Network Theorems

1 Superposition Theorem

The basic principle of superposition states that, if the effect produced in a system is directly proportional to the cause, then the overall effect produced in the system, due to a number of causes acting jointly, can be determined by superposing (adding) the effects of each source acting separately. This principle is only applicable to "linear" networks and system.

Consider the circuits below. In Fig. 1, clearly the voltage across the resistor and the current are given by

$V_R = V_1 - V_2; I = V_R / R = V_1 / R - V_2 / R$.

Now, applying the superposition theorem yields

$I = I_1 + I_2 = V_1 / R - V_2 / R$,

i.e., the sum of the currents due to two sources.

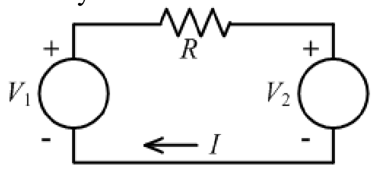

Fig. 1: A circuit

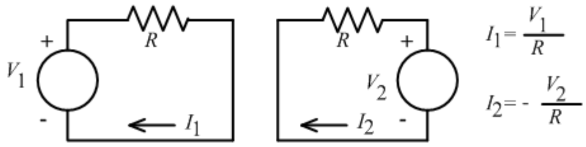

Fig. 2: Illustration of Superposition Theorem

<u>Question</u> Is the superposition theorem applicable to the power as well?

<u>Example 1</u> Verify the superposition theorem.

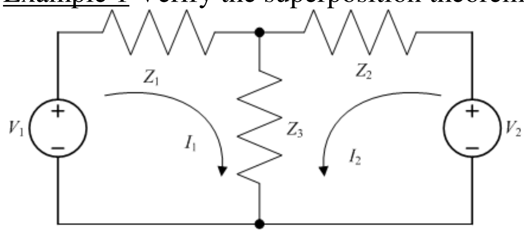

Fig. 3: Example 1 problem

<u>Example 2</u> Verify the superposition theorem.

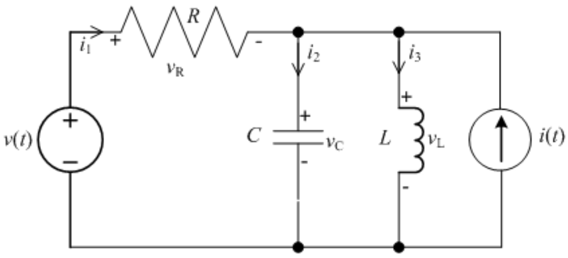

Fig. 4: Example 2 problem

2 Reciprocity Theorem

Consider two loops *A* and *B* of a network *N* where an ideal voltage source *V* in loop *A* produces a current *I* in loop *B*, then the network is said to be <u>reciprocal</u> if an identical source in loop *B* produces the same current *I* in loop *A*. In short, a linear network is said to be reciprocal if it remains invariant due to the interchange of position of cause (source) and effect (linear elements) in the network.

Example 3 Verify the reciprocity theorem.

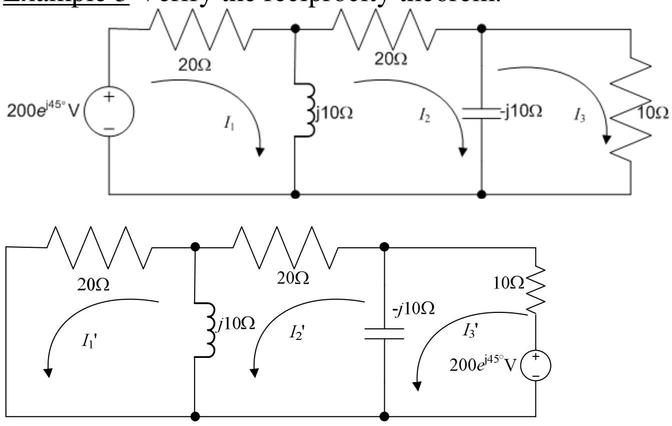

Fig. 5: Example 3 problem

3 Thevenin's Theorem

This theorem states that a linear circuit containing one or more sources and other linear elements can be represented by a voltage source V_{TH} in series with an impedance Z_{TH}. V_{TH} is the open-circuit voltage between the terminals of the network and Z_{TH} is the impedance measured between the terminals of the network with all sources removed(but not their impedances). This is also called the *voltage source equivalent circuit*.

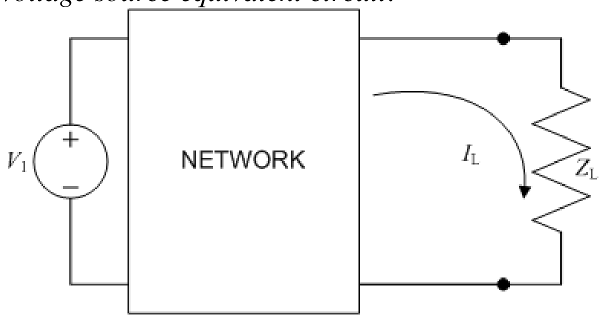

Fig. 6: General network

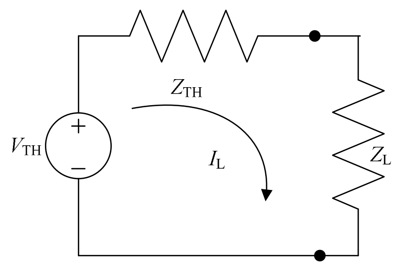

Fig. 7: Equivalent circuit

Example 4 Find the Thevenin's equivalent circuit.

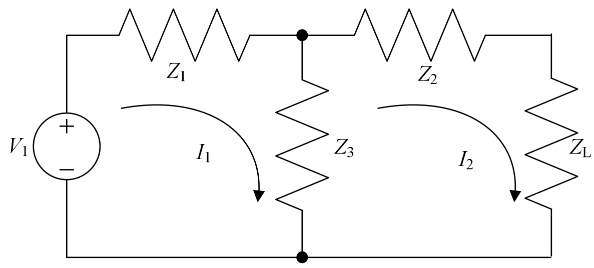

Fig. 8: Example 4 problem

Example 5 Find the Thevenin's equivalent circuit.

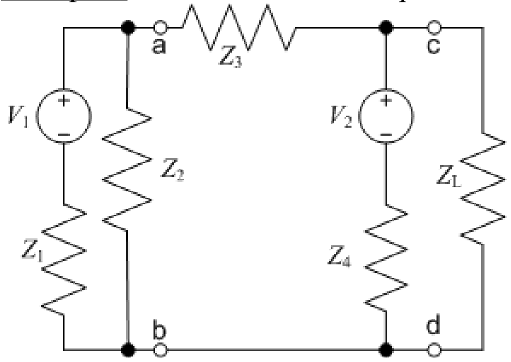

Fig. 9: Example 5 problem

4 Norton's Theorem

Norton's theorem says that the linear network consisting of one or more independent sources and linear elements can be represented by a current source I_{SC} and an equivalent impedance Z_{TH} in parallel with the current source. I_{SC} is the short-circuit current between the terminals of the network and Z_{TH} is the impedance measured between the terminals with all sources removed (but not their impedances). This is also called the *current source equivalent circuit*.

Example Repeat examples 4, 5 using Norton's equivalent circuits.

5 Millman's Theorem

Let V_i (i=1,2,…,n) be the open-circuit voltages of n voltage sources having internal impedances Z_i in series, respectively, as shown in Fig.10. Suppose these sources are connected in parallel, then they may be replaced by a single ideal voltage source V in series with an impedance Z, where

$$V = \frac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i}; Z = \frac{1}{\sum_{i=1}^{n} Y_i}.$$

Proof

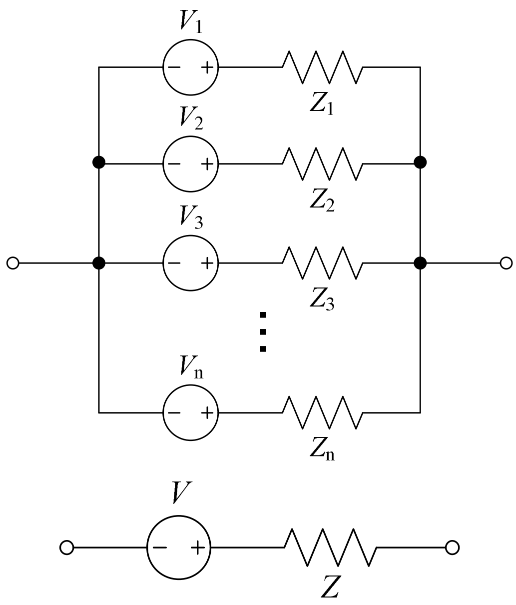

Fig. 10: Millman's Theorem

Example 6 Find the current I in Fig. 11.

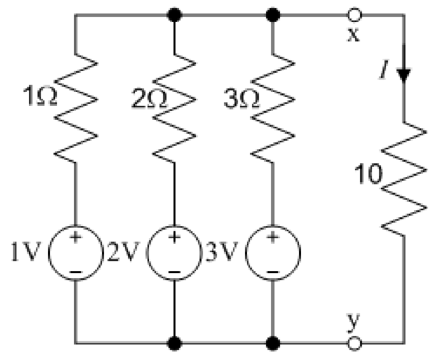

Fig. 11: Example 6 problem

6 Maximum Power Transfer Theorem

Maximum power will be delivered to a network, to an impedance Z_L if the impedance of Z_L is the complex conjugate of the impedance Z of the network, measured looking back into the terminals of the network.

Derivation

Example 7 A circuit model of a transistor driven by a current source $i(t)$ is shown in Fig. 12, where R_S is the source internal impedance and h_i,h_r,h_f and $1/h_0$ are transistor parameters. Find Thevenin's and Norton's equivalent circuits and derive the condition of maximum power transfer.

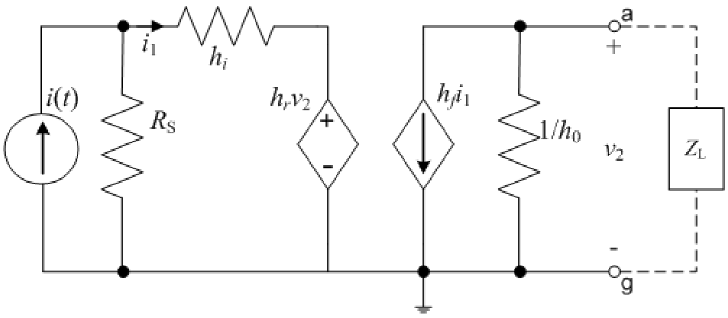

Fig. 12: Example 7 problem

7 Substitution Theorem

Sometimes, it is convenient to replace an impedance branch by another branch with different circuit components, without disturbing the voltage-current relationship in the network. The condition under which, branch replacement is possible, is given by the substitution theorem. It states that any branch in a network may be substituted by a different branch without disturbing the voltages and currents in the entire network, provided the new branch has the same set of terminal voltage and current as the original branch.

The substitution theorem is a general theorem and is applicable for any arbitrary network. It is very useful in circuit analysis of networks having one non-linear element. Also, it is often used to replace the effect of mutual inductance.

Example 8 Find the substitutions for xy branch.

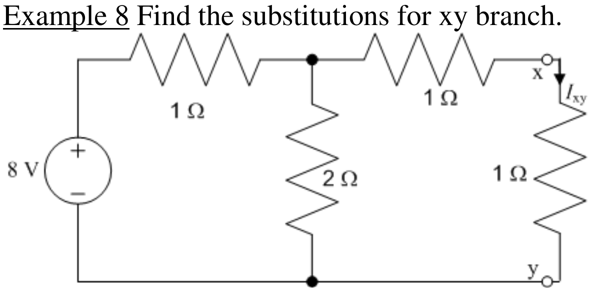

Fig. 13: Example 8 problem

8 Compensation Theorem

In some problems, we are interested in finding the corresponding changes in various voltages and currents of a network subjected to a change in one of its branches. The compensation theorem provides us a convenient method for determining such effects.

In a linear network *N*, if the current in a branch is *I* and the impedance *Z* of the branch is increased by Δ*Z*, then the increment of voltage and current in each branch of the network is that voltage or current that would be produced by an opposing voltage source of value V_c (=$I\Delta Z$) introduced into the altered branch after the modification. The compensation theorem is based on the superposition principle, and the network is required to be linear.

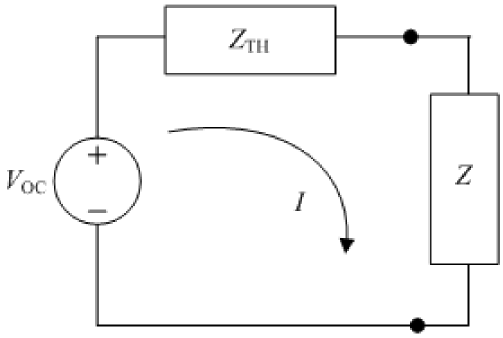

Fig. 14: Original Thevenin's equivalent circuit

Consider the network *N* in Fig. 14, having branch impedance *Z*, then

$$I = \frac{V_{OC}}{Z + Z_{TH}}.$$

Let δZ be the change in *Z*. Then *I*' (the new current) can be written as

$$I = \frac{V_{OC}}{Z + \delta Z + Z_{TH}},$$

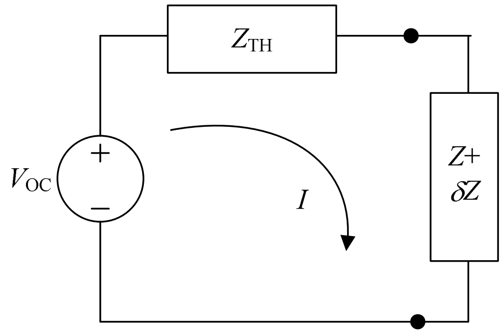

Fig. 15: Load is changed by δZ.

as shown in Fig. 15. It follows that

$$\delta I = I' - I = \frac{V_{OC}}{Z + \delta Z + Z_{TH}} - \frac{V_{OC}}{Z + Z_{TH}}$$

$$= -\left(\frac{V_{OC}}{Z + Z_{TH}}\right)\frac{\delta Z}{Z + \delta Z + Z_{TH}} = -\frac{I\delta Z}{Z + \delta Z + Z_{TH}}$$

$$= -\frac{V_C}{Z + \delta Z + Z_{TH}}$$

where $V_C = I\delta Z$, which is shown in Fig. 16.

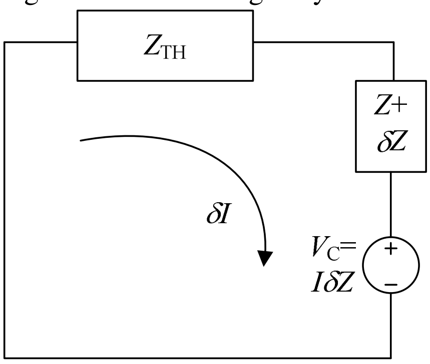

Fig. 16: Equivalent circuit by the Compensation theorem

<u>Example 9</u> Verify the compensation theorem when *R* is changed from 4 to 2 Ω.

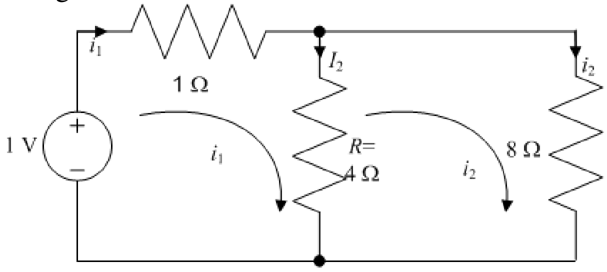

Fig. 17: Example 9 problem

9 Tellegen's Theorem

Tellegen's theorem is based on two Kirchhoff's laws and is applicable for any lumped network having elements which are linear or non-linear, active or passive, time-varying or time-invariant. It is completely independent of the nature of elements and is only concerned with the graph of the network. Consider an arbitrary lumped network whose graph *G* has *b* branches and *n* nodes. Suppose, to each branch of the graph, we assign arbitrarily a branch voltage v_k and a branch current i_k for $k = 1,2,\ldots,b$ and suppose that they are measured with respect to arbitrarily chosen associated reference directions. If the branch voltages $v_1, v_2, \ldots, v_b$ satisfy all the conditions imposed by KVL and if the branch currents $i_1, i_2, \ldots, i_b$ satisfy all the constraints imposed by KCL, then

$$\sum_{k=1}^{b} v_k(t) i_k(t') = 0, \forall t, t'.$$

It is noted that in a linear time-invariant network composed of energy sources and passive elements under steady-state sine-wave excitation, the conservation of power is depicted by Tellegen's theorem with $t = t'$.

Proof Consider a network *N* consisting of *b* branches and *n* nodes, let $\mathbf{A} = [a_{kj}]$ be the *incidence matrix*, whose elements are given by

$$a_{kj} = \begin{cases} +1, & \text{if branch } i_j \text{ leaves node } k \\ -1, & \text{if branch } i_j \text{ enters node } k \\ 0, & \text{otherwise.} \end{cases}$$

Then KCL can be written as

$$\mathbf{A}\underline{\mathbf{i}} = \underline{\mathbf{0}},$$

where $\underline{\mathbf{i}} = [i_1\ i_2\ \ldots\ i_b]^T$ denotes the *branch-current vector*. Note that $\mathbf{A}$ is an $n \times b$ matrix. Also, the branch voltages $\underline{\mathbf{v}} = [v_1\ v_2\ \ldots\ v_b]^T$ are related to the node voltages $\underline{\mathbf{v}_n} = [v_{n1}\ v_{n2}\ \ldots\ v_{nn}]^T$ by

$$\underline{\mathbf{v}} = \mathbf{A}^T \underline{\mathbf{v}_n}.$$

Therefore, $\sum_{k=1}^{b} v_k(t) i_k(t') = \underline{\mathbf{v}}^T \underline{\mathbf{i}} = \left(\mathbf{A}^T \underline{\mathbf{v}_n}\right)^T \underline{\mathbf{i}} = \underline{\mathbf{v}_n}^T \mathbf{A} \underline{\mathbf{i}} = 0$. (QED)

Furthermore, consider another network *N′*, which has the same *topological* configuration, the same references for branch currents and voltages, and the same numbering for the branches as the network *N*. Consequently, both networks have the same incidence matrix $\mathbf{A}$, and it follows that

$$\mathbf{A}\underline{\mathbf{i}'} = \underline{\mathbf{0}};\ \underline{\mathbf{v}'} = \mathbf{A}^T \underline{\mathbf{v}'_n},$$

where $\underline{\mathbf{i}'}$, $\underline{\mathbf{v}'}$, $\underline{\mathbf{v}'_n}$ denote branch current vector, branch voltage vector and node voltage vector, of the network *N′*, respectively. Therefore,

$$\sum_{k=1}^{b} v_k(t) i'_k(t') = \underline{\mathbf{v}}^T \underline{\mathbf{i}'} = \left(\mathbf{A}^T \underline{\mathbf{v}_n}\right)^T \underline{\mathbf{i}'} = \underline{\mathbf{v}_n}^T \mathbf{A} \underline{\mathbf{i}'} = 0,$$

which implies that Tellegen's theorem is applicable to two networks with the same topological configuration as well. Furthermore, it can be easily shown that

$$\underline{\mathbf{v}}^T \underline{\mathbf{i}} = \underline{\mathbf{v}}^T \underline{\mathbf{i}'} = \underline{\mathbf{v}'}^T \underline{\mathbf{i}} = \underline{\mathbf{v}'}^T \underline{\mathbf{i}'} = 0.$$

Example 10 Find all branch currents and voltages for both networks N_1, N_2 in Fig. 18, 19. Then verify Tellegen's theorem.

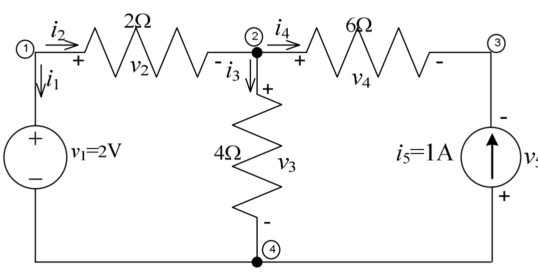

Fig. 18: Network N_1 in example 10 problem

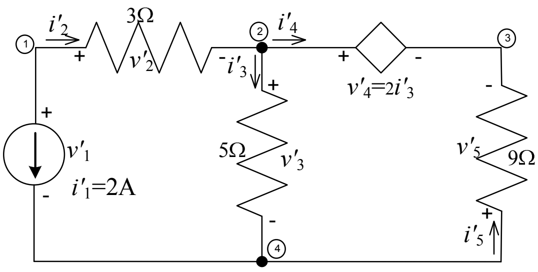

Fig. 19: Network N_2 in example 10 problem

Example 11 Verify Tellegen's theorem for networks N_1, N_2 in Fig. 20, 21. Assume steady-state conditions.

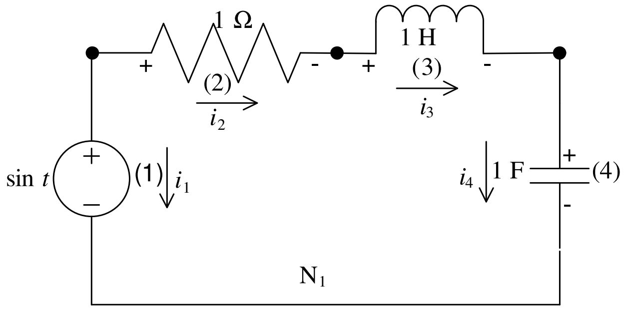

Fig. 20: Network N_1 in example 11 problem

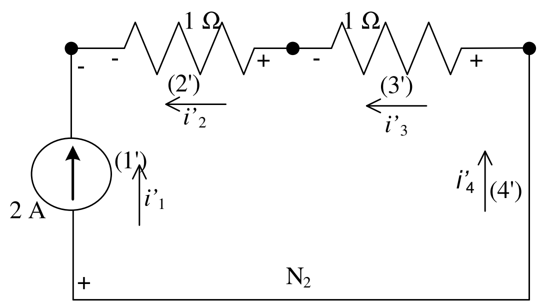

Fig. 21: Network N_2 in example 11 problem

Example 12 Consider two networks with the same topology and, inside their respective two-port boxes, the same set of elements—passive complex impedances $z_n(s)$. The outside elements differ—an open circuit at port 0 and a source at port 1 in one case, and a source at port 0 and a short circuit at port 1 in the other case.

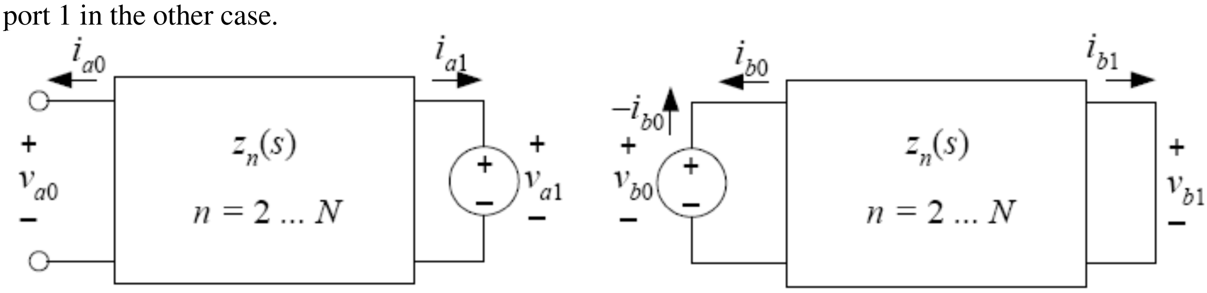

Fig. 20: Networks for example 12

Choosing the currents from network a and the voltages from network b for Tellegen's Theorem,

$$0 = \sum_{k=0}^{N} v_{bk} i_{ak} = v_{b0} i_{a0} + v_{b1} i_{a1} + \sum_{k=2}^{N} v_{bk} i_{ak} = \sum_{k=2}^{N} v_{bk} i_{ak} = \sum_{k=2}^{N} i_{ak} i_{bk} z_k(s)$$

Choosing the voltages from network a and the currents from network b for Tellegen's Theorem,

$$0 = \sum_{k=0}^{N} v_{ak} i_{bk} = v_{a0} i_{b0} + v_{a1} i_{b1} + \sum_{k=2}^{N} v_{ak} i_{bk} = v_{a0} i_{b0} + v_{a1} i_{b1} + \sum_{k=2}^{N} i_{ak} i_{bk} z_k(s) = v_{a0} i_{b0} + v_{a1} i_{b1}$$

Therefore we have a reciprocity of the reverse open-circuit voltage transfer equaling the forward short-circuit current transfer: $\dfrac{v_{a0}}{v_{a1}} = -\dfrac{i_{b1}}{i_{b0}}$.