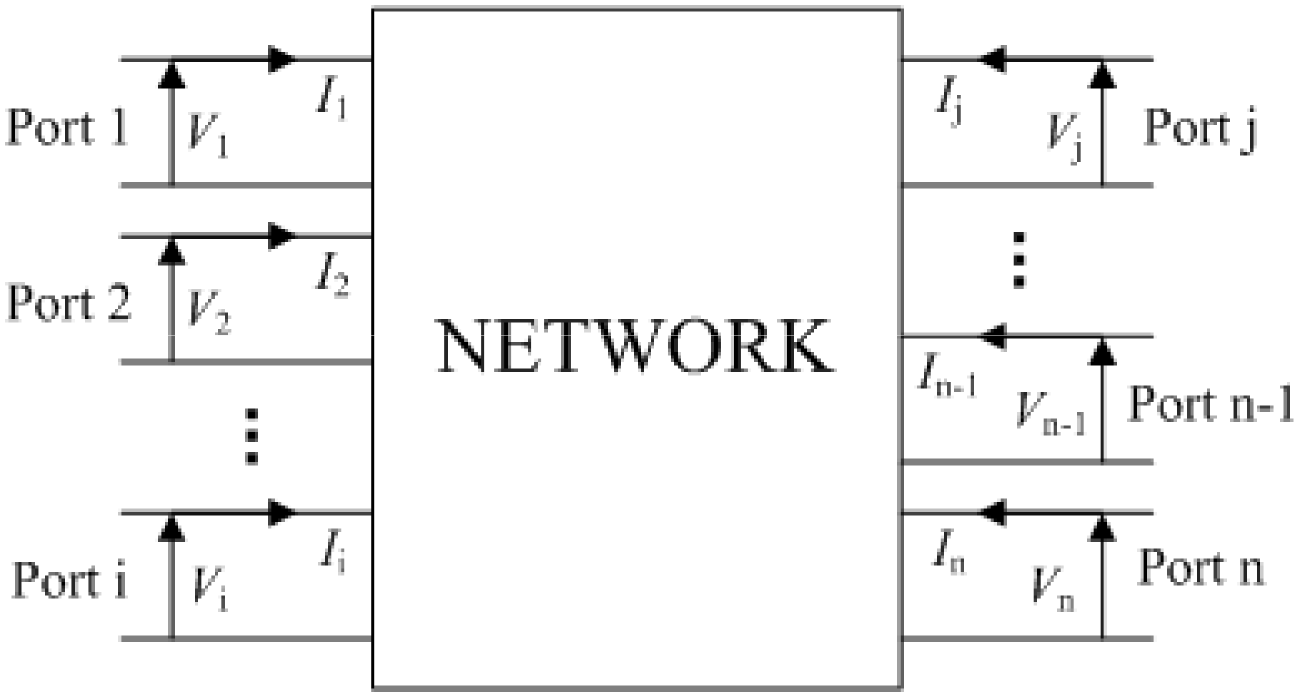
Two – Port Networks

N-Port Networks

Generalities:

The standard configuration of an *N*-port:



1-port Example : Resonator, Transmitter, Receiver
2-port Example : Filter, Amplifier
3-port Example : Coupler, Power Divider
N-port Example : Antenna Arrays, Switch

Z Matrix or Impedance Matrix

General Element

$$Z_{ij} = \left. \frac{V_i}{I_j} \right|_{I_k = 0 \text{ for } k \neq j}$$

Once the Z matrix is found, the network can be written systematically as

$$\underline{V} = [Z]\underline{I}$$

where

$$\underline{V} = [V_1 \, V_2 \cdots V_n]^T$$

$$\underline{I} = [I_1 \, I_2 \cdots I_n]^T$$

Then the currents in all ports can be obtained if all port voltages are known.

Y Matrix or Admittance Matrix

General Element

$$Y_{ij} = \left. \frac{I_i}{V_j} \right|_{V_k = 0 \text{ for } k \neq j}$$

Once the Y matrix is found, the network can be written systematically as

$$\underline{I} = [Y]\underline{V}$$

where

$$\underline{V} = [V_1 \, V_2 \cdots V_n]^T$$

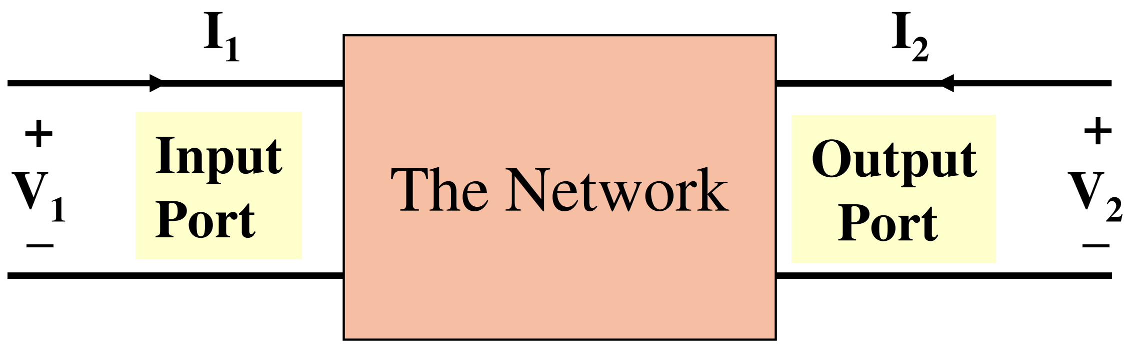
$$\underline{I} = [I_1 \, I_2 \cdots I_n]^T$$

Then the voltages in all ports can be obtained if all port currents are known.
The other *N*-port matrix is the scattering matrix (S matrix), which is used in microwave circuit analysis.

Two Port Networks

Generalities:

The standard configuration of a two port:



The network ?

The voltage and current convention ?

Two Port Networks

Network Equations:

Impedance Z parameters

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

Admittance Y parameters

$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

Transmission A, B, C, D parameters

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

Inverse Transmission T' parameters

$$V_2 = b_{11}V_1 - b_{12}I_1$$
$$I_2 = b_{21}V_1 - b_{22}I_1$$

Hybrid h parameters

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

Inverse Hybrid g parameters

$$I_1 = g_{11}V_1 + g_{12}I_2$$
$$V_2 = g_{21}V_1 + g_{22}I_2$$

Two Port Networks

Z parameters:

$$z_{11} = \frac{V_1}{I_1}\Bigg|_{I_2=0}$$

z_{11} is the impedance seen looking into port 1 when port 2 is open.

$$z_{12} = \frac{V_1}{I_2}\Bigg|_{I_1=0}$$

z_{12} is a transfer impedance. It is the ratio of the voltage at port 1 to the current at port 2 when port 1 is open.

$$z_{21} = \frac{V_2}{I_1}\Bigg|_{I_2=0}$$

z_{21} is a transfer impedance. It is the ratio of the voltage at port 2 to the current at port 1 when port 2 is open.

$$z_{22} = \frac{V_2}{I_2}\Bigg|_{I_1=0}$$

z_{22} is the impedance seen looking into port 2 when port 1 is open.

Two Port Networks

Y parameters:

$$y_{11} = \frac{I_1}{V_1}\Bigg|_{V_2=0}$$

y_{11} is the admittance seen looking into port 1 when port 2 is shorted.

$$y_{12} = \frac{I_1}{V_2}\Bigg|_{V_1=0}$$

y_{12} is a transfer admittance. It is the ratio of the current at port 1 to the voltage at port 2 when port 1 is shorted.

$$y_{21} = \frac{I_2}{V_1}\Bigg|_{V_2=0}$$

y_{21} is a transfer impedance. It is the ratio of the current at port 2 to the voltage at port 1 when port 2 is shorted.

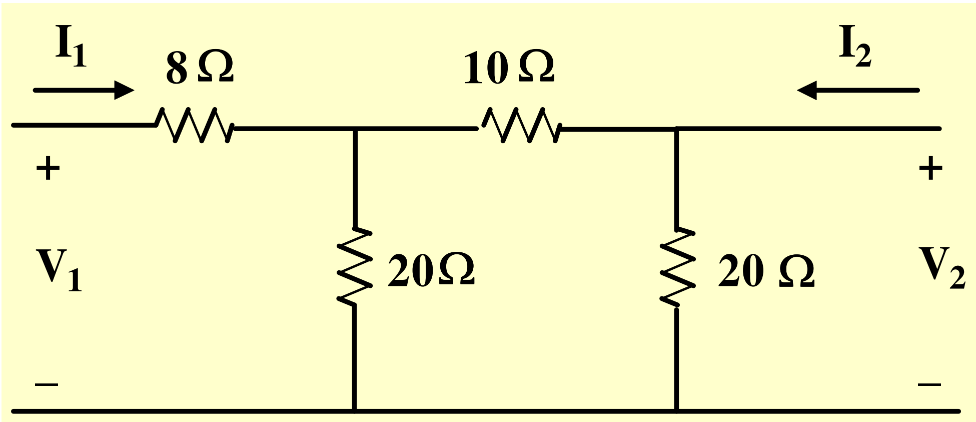
$$y_{22} = \frac{I_2}{V_2}\Bigg|_{V_1=0}$$

y_{22} is the admittance seen looking into port 2 when port 1 is shorted.

Two Port Networks

Z parameters: **Example 1**

Given the following circuit. Determine the Z parameters.



Find the Z parameters for the above network.

Two Port Networks

Z parameters:

Example 1 (cont'd)

For z_{11}:

$Z_{11} = 8 + 20||30 = 20\ \Omega$

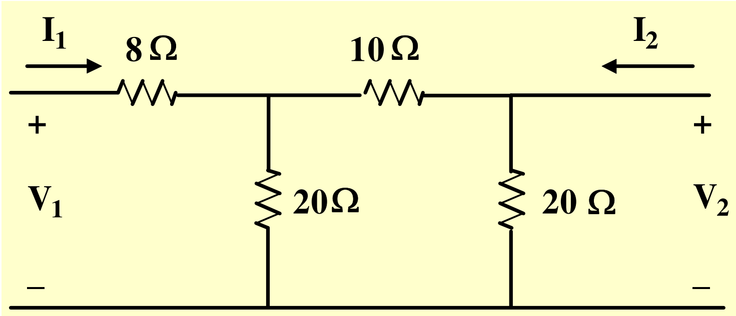
For z_{12}:

$$z_{12} = \frac{V_1}{I_2}\bigg|_{I_1=0}$$

$$V_1 = \frac{20 \times I_2 \times 20}{20+30} = 8I_2$$

For z_{22}:

$Z_{22} = 20||30 = 12\ \Omega$



Therefore:

$$z_{12} = \frac{8I_2}{I_2} = 8\ \ \Omega = z_{21}$$

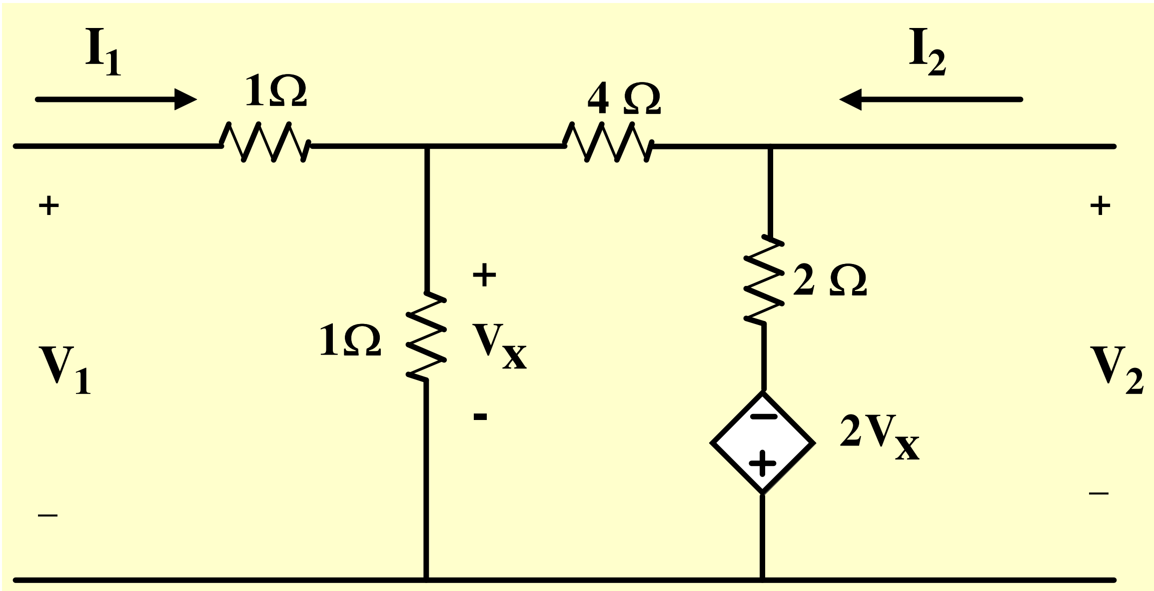
Thus:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 8 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Two Port Networks

Z parameters: **Example 2**

You are given the following circuit. Find the Z parameters.

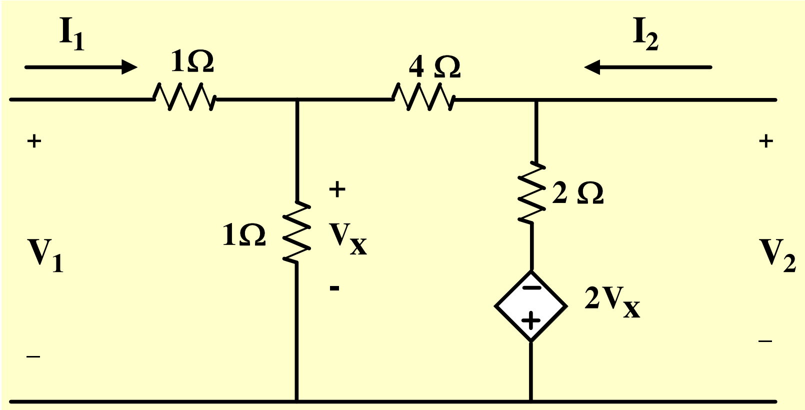


Two Port Networks

Z parameters:

Example 2 (continue p2)

$$z_{11} = \frac{V_1}{I_1}\Bigg|_{I_2=0}$$

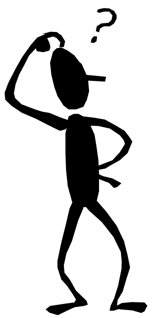


$$I_1 = \frac{V_x}{1} + \frac{V_x + 2V_x}{6} = \frac{6V_x + V_x + 2V_x}{6}$$

$$I_1 = \frac{3V_x}{2} \quad ; \quad \text{but } V_x = V_1 - I_1$$

Substituting gives;

$$I_1 = \frac{3(V_1 - I_1)}{2} \quad \text{or} \quad \frac{V_1}{I_1} = z_{11} = \frac{5}{3}\ \Omega$$



Other Answers

$Z_{21} = -0.667\ \Omega$

$Z_{12} = 0.222\ \Omega$

$Z_{22} = 1.111\ \Omega$

Two Port Networks

Transmission parameters (A,B,C,D):

The defining equations are:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$A = \frac{V_1}{V_2} \Bigg|_{I_2 = 0} \qquad B = \frac{V_1}{-I_2} \Bigg|_{V_2 = 0}$$

$$C = \frac{I_1}{V_2} \Bigg|_{I_2 = 0} \qquad D = \frac{I_1}{-I_2} \Bigg|_{V_2 = 0}$$

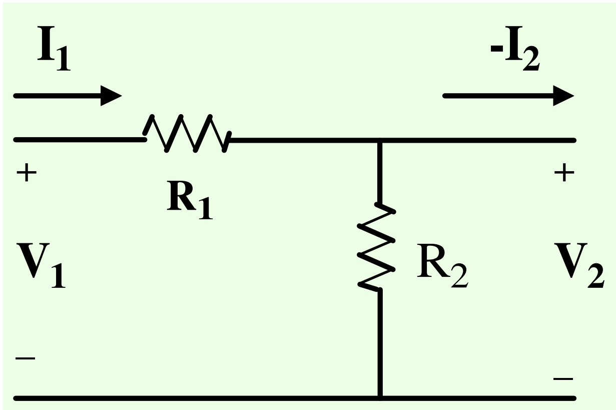
Note the direction of I_2

Two Port Networks

Transmission parameters (A,B,C,D):

Example

Given the network below with assumed voltage polarities and Current directions compatible with the A,B,C,D parameters.



We can write the following equations.

$$V_1 = (R_1 + R_2)I_1 + R_2 I_2$$

$$V_2 = R_2 I_1 + R_2 I_2$$

It is not always possible to write 2 equations in terms of the V's and I's Of the parameter set.

Two Port Networks

Transmission parameters (A,B,C,D):

Example (cont.)

$$V_1 = (R_1 + R_2)I_1 + R_2 I_2$$

$$V_2 = R_2 I_1 + R_2 I_2$$

From these equations we can directly evaluate the A,B,C,D parameters.

$$A = \left.\frac{V_1}{V_2}\right|_{I_2 = 0} = \square \qquad B = \left.\frac{V_1}{-I_2}\right|_{V_2 = 0} = \square$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2 = 0} = \square \qquad D = \left.\frac{I_1}{-I_2}\right|_{V_2 = 0} = \square$$

Later we will see how to interconnect two of these networks together for a final answer

Two Port Networks

Hybrid Parameters:

The equations for the hybrid parameters are:

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Short-circuit input impedance

$$h_{11} = \frac{V_1}{I_1} \bigg|_{V_2 = 0}$$

Open-circuit reverse voltage gain

$$h_{12} = \frac{V_1}{V_2} \bigg|_{I_1 = 0}$$

Short-circuit forward current gain

$$h_{21} = \frac{I_2}{I_1} \bigg|_{V_2 = 0}$$

Open-circuit output admittance

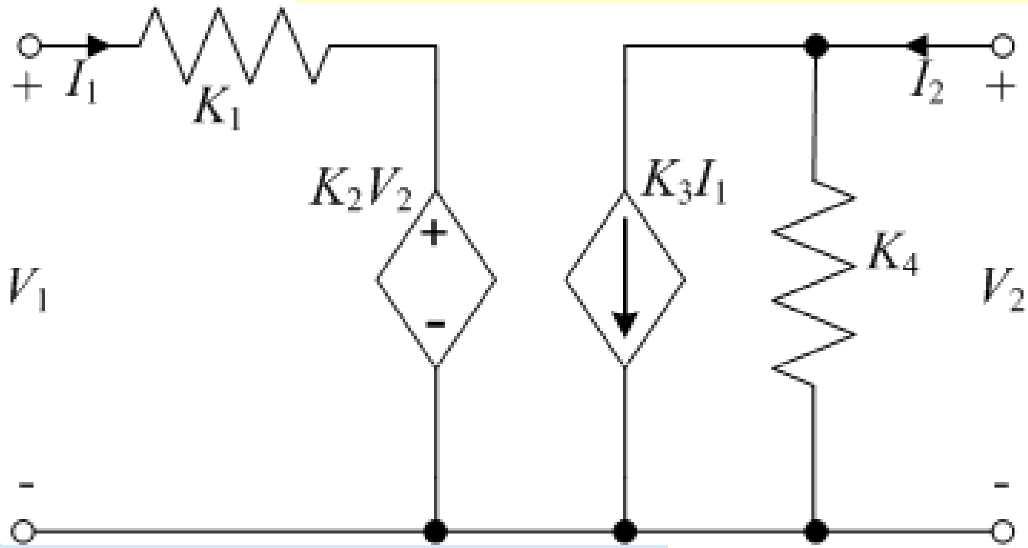
$$h_{22} = \frac{I_2}{V_2} \bigg|_{I_1 = 0}$$

<u>Note</u> : The h parameters are used mostly in electronics. For example, they are used in the equivalent circuit of a transistor.

Two Port Networks

Hybrid Parameters:

The following is a popular model used to represent a particular variety of transistors.



We can write the following equations:

$$V_1 = AI_1 + BV_2$$

$$I_2 = CI_1 + \frac{V_2}{D}$$

Two Port Networks

Hybrid Parameters:

$$V_1 = AI_1 + BV_2$$

$$I_2 = CI_1 + \frac{V_2}{D}$$

We want to evaluate the H parameters from the above set of equations.

$$h_{11} = \frac{V_1}{I_1}\bigg|_{V_2 = 0} =$$

$$h_{12} = \frac{V_1}{V_2}\bigg|_{I_1 = 0} =$$

$$h_{21} = \frac{I_2}{I_1}\bigg|_{V_2 = 0} =$$

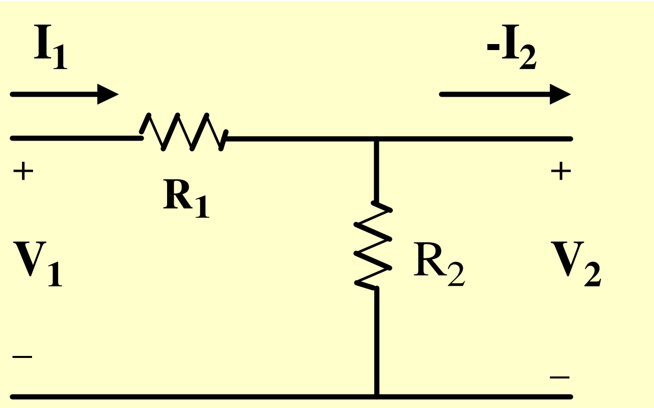
$$h_{22} = \frac{I_2}{V_2}\bigg|_{I_1 = 0} =$$

Two Port Networks

Hybrid Parameters:

Another example with hybrid parameters.

Given the circuit below.



The equations for the circuit are:

$$V_1 = (R_1 + R_2)I_1 + R_2 I_2$$

$$V_2 = R_2 I_1 + R_2 I_2$$

The H parameters are as follows.

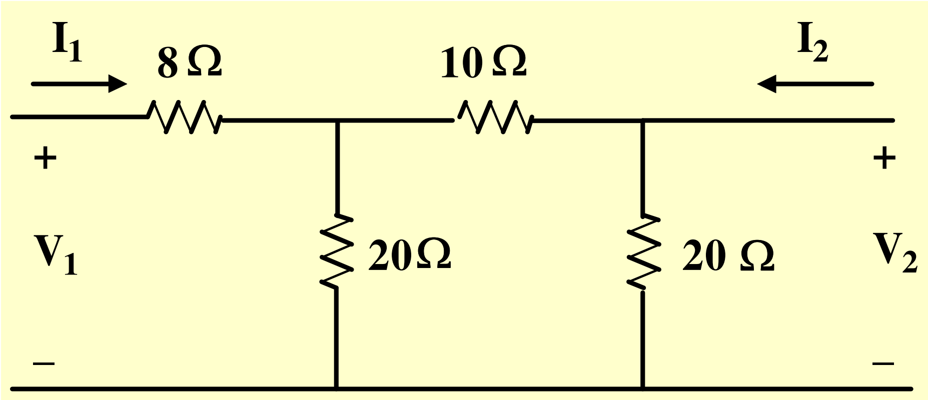
$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \qquad \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} =$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \qquad \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} =$$

Two Port Networks

Modifying the two port network:

Earlier we found the z parameters of the following network.

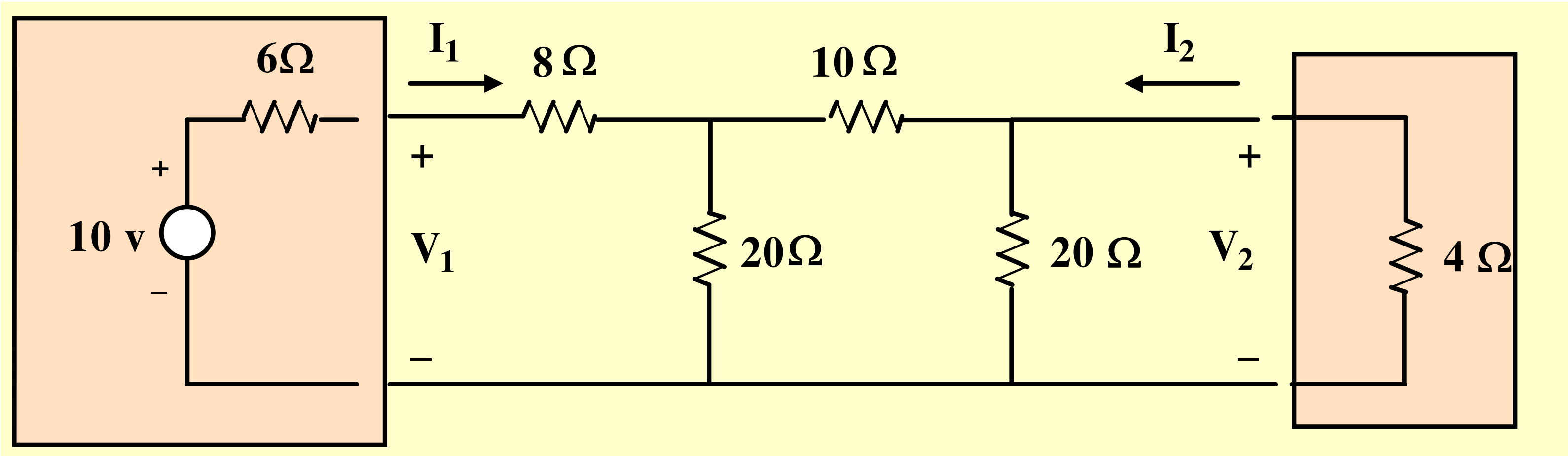


$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 8 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Two Port Networks

Modifying the two port network:

We modify the network as shown be adding elements outside the two ports



We now have:

$$V_1 = 10 - 6I_1$$

$$V_2 = -4I_2$$

Two Port Networks

Modifying the two port network:

We take a look at the original equations and the equations describing the new port conditions.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 8 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$V_1 = 10 - 6I_1$$

$$V_2 = -4I_2$$

So we have,

$$10 - 6I_1 = 20I_1 + 8I_2$$

$$-4I_2 = 8I_1 + 12I_2$$

*** notes**

Two Port Networks

Modifying the two port network:

Rearranging the equations gives,

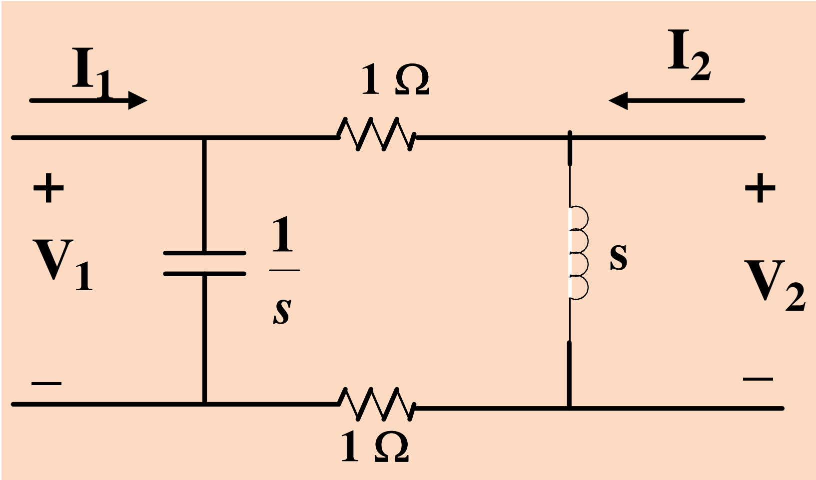
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}^{-1} \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Two Port Networks

Y Parameters and Beyond:

Given the following network.



(a) Find the Y parameters for the network.

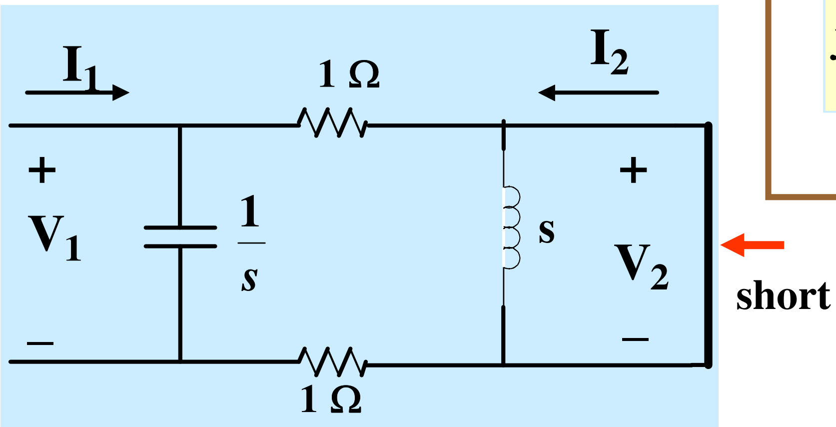
(b) From the Y parameters find the z parameters

Two Port Networks

Y Parameter Example

$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$



To find y_{11}

$$y_{11} = \frac{I_1}{V_1}\Bigg|_{V_2=0} \qquad y_{12} = \frac{I_1}{V_2}\Bigg|_{V_1=0}$$

$$y_{21} = \frac{I_2}{V_1}\Bigg|_{V_2=0} \qquad y_{22} = \frac{I_2}{V_2}\Bigg|_{V_1=0}$$

We use the above equations to evaluate the parameters from the network.

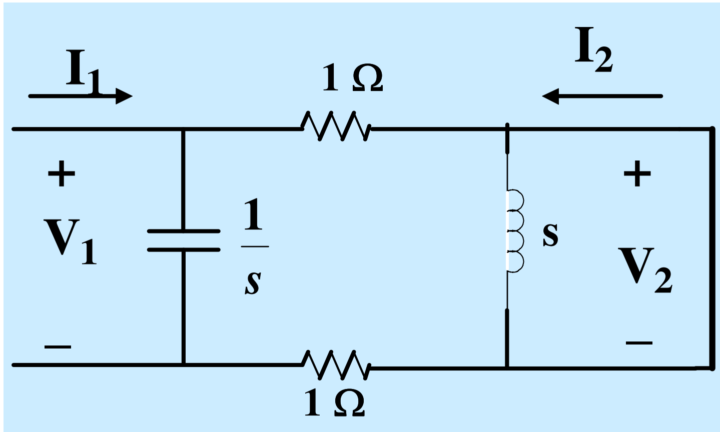
$$V_1 = I_1\left(\frac{2/s}{2+1/s}\right) = I_1\left[\frac{2}{2s+1}\right]$$

so $y_{11} = \frac{I_1}{V_1}\Big|_{V_2=0}$ = $s + 0.5$

Two Port Networks

Y Parameter Example

$$y_{21} = \frac{I_2}{V_1}\Bigg|_{V_2=0}$$



We see

$$V_1 = -2I_2$$

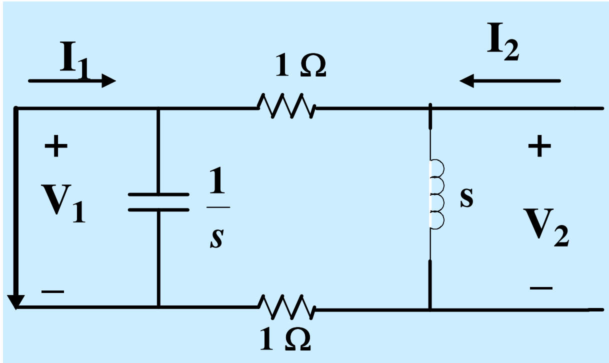
$$y_{21} = \frac{I_2}{V_1} = -0.5 \text{ S}$$

Two Port Networks

Y Parameter Example

To find y_{12} and y_{21} we reverse things and short V_1

short



$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

We have

$$V_2 = -2I_1$$

$$y_{12} = \frac{I_1}{V2} = -0.5 \text{ S}$$

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

We have

$$V_2 = I_2 \frac{2s}{(s+2)}$$

$$y_{22} = 0.5 + \frac{1}{s}$$

Two Port Networks

Y Parameter Example

Summary:

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} s+0.5 & -0.5 \\ -0.5 & 0.5+1/s \end{bmatrix}$$

Now suppose you want the Z parameters for the same network.

Two Port Networks

Going From Y to Z Parameters

For the Y parameters we have:

$$I = Y\,V$$

For the Z parameters we have:

$$V = Z\,I$$

From above;

$$V = Y^{-1} I = Z\,I$$

Therefore

$$Z = Y^{-1} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} \frac{y_{22}}{\Delta_Y} & \frac{-y_{12}}{\Delta_Y} \\ \frac{-y_{21}}{\Delta_Y} & \frac{y_{11}}{\Delta_Y} \end{bmatrix}$$

where

$$\Delta_Y = \det|Y|$$

Two Port Parameter Conversions:

$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{y_{22}}{\Delta_Y} & \frac{-y_{12}}{\Delta_Y} \\ \frac{-y_{21}}{\Delta_Y} & \frac{y_{11}}{\Delta_Y} \end{bmatrix}$	$\begin{bmatrix} \frac{A}{C} & \frac{\Delta_T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix}$	$\begin{bmatrix} \frac{\Delta_H}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$
$\begin{bmatrix} \frac{z_{22}}{\Delta_Z} & \frac{-z_{12}}{\Delta_Z} \\ \frac{-z_{21}}{\Delta_Z} & \frac{z_{11}}{\Delta_Z} \end{bmatrix}$	$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{D}{B} & \frac{-\Delta_T}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_H}{h_{11}} \end{bmatrix}$
$\begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta_Z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$	$\begin{bmatrix} \frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ \frac{-\Delta_Y}{y_{21}} & \frac{-y_{11}}{y_{21}} \end{bmatrix}$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$	$\begin{bmatrix} \frac{-\Delta_H}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{bmatrix}$
$\begin{bmatrix} \frac{\Delta_Z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta_Y}{y_{11}} \end{bmatrix}$	$\begin{bmatrix} \frac{B}{D} & \frac{\Delta_T}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$

Two Port Parameter Conversions:

To go from one set of parameters to another, locate the set of parameters you are in, move along the vertical until you are in the row that contains the parameters you want to convert to – then compare element for element

$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{y_{22}}{\Delta_Y} & \frac{-y_{12}}{\Delta_Y} \\ \frac{-y_{21}}{\Delta_Y} & \frac{y_{11}}{\Delta_Y} \end{bmatrix}$	$\begin{bmatrix} \frac{A}{C} & \frac{\Delta_T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix}$	$\begin{bmatrix} \frac{\Delta_H}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$
$\begin{bmatrix} \frac{z_{22}}{\Delta_Z} & \frac{-z_{12}}{\Delta_Z} \\ \frac{-z_{21}}{\Delta_Z} & \frac{z_{11}}{\Delta_Z} \end{bmatrix}$	$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{D}{B} & \frac{-\Delta_T}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_H}{h_{11}} \end{bmatrix}$
$\begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta_Z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$	$\begin{bmatrix} \frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ \frac{-\Delta_Y}{y_{21}} & \frac{-y_{11}}{y_{21}} \end{bmatrix}$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$	$\begin{bmatrix} \frac{-\Delta_H}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{bmatrix}$
$\begin{bmatrix} \frac{\Delta_Z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta_Y}{y_{11}} \end{bmatrix}$	$\begin{bmatrix} \frac{B}{D} & \frac{\Delta_T}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$

$$z_{11} = \frac{\Delta_H}{h_{22}}$$

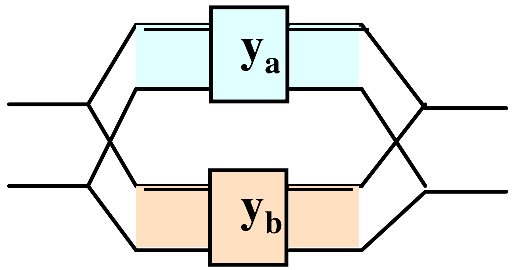
Conditions for reciprocity and symmetry

Parameters	Reciprocity	Symmetry
[Z]	$z_{12}=z_{21}$	$z_{11}=z_{22}$
[Y]	$y_{12}=y_{21}$	$y_{11}=y_{22}$
[ABCD]	$AD-BC=1$	$A=D$
[h]	$h_{12}=-h_{21}$	$h_{11}h_{22}-h_{12}h_{21}=1$
[A’B’C’D’]	$A'D'-B'C'=1$	$A'=D'$
[g]	$g_{12}=-g_{21}$	$g_{11}g_{22}-g_{12}g_{21}=1$

Interconnection Of Two Port Networks

Three ways that two ports are interconnected:

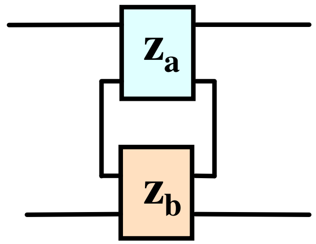
* **Parallel**



Y parameters

$$[Y]=[Y_a]+[Y_b]$$

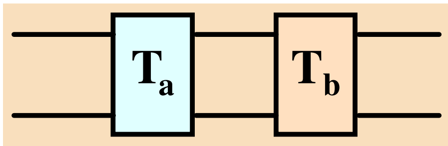
* **Series**



Z parameters

$$[Z]=[Z_a]+[Z_b]$$

* **Cascade**

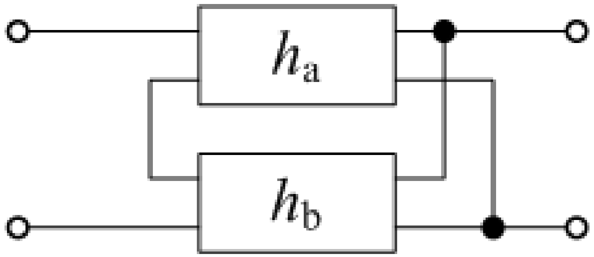


ABCD parameters

$$[T]=[T_a][T_b]$$

Interconnection Of Two Port Networks (2)

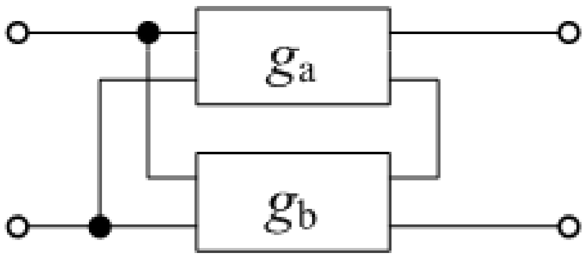
*** Series-Parallel**



***h* parameters**

$[h]=[h_a]+[h_b]$

*** Parallel-Series**



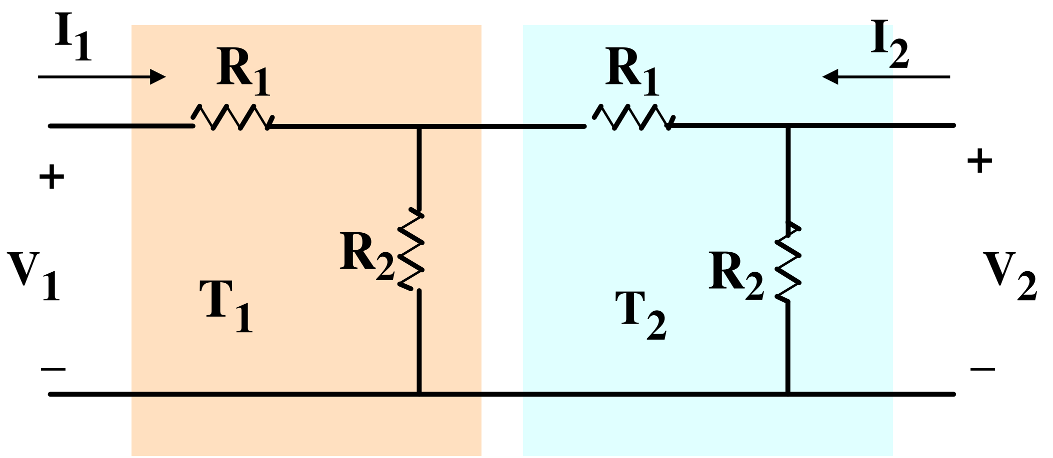
***g* parameters**

$[g]=[g_a]+[g_b]$

Interconnection Of Two Port Networks

Consider the following network:

Find $\dfrac{V_2}{V_1}$



Referring to slide 15 we have;

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \dfrac{R_1 + R_2}{R_2} & R_1 \\ \dfrac{1}{R_2} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{R_1 + R_2}{R_2} & R_1 \\ \dfrac{1}{R_2} & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Interconnection Of Two Port Networks

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \dfrac{R_1 + R_2}{R_2} & R_1 \\ \dfrac{1}{R_2} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{R_1 + R_2}{R_2} & R_1 \\ \dfrac{1}{R_2} & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Multiply out the first row:

$$V_1 = \left[\left[\left(\frac{R_1 + R_2}{R_2} \right)^2 + \frac{R_1}{R_2} \right] V_2 + \left[\left(\frac{R_1 + R_2}{R_2} \right) R_1 + R_1 \right] (-I_2) \right]$$

Set $I_2 = 0$ (as in the diagram)

$$\frac{V_2}{V_1} = \frac{R_2^{\,2}}{R_1^{\,2} + 3R_1R_2 \;\; R_2^{\,2}}$$

Can be verified directly by solving the circuit

The Scattering Matrix

- The scattering matrix relates the voltage waves incident on the ports to those reflected from the ports.
- The scattering parameters can be calculated using network analysis technique. Otherwise, they can be measured directly with a vector network analyzer.
- Once the scattering matrix is known, conversion to other matrices can be performed.

$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & & & \vdots \\ \vdots & & & \vdots \\ S_{N1} & \cdots & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix}$$

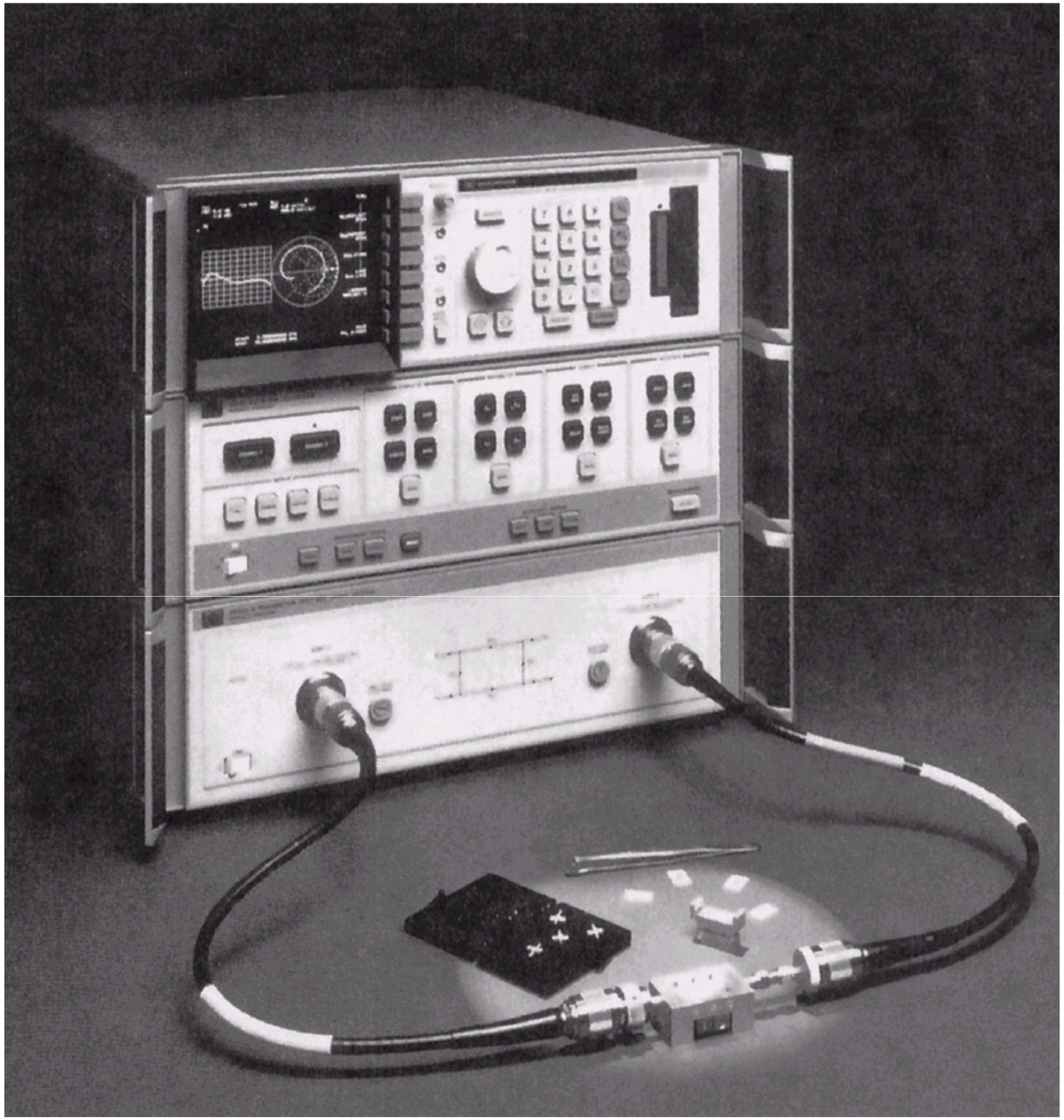
or

$$\left[V^-\right] = [S]\left[V^+\right]$$

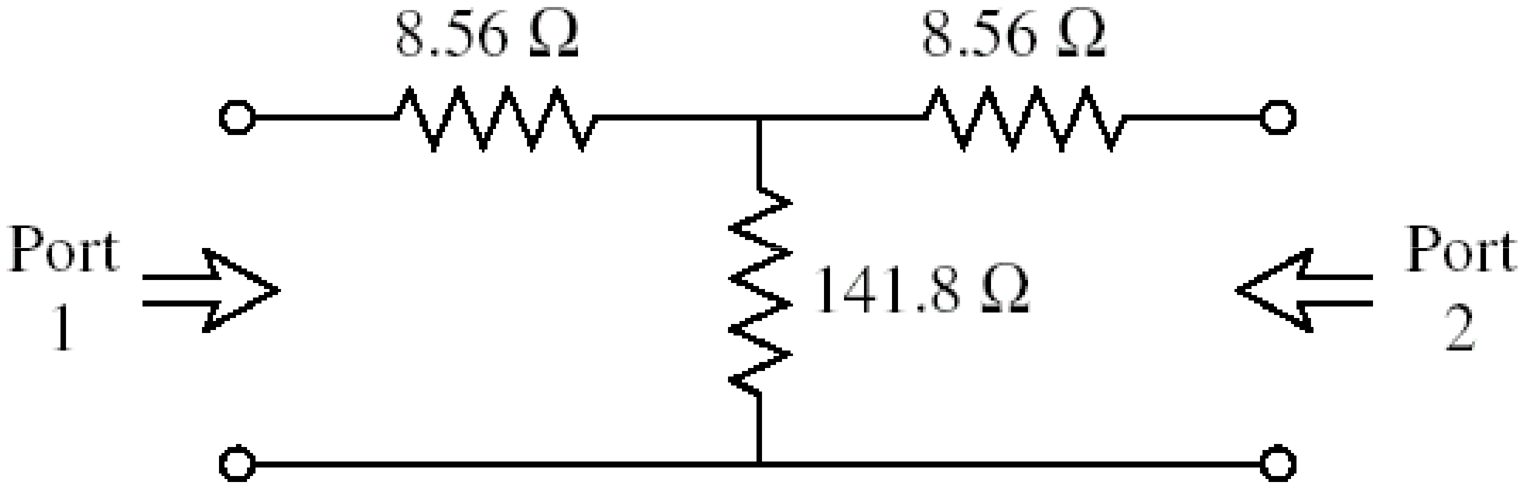
$$S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+ = 0 \text{ for } k \neq j}$$

- S_{ii} → the reflection coefficient seen looking into port i when all other ports are terminated in matched loads,
- S_{ij} → the transmission coefficient from port j to port i when all other ports are terminated in matched loads.

Figure 4.7 (p. 175)
A photograph of the Hewlett-Packard HP8510B Network Analyzer. This test instrument is used to measure the scattering parameters (magnitude and phase) of a one- or two-port microwave network from 0.05 GHz to 26.5 GHz. Built-in microprocessors provide error correction, a high degree of accuracy, and a wide choice of display formats. This analyzer can also perform a fast Fourier transform of the frequency domain data to provide a time domain response of the network under test.
Courtesy of Agilent Technologies.



Ex.4, Evaluation of Scattering Parameters



A matched 3B attenuator with a 50 Ω Characteristic impedance

- Show how [S] ⇔ [Z] or [Y]. Assume Z_{0n} are all identical, for convenience $Z_{0n} = 1$.

$$V_n = V_n^+ + V_n^-, I_n = I_n^+ - I_n^- = V_n^+ - V_n^-$$

$$[Z][I] = [Z][V^+] - [Z][V^-] = [V] = [V^+] + [V^-]$$
$$([Z] + [U])[V^-] = ([Z] - [U])[V^+]$$

where $$[U] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & & \cdots & 1 \end{bmatrix}$$

$$[S] = [V^-][V^+]^{-1} = ([Z] + [U])^{-1}([Z] - [U])$$

Therefore,

- For a one-port network,

$$S_{11} = \frac{z_{11} - 1}{z_{11} + 1}$$

- To find [Z],

$$[Z][S]+[U][S]=[Z]-[U]$$
$$[Z]=([U]+[S])([U]-[S])^{-1}$$

Reciprocal Networks and Lossless Networks

- As mentioned previously, the [Z] and [Y] are symmetric for reciprocal networks, and purely imaginary for lossless networks.
- From

$$V_n^+ = \frac{1}{2}(V_n + I_n)$$
$$[V^+] = \frac{1}{2}([Z]+[U])[I]$$

$$V_n^- = \frac{1}{2}(V_n - I_n)$$
$$[V^-] = \frac{1}{2}([Z]-[U])[I]$$

$$[V^-]=([Z]-[U])([Z]+[U])^{-1}[V^+]$$
$$[S]=([Z]-[U])([Z]+[U])^{-1}$$
$$[S]^t=\left\{([Z]+[U])^{-1}\right\}^t([Z]-[U])^t$$

- If the network is reciprocal, $[Z]^t = [Z]$.

$$[S]^t = ([Z]+[U])^{-1}([Z]-[U])$$

$$[S] = [S]^t$$

- If the network is lossless, no real power delivers to the network.

$$P_{av} = \frac{1}{2}\mathrm{Re}\{[V]^t[I]^*\} = \frac{1}{2}\mathrm{Re}\{([V^+]^t + [V^-]^t)([V^+]^* + [V^-]^*)\}$$

$$= \frac{1}{2}\mathrm{Re}\{([V^+]^t[V^+]^* - [V^+]^t[V^-]^* + [V^-]^t[V^+]^* - [V^-]^t[V^-]^*)\}$$

$$= \frac{1}{2}[V^+]^t[V^+]^* - \frac{1}{2}[V^-]^t[V^-]^* = 0$$

$$\Rightarrow \quad [V^+]^t[V^+]^* = [V^-]^t[V^-]^*$$

$$= ([S][V^+])^t([S][V^+])^*$$

$$= [V^+]^t[S]^t[S]^*[V^-]^*$$

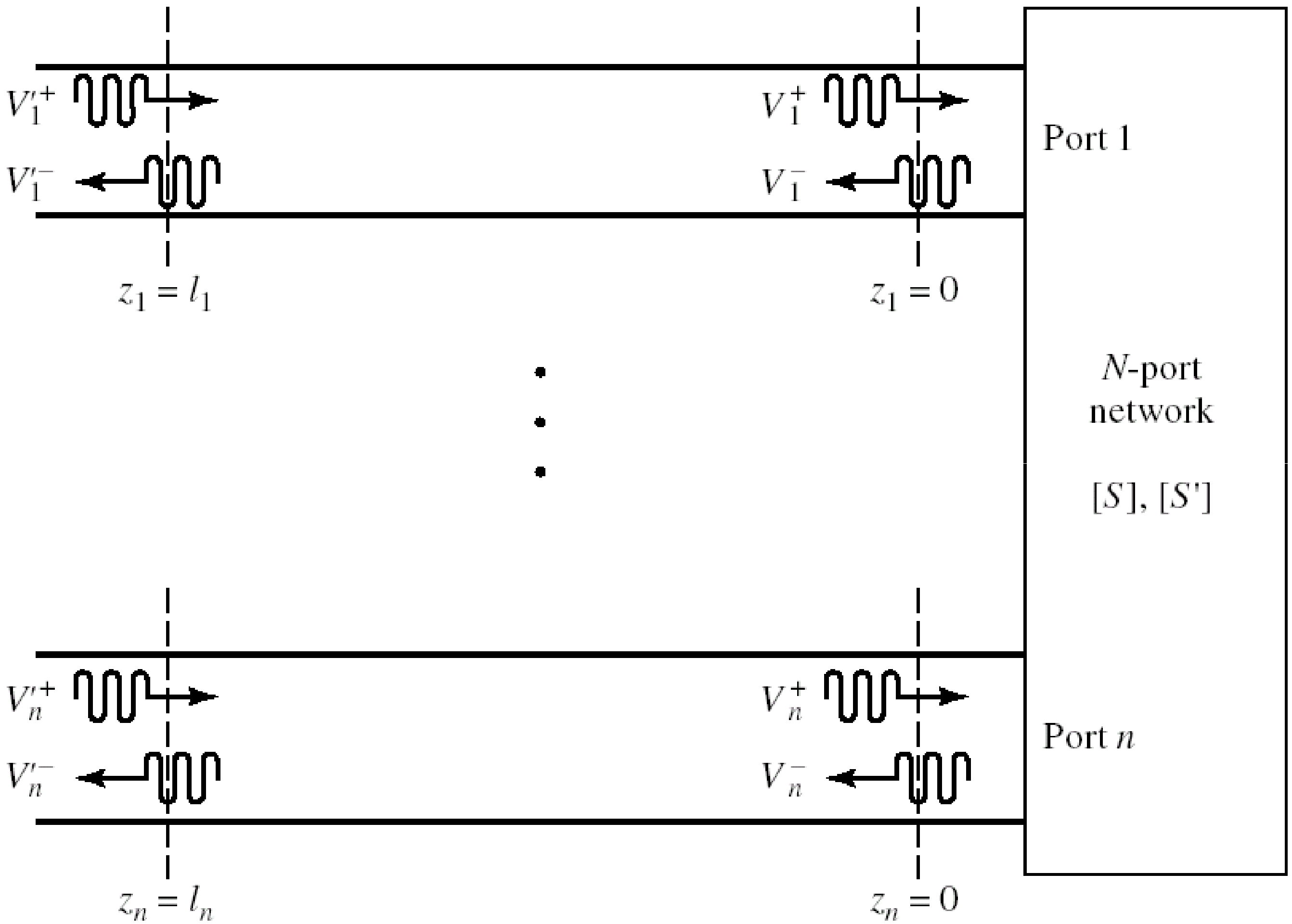
- For nonzero $[V^{+}]$, $[S]^{t}[S]^{*}=[U]$, or $[S]^{*}=\{[S]^{t}\}^{-1}$.

➔ Unitary matrix → $\sum_{k=1}^{N} S_{ki}S_{kj}^{*} = \delta_{ij}, \quad$ for all i, j

where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$.

- If $i = j$, $\quad \sum_{k=1}^{N} S_{ki}S_{ki}^{*} = 1$
- If $i \neq j$, $\quad \sum_{k=1}^{N} S_{ki}S_{kj}^{*} = 0.$
- The S parameters of a network are properties only of the network itself (assuming the network in linear), and are defined under the condition that all ports are matched.

A Shift in Reference Planes



Shifting reference planes for an *N*-port network.

- [S]: the scattering matrix at $z_n = 0$ plane.
- [S']: the scattering matrix at $z_n = \ell_n$ plane.

$$[V^-] = [S][V^+],$$

$$[V'^-] = [S'][V'^+]$$

$$V_n'^+ = V_n^+ e^{j\theta_n}, V_n'^- = V_n^- e^{-j\theta_n}$$

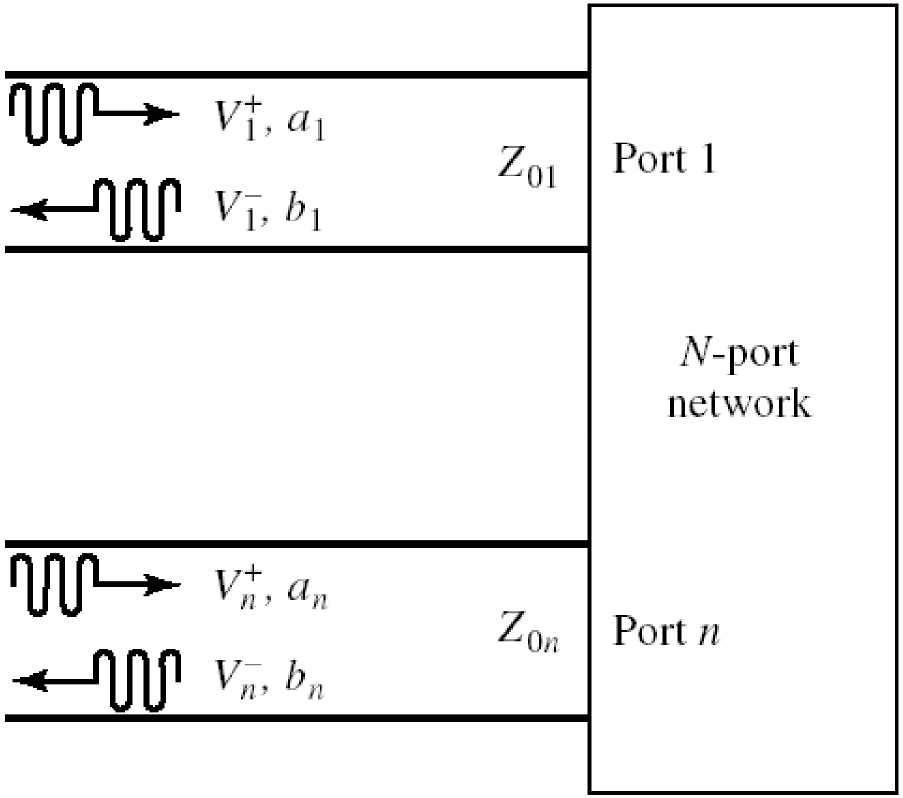
$$\theta_n = \beta_n \ell_n$$

$$\begin{bmatrix} e^{j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{j\theta_N} \end{bmatrix} \left[V'^-\right] = [S] \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \left[V'^+\right]$$

$$\left[V'^-\right] = \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} [S] \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \left[V'^+\right]$$

$$[S'] = \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} [S] \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix}$$

Generalized Scattering Parameters



An *N*-port network with different characteristic impedances.

$$a_n = V_n^+ / \sqrt{Z_{0n}}, b_n = V_n^- / \sqrt{Z_{0n}}$$

$$V_n = V_n^+ + V_n^- = \sqrt{Z_{0n}}(a_n + b_n)$$

$$I_n = \frac{1}{Z_{0n}}\left(V_n^+ - V_n^-\right) = \sqrt{Z_{0n}}(a_n - b_n)$$

$$P_n = \frac{1}{2}\mathrm{Re}\left\{V_n I_n^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\left|a_n\right|^2 - \left|b_n\right|^2 + \left(b_n a_n^* - b_n^* a_n\right)\right\}$$

$$= \frac{1}{2}\left|a_n\right|^2 - \frac{1}{2}\left|b_n\right|^2$$

- The generalized scattering matrix can be used to relate the incident and reflected waves,

$$[b]=[S][a]$$

$$S_{ij}=\left.\frac{b_i}{a_j}\right|_{a_k=0 \text{ for } k\neq j}$$

$$S_{ij}=\left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0 \text{ for } k\neq j}$$