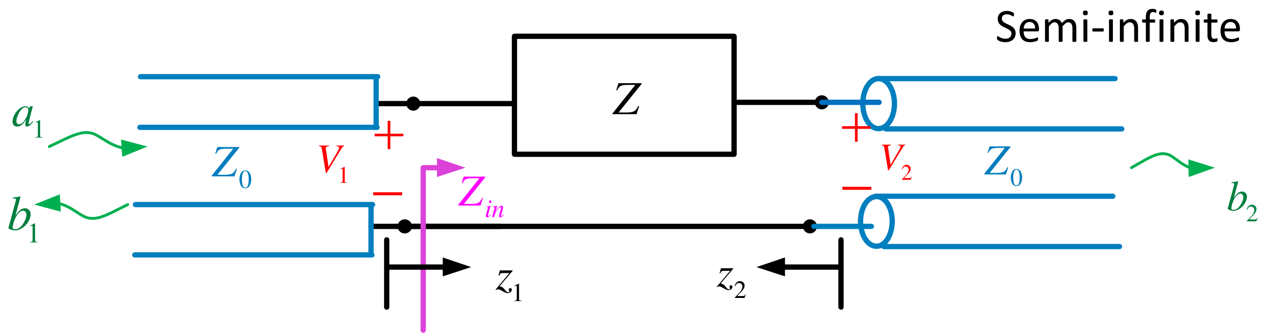
$$\Rightarrow \quad S_{11} = \frac{Z}{Z + 2Z_0}$$

By symmetry:

$$S_{22} = S_{11}$$

Example (cont.)

S_{21} Calculation:



$$S_{21} = \left. \frac{b_2(0)}{a_1(0)} \right|_{a_2=0}$$

$$= \left. \frac{V_2^-(0)}{V_1^+(0)} \right|_{a_2=0}$$

$$V_1^+(0) = a_1(0)\sqrt{Z_0}$$

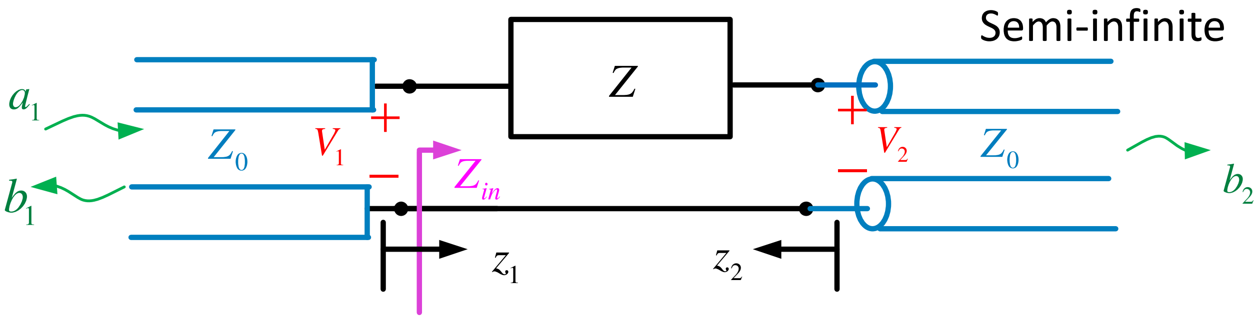
$$a_2 = 0 \Rightarrow V_2^-(0) = V_2(0)$$

$$V_2(0) = V_1(0)\left(\frac{Z_0}{Z + Z_0}\right)$$

$$V_1(0) = a_1\sqrt{Z_0}\,(1 + S_{11})$$

$$\Rightarrow V_2^-(0) = V_2(0) = a_1\sqrt{Z_0}\,(1 + S_{11})\left(\frac{Z_0}{Z + Z_0}\right)$$

Example (cont.)



$$S_{21} = \frac{a_1(0)\sqrt{Z_0}\,(1+S_{11})\left(\dfrac{Z_0}{Z+Z_0}\right)}{a_1(0)\sqrt{Z_0}}$$

$$= (1+S_{11})\left(\frac{Z_0}{Z+Z_0}\right) = \left(1+\frac{Z}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right) = \left(\frac{2Z+2Z_0}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right)$$

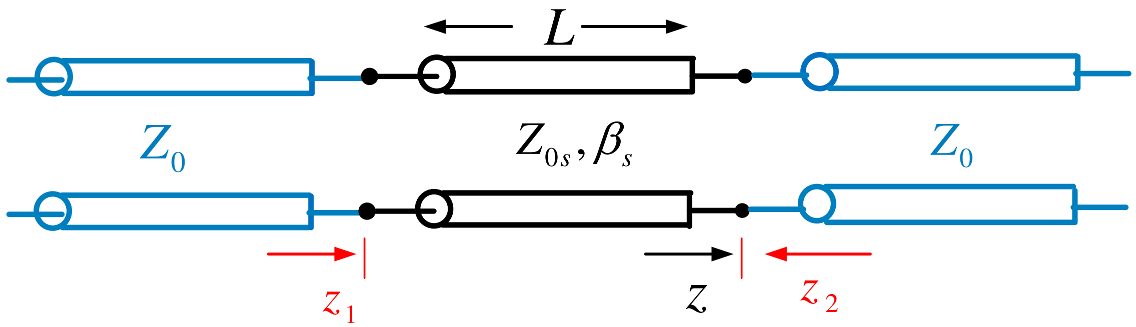
Hence

$$S_{21} = \frac{2Z_0}{Z+2Z_0}$$

$$S_{12} = S_{21}$$

Example

Find the S parameters for a length L of transmission line.

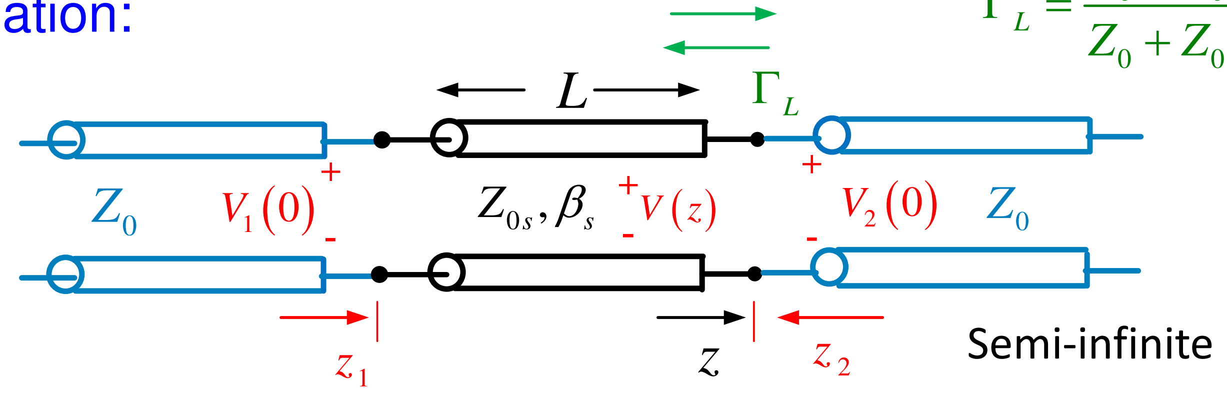


Note that three different coordinate systems are being used here!

Example (cont.)

S_{11} Calculation:

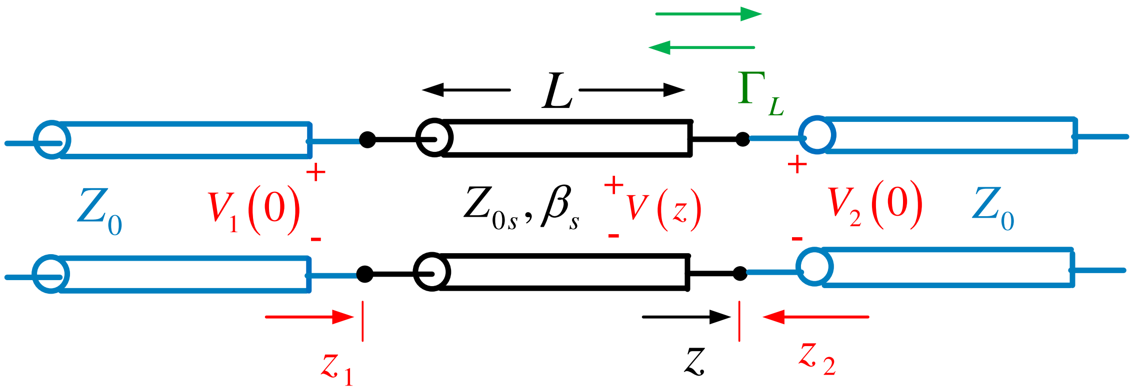
$$\Gamma_L = \frac{Z_0 - Z_{0s}}{Z_0 + Z_{0s}}$$



$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \frac{Z_{in}|_{a_2=0} - Z_0}{Z_{in}|_{a_2=0} + Z_0} = S_{22} \text{ (by symmetry)}$$

$$Z_{in}|_{a_2=0} = Z_{0s}\frac{(Z_0 + jZ_{0s}\tan\beta_s L)}{(Z_{0s} + jZ_0\tan\beta_s L)} = Z_{0s}\frac{\left(1+\Gamma_L e^{-j2\beta_s L}\right)}{\left(1-\Gamma_L e^{-j2\beta_s L}\right)}$$

Example (cont.)



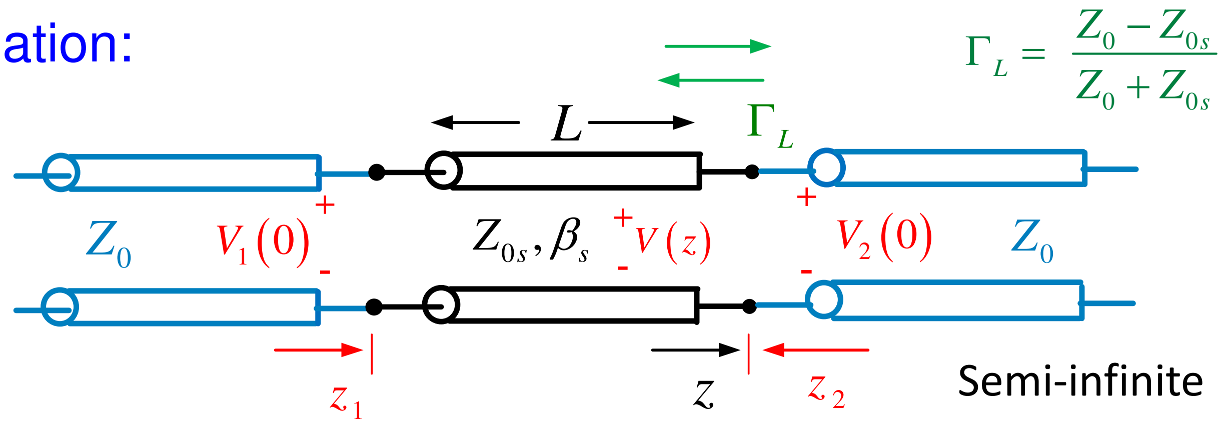
Hence

$$S_{11} = S_{22} = \frac{Z_{0s}\dfrac{\left(Z_0 + jZ_{0s}\tan\beta_s L\right)}{\left(Z_{0s} + jZ_0\tan\beta_s L\right)} - Z_0}{Z_{0s}\dfrac{\left(Z_0 + jZ_{0s}\tan\beta_s L\right)}{\left(Z_{0s} + jZ_0\tan\beta_s L\right)} + Z_0}$$

Note: If $Z_{0s} = Z_0 \Rightarrow Z_{in}\big|_{a_2=0} = Z_0 \Rightarrow S_{11} = S_{22} = 0$

Example (cont.)

S_{21} Calculation:



$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \left.\frac{V_2^-(0)/\sqrt{Z_0}}{V_1^+(0)/\sqrt{Z_0}}\right|_{a_2=0}$$

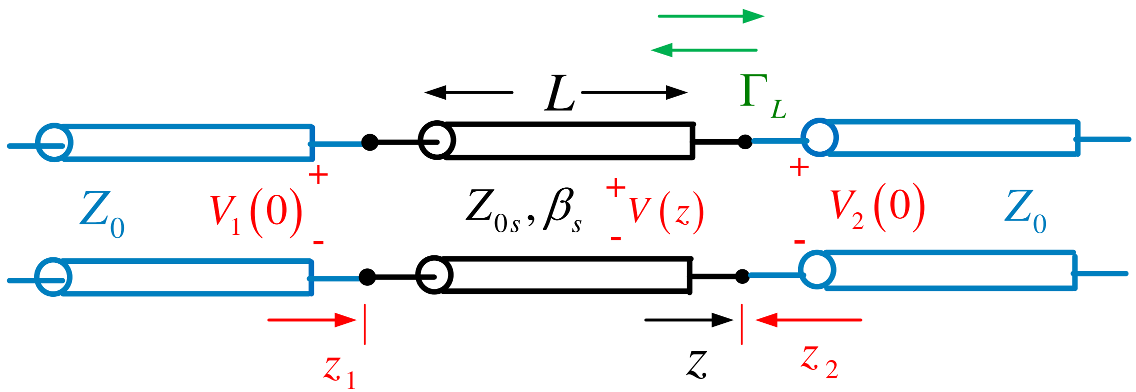
$$V_1(0) = V_1^+(0)\left(1 + S_{11}\right)$$

Hence, for the denominator of the S_{21} equation we have

$$V_1^+(0) = \frac{V_1(0)}{1 + S_{11}}$$

We now try to put the numerator of the S_{21} equation in terms of $V_1(0)$.

Example (cont.)



$$V_2^-(0) = V_2(0) = V(0) = V^+(0)(1+\Gamma_L)$$

Next, use

$$V(z) = V^+(0)e^{-j\beta_s z}\left(1+\Gamma_L e^{+j2\beta_s z}\right)$$

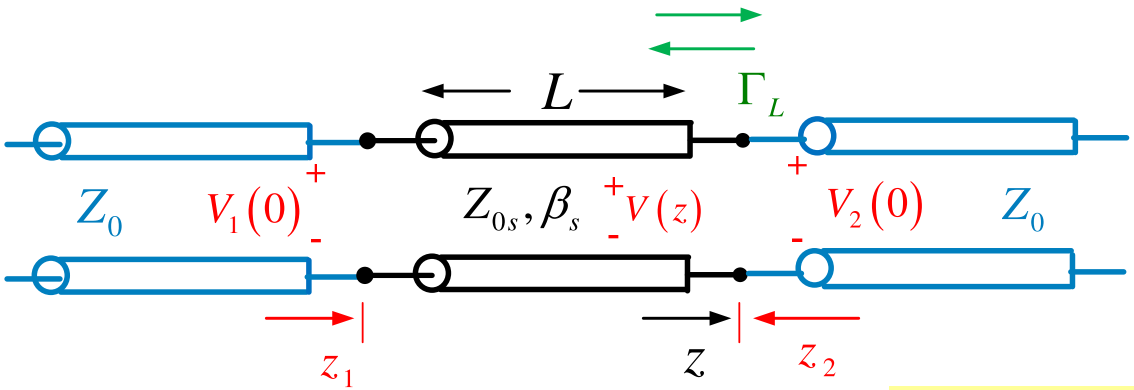
$$\Rightarrow V_1(0) = V(-L) = V^+(0)e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)$$

$$\Rightarrow V^+(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}$$

Hence, we have

$$V_2^-(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}(1+\Gamma_L)$$

Example (cont.)



$$V_2^-(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}\left(1+\Gamma_L\right)$$

$$V_1^+(0) = \frac{V_1(0)}{1+S_{11}}$$

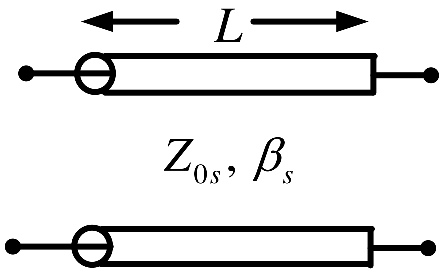
Therefore, we have

$$S_{21} = \left.\frac{V_2^-(0)}{V_1^+(0)}\right|_{a_2=0} = \frac{\left(1+S_{11}\right)\left(1+\Gamma_L\right)e^{-j\beta_s L}}{1+\Gamma_L e^{-j2\beta_s L}}$$

so

$$S_{21} = \frac{\left(1+S_{11}\right)\left(1+\Gamma_L\right)e^{-j\beta_s L}}{1+\Gamma_L e^{-j2\beta_s L}} = S_{12} \text{ by symmetry}$$

Example (cont.)



Special cases:

$$a) \quad Z_{0s} = Z_0 \Rightarrow S_{11} = S_{22} = 0, \quad \Gamma_L = 0$$

$$S_{21} = S_{12} = e^{-j\beta_s L}$$

$$[S] = \begin{bmatrix} 0 & e^{-j\beta_s L} \\ e^{-j\beta_s L} & 0 \end{bmatrix}$$

$$b) \quad L = \frac{\lambda_g}{2} \Rightarrow \beta_s L = \frac{2\pi}{\lambda_g}\frac{\lambda_g}{2} = \pi$$

$$\Rightarrow Z_{in}\big|_{a_2=0} = Z_0 \quad \Rightarrow S_{11} = S_{22} = 0$$

$$e^{-j\beta_s L} = -1 \quad \Rightarrow \quad S_{21} = -1$$

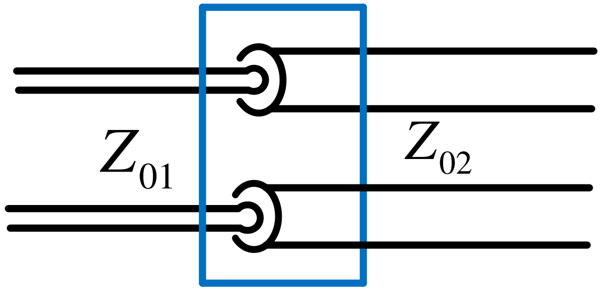
$$[S] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Example

Find the S parameters for a step-impedance discontinuity.

$$S_{11} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

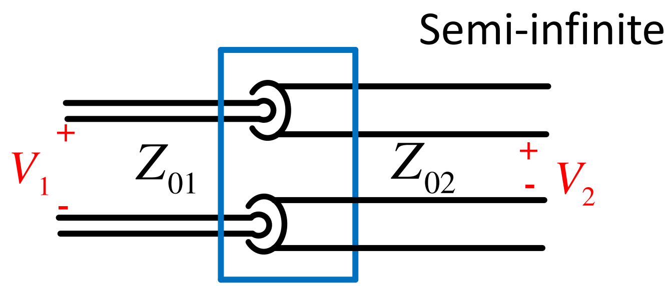
$$S_{22} = \frac{Z_{01} - Z_{02}}{Z_{02} + Z_{01}} = -S_{11}$$



$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0} = \left.\frac{\dfrac{V_2^-(0)}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0}$$

Example (cont.)

S_{21} Calculation:



Because of continuity of the voltage across the junction, we have:

$$V_2^-(0)\Big|_{a_2=0} = V_2(0)\Big|_{a_2=0} = V_1(0)\Big|_{a_2=0} = V_1^+(0)\left(1+S_{11}\right)$$

$$S_{21} == \left.\frac{\dfrac{V_2^-(0)}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0} = \left.\frac{\dfrac{V_1^+(0)\left(1+S_{11}\right)}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0}$$

$$1+S_{11} = 1+\frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}} = \frac{2Z_{02}}{Z_{02}+Z_{01}}$$

so

$$S_{21} = \left(1+S_{11}\right)\sqrt{\frac{Z_{01}}{Z_{02}}}$$

Hence

$$S_{21} = S_{12} = 2\frac{\sqrt{Z_{01}Z_{02}}}{Z_{01}+Z_{02}}$$

Properties of the S Matrix

❖For reciprocal networks, the S-matrix is symmetric.

$$\Rightarrow \quad [S] = [S]^T$$

Note :

If $[A][B] = [U]$

then

$[B][A] = [U]$

❖For lossless networks, the S-matrix is unitary.

$$\Rightarrow \quad [S]^T [S]^* = [S]^* [S]^T = [U]$$

Identity matrix

Equivalently,

$$[S]^{T*} = [S]^{-1}$$

Notation: $[S]^\dagger = [S]^H = [S]^{T*}$

so $[S]^\dagger = [S]^{-1}$

N-port network

Take (i, j) element $\Rightarrow \sum_{k=1}^{N} S_{ik}^T \, S_{kj}^* = \sum_{k=1}^{N} S_{ki} \, S_{kj}^* = \delta_{ij}$ $\quad \delta_{ij} = \begin{cases} 1\,;\, i = j \\ 0;\, i \neq j \end{cases}$

Properties of the S Matrix (cont.)

Example:

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Unitary $\Rightarrow$

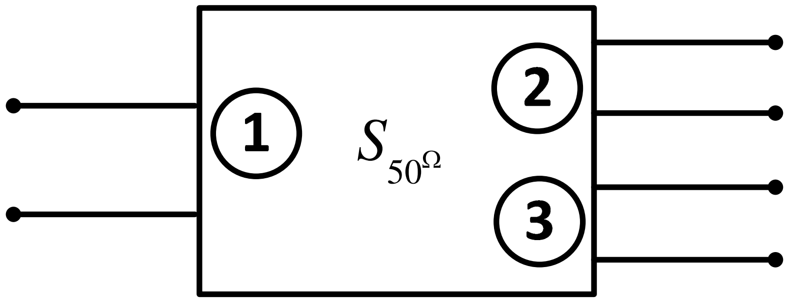
$$S_{11}S_{11}^* + S_{21}S_{21}^* + S_{31}S_{31}^* = 1$$
$$S_{12}S_{12}^* + S_{22}S_{22}^* + S_{32}S_{32}^* = 1$$
$$S_{13}S_{13}^* + S_{23}S_{23}^* + S_{33}S_{33}^* = 1$$
$$S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^* = 0$$
$$S_{11}S_{13}^* + S_{21}S_{23}^* + S_{31}S_{33}^* = 0$$
$$S_{12}S_{13}^* + S_{22}S_{23}^* + S_{32}S_{33}^* = 0$$

The column vectors form an orthogonal set.

The rows also form orthogonal sets (see the note on the previous slide).

Example

$$[S_{50\Omega}] = \begin{bmatrix} 0 & \frac{-j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & 0 & 0 \\ \frac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$



Not unitary $\rightarrow$ Not lossless

(For example, column 2 doted with the conjugate of column three is not zero.

1) Find the input impedance looking into port 1 when ports 2 and 3 are terminated in $50\ [\Omega]$ loads.

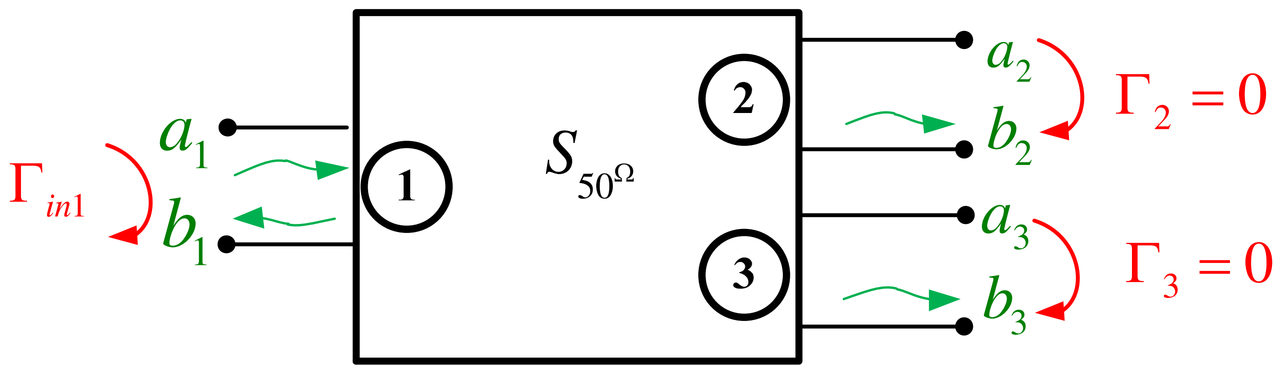
2) Find the input impedance looking into port 1 when port 2 is terminated in a $75\ [\Omega]$ load and port 3 is terminated in a $50\ [\Omega]$ load.

Example (cont.)

1 If ports 2 and 3 are terminated in 50 [Ω]: $(a_2 = a_3 = 0)$

$$b_1 = S_{11}a_1 + S_{12}\cancel{a_2} + S_{13}\cancel{a_3}$$

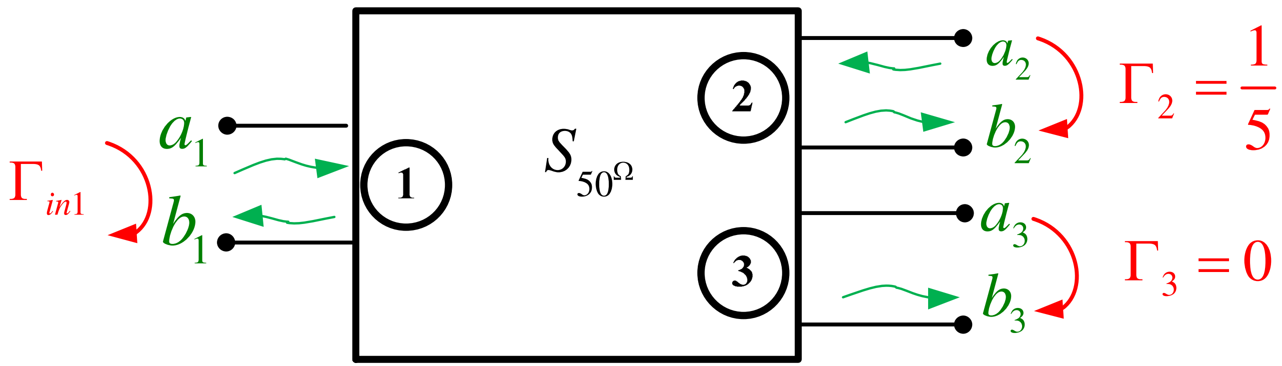
$$\Rightarrow \Gamma_{in1} = \frac{b_1}{a_1} = S_{11} = 0 \quad \Rightarrow \quad Z_{in1} = 50[\Omega]$$



Example (cont.)

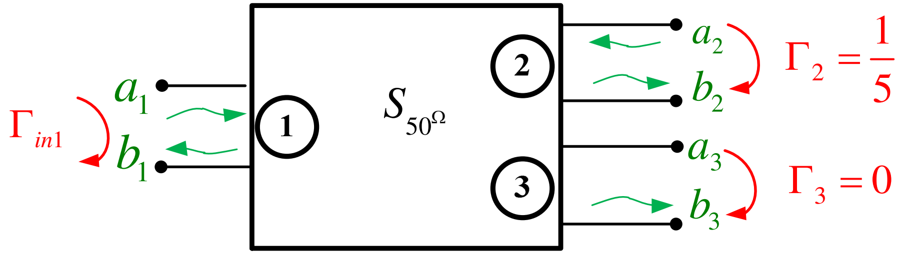
2) If port 2 is terminated in 75 [Ω] and port 3 in 50 [Ω]:

$$\Rightarrow \quad \Gamma_2 = \frac{a_2}{b_2} = \frac{75-50}{75+50} = \frac{1}{5}$$



Example (cont.)

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & 0 & 0 \\ \frac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$



$$b_2 / a_1 = S_{21} + \cancel{S_{22}} \left(\frac{a_2}{a_1}\right) + \cancel{S_{23}} \left(\frac{a_3}{a_1}\right)$$

$$\Rightarrow \Gamma_{in1} = \frac{b_1}{a_1} = \cancel{S_{11}} + S_{12}\frac{a_2}{a_1} + \cancel{S_{13}\frac{a_3}{a_1}}$$

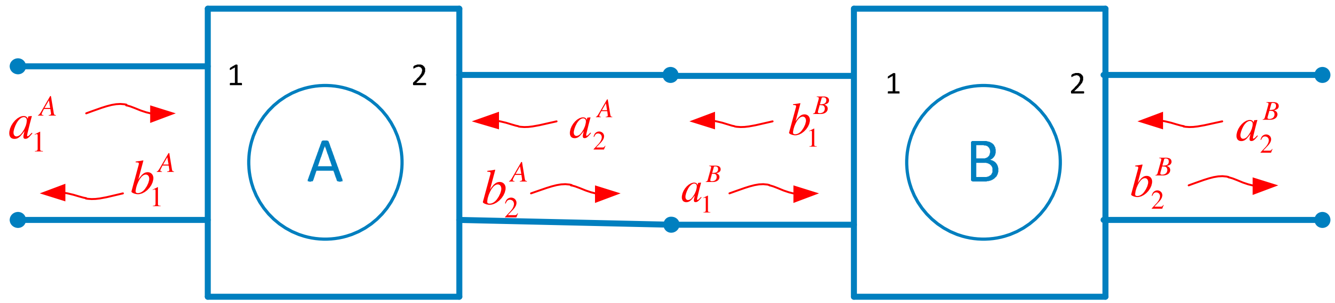
$$= S_{12}\left(\Gamma_2 \frac{b_2}{a_1}\right) = S_{12}\left(\Gamma_2 S_{21}\right) = \left(\frac{-j}{\sqrt{2}}\right)\left(\frac{1}{5}\right)\left(\frac{-j}{\sqrt{2}}\right) = -\frac{1}{10}$$

$$a_2 = \Gamma_2 b_2$$

$$\Rightarrow Z_{in1} = 50\left(\frac{1+\Gamma_{in_1}}{1-\Gamma_{in_1}}\right) = 44.55[\Omega]$$

Transfer (T) Matrix

For cascaded 2-port networks:



T Matrix:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

$$= [T] \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

$$[T] = \begin{bmatrix} \dfrac{1}{S_{21}} & \dfrac{-S_{22}}{S_{21}} \\ \dfrac{S_{11}}{S_{21}} & S_{12} - \dfrac{S_{11}S_{22}}{S_{21}} \end{bmatrix}$$

$$[S] = \begin{bmatrix} \dfrac{-T_{21}}{T_{22}} & \dfrac{1}{T_{22}} \\ T_{11} - \dfrac{T_{12}^2}{T_{22}} & \dfrac{T_{12}}{T_{22}} \end{bmatrix}$$

(Derivation omitted)

Transfer (T) Matrix (cont.)

$$\Rightarrow \begin{bmatrix} a_1^A \\ b_1^A \end{bmatrix} = \begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} b_2^A \\ a_2^A \end{bmatrix}$$

But $$\begin{bmatrix} b_2^A \\ a_2^A \end{bmatrix} = \begin{bmatrix} a_1^B \\ b_1^B \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1^A \\ b_1^A \end{bmatrix} = \begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} a_1^B \\ b_1^B \end{bmatrix}$$

Hence $$\begin{bmatrix} a_1^A \\ b_1^A \end{bmatrix} = \underbrace{\begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} T^B \end{bmatrix}}_{\begin{bmatrix} T^{AB} \end{bmatrix}} \begin{bmatrix} b_2^B \\ a_2^B \end{bmatrix}$$

The T matrix of a cascaded set of networks is the product of the T matrices.

Conversion Between Parameters

TABLE 4.2 Conversions Between Two-Port Network Parameters

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\lvert Y\rvert}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\lvert Y\rvert}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\lvert Y\rvert}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\lvert Y\rvert}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\lvert Z\rvert}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\lvert Z\rvert}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\lvert Z\rvert}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\lvert Z\rvert}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\lvert Z\rvert}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\lvert Y\rvert}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

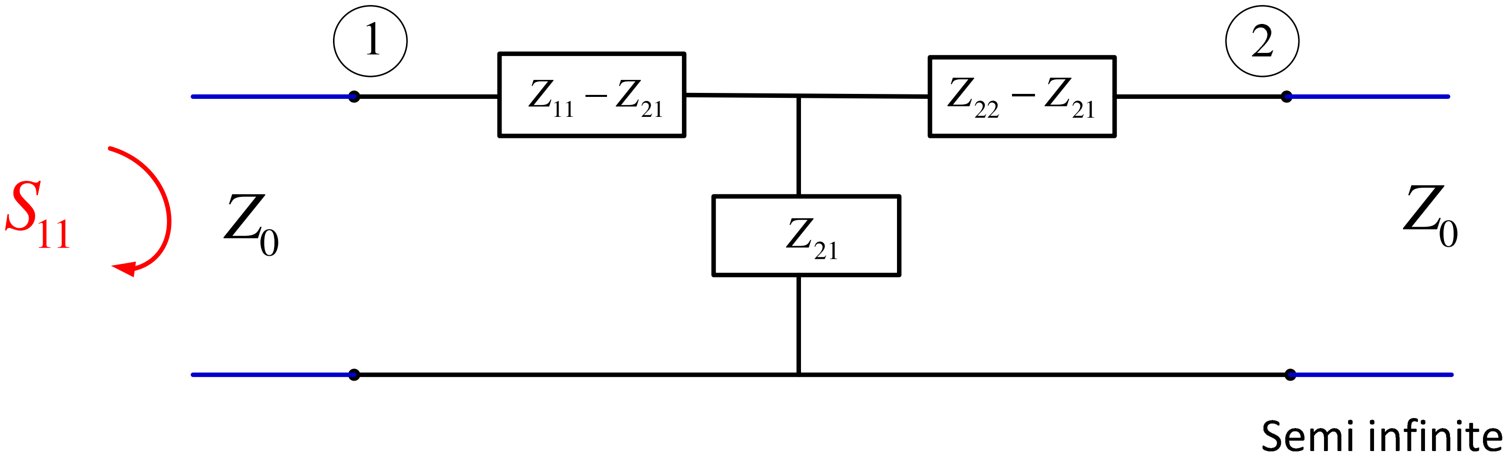
$\lvert Z\rvert = Z_{11}Z_{22} - Z_{12}Z_{21}$; $\lvert Y\rvert = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$

Example

Derive S_{ij} from the Z parameters.

(The result is given inside row 1, column 2, of the previous table.)

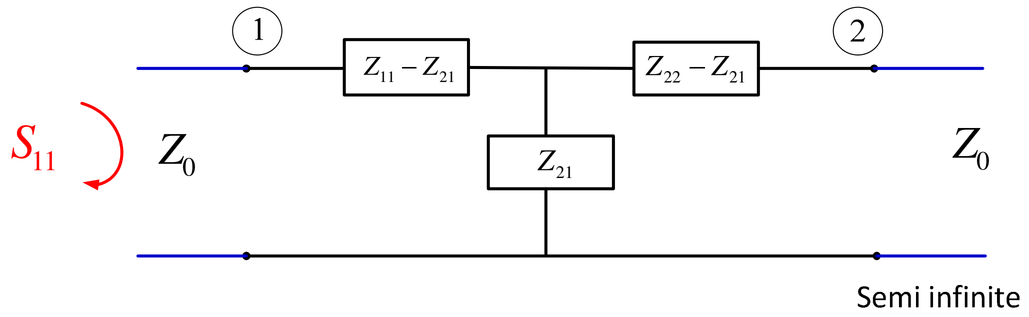
S_{11} Calculation:



$$S_{11} = \Gamma_{in1} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

$$Z_{in} = (Z_{11} - Z_{21}) + (Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0 \right]$$

Example (cont.)



$$Z_{in} = (Z_{11} - Z_{21}) + (Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0 \right]$$

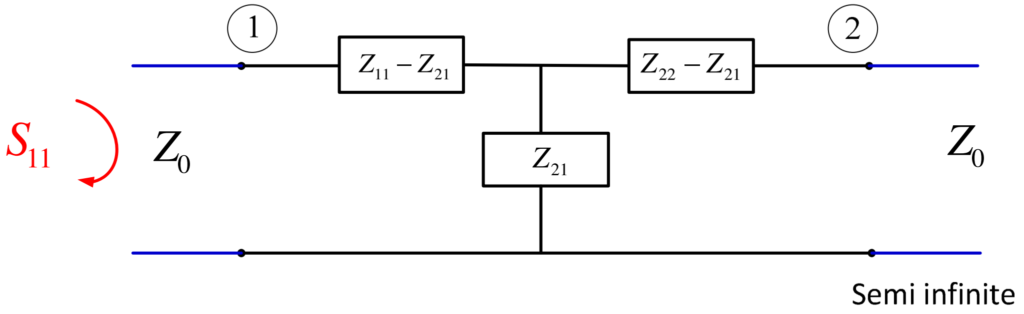
$$= (Z_{11} - Z_{21}) + \frac{Z_{21}(Z_{22} + Z_0 - Z_{21})}{Z_{22} + Z_0}$$

$$= \frac{(Z_{11} - Z_{21})(Z_{22} + Z_0) + Z_{21}(Z_{22} + Z_0 - Z_{21})}{Z_{22} + Z_0}$$

$$= \frac{Z_{11}Z_{22} + Z_{11}Z_0 - Z_{21}Z_{22} - Z_{21}Z_0 + Z_{21}Z_{22} + Z_{21}Z_0 - Z_{21}^2}{Z_{22} + Z_0}$$

$$= \frac{Z_{11}Z_{22} + Z_{11}Z_0 - Z_{21}^2}{Z_{22} + Z_0}$$

Example (cont.)



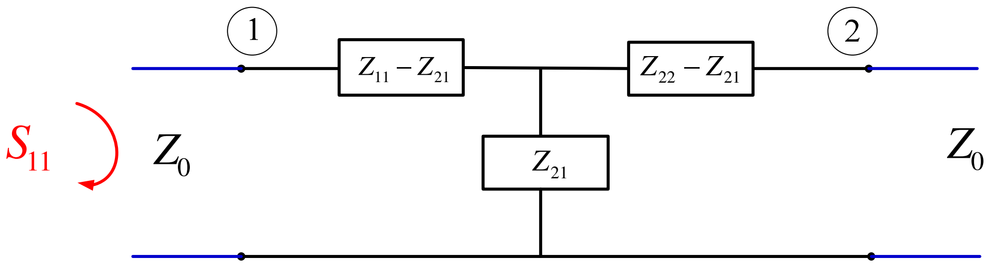
$$Z_{in} = \frac{Z_{11}(Z_0 + Z_{22}) - Z_{21}^2}{Z_{22} + Z_0}$$

$$S_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

so

$$S_{11} = \frac{Z_{11}(Z_0 + Z_{22}) - Z_{21}^2 - Z_0(Z_0 + Z_{22})}{Z_{11}(Z_0 + Z_{22}) - Z_{21}^2 + Z_0(Z_0 + Z_{22})}$$

Example (cont.)

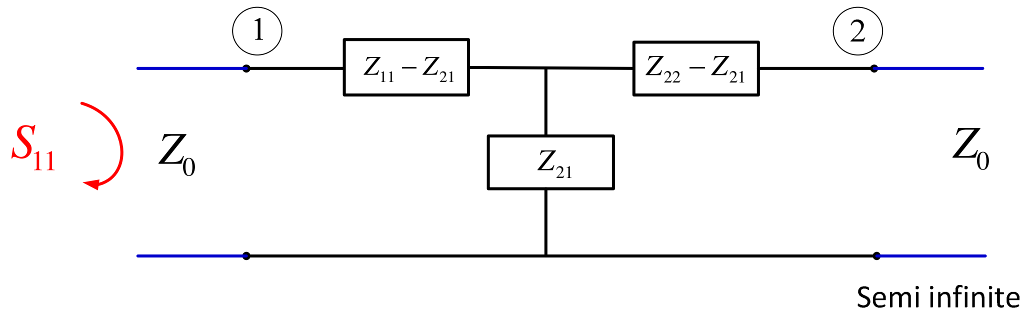


$$S_{11}=\frac{Z_{11}\left(Z_0+Z_{22}\right)-Z_{21}^2-Z_0\left(Z_0+Z_{22}\right)}{Z_{11}\left(Z_0+Z_{22}\right)-Z_{21}^2+Z_0\left(Z_0+Z_{22}\right)}$$

$$=\frac{Z_{11}Z_0+Z_{11}Z_{22}-Z_{21}^2-Z_0^2-Z_0Z_{22}}{Z_{11}Z_0+Z_{11}Z_{22}-Z_{21}^2+Z_0^2+Z_0Z_{22}}$$

$$=\frac{\left(Z_0+Z_{22}\right)\left(Z_{11}-Z_0\right)-Z_{21}^2}{\left(Z_0+Z_{22}\right)\left(Z_{11}+Z_0\right)-Z_{21}^2}$$

Example (cont.)



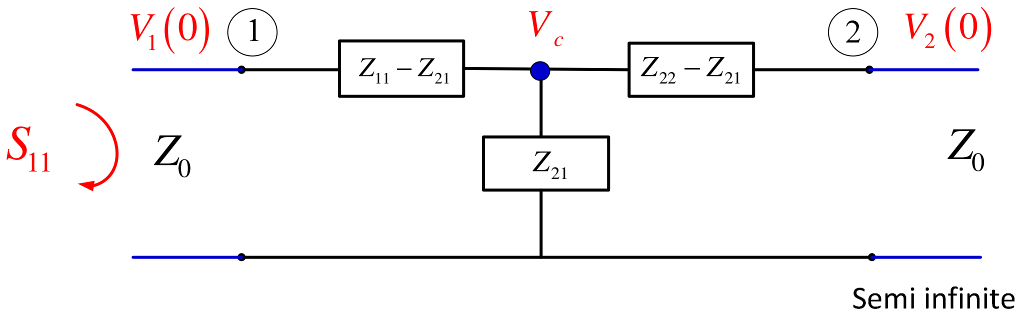
$$S_{11} = \frac{(Z_0 + Z_{22})(Z_{11} - Z_0) - Z_{21}^2}{(Z_0 + Z_{22})(Z_{11} + Z_0) - Z_{21}^2}$$

Note: to get S_{22}, simply let $Z_{11} \to Z_{22}$ in the previous result.

$$S_{22} = \frac{(Z_0 + Z_{11})(Z_{22} - Z_0) - Z_{21}^2}{(Z_0 + Z_{11})(Z_{22} + Z_0) - Z_{21}^2}$$

Example (cont.)

S_{21} Calculation:



Assume $V_1^+(0) = 1 \text{ [V]}$

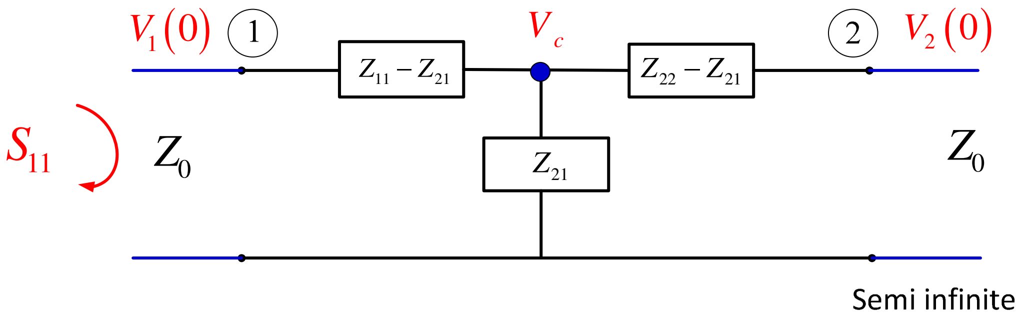
$$V_1(0) = 1 + S_{11} \qquad S_{21} = V_2^-(0) = V_2(0)$$

Use voltage divider equation twice:

$$V_c = V_1(0)\left(\frac{(Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0\right]}{(Z_{11} - Z_{21}) + (Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0\right]}\right)$$

$$V_2(0) = V_c\left(\frac{Z_0}{(Z_{22} - Z_{21}) + Z_0}\right)$$

Example (cont.)



Hence

$$S_{21}=(1+S_{11})\left(\frac{(Z_{21})\,||\,\left[(Z_{22}-Z_{21})+Z_0\right]}{(Z_{11}-Z_{21})+(Z_{21})\,||\,\left[(Z_{22}-Z_{21})+Z_0\right]}\right)\left(\frac{Z_0}{(Z_{22}-Z_{21})+Z_0}\right)$$

After simplifying, we should get the result in the table.

(You are welcome to check it!)