Impedance Matching

1 Introduction

Impedance matching is the process to match the load Z_L to a transmission line by a matching network, as depicted in Fig. 1. Recall that the reflections are eliminated under the matched condition. Impedance matching is important for the following reasons:

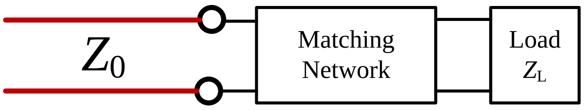


Fig. 1: Impedance matching

- To achieve maximum power transfer and minimize power loss.
- To improve signal-to-noise ratio.
- To reduce amplitude and phase errors for power distribution networks, e.g., antenna arrays.

There are many choices regarding matching network design, but the following factors must be considered in the selection of the network:

- Complexity
- Bandwidth
- Implementation
- Adjustibility

2 Matching with Lumped Elements (L Networks)

The L-section is considered the simplest type of matching network. There are two possible configurations, as depicted in Fig. 2. (a) is the network for Re[Z_L] > Z_0, while (b) is the network for Re[Z_L] < Z_0. Note that in both configurations, two components (jX, jB) are required in order to have degree of freedom 2, since the load impedance is generally complex.

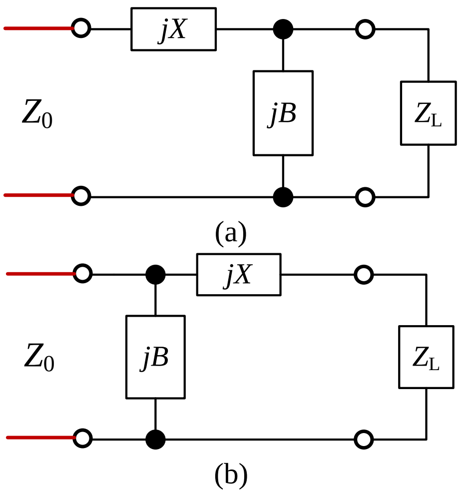


Fig. 2: L-section matching networks.

Consider Fig. 2(a). Let $Z_L = R_L + jX_L$, then the impedance seen looking into the matching network followed by the load impedance must be equal to Z_0, i.e.,

$$Z_0 = jX + \frac{1}{jB + 1/(R_L + jX_L)}.$$

Rearranging and separating into real and imaginary parts yield

$$B(XR_L - X_L Z_0) = R_L - Z_0;\ X(1 - BX_L) = BZ_0 R_L - X_L$$

Solving the above equations yields

$$B = \frac{X_L \pm \sqrt{R_L / Z_0}\sqrt{R_L^2 + X_L^2 - Z_0 R_L}}{R_L^2 + X_L^2}.$$

Note that the argument inside the second square root is always positive since $R_L > Z_0$. The series reactance can be found as

$$X = \frac{1}{B} + \frac{X_L Z_0}{R_L} - \frac{Z_0}{BR_L}.$$

Note also that two solutions are generally possible. One must consider the above factors in deciding which L network to use.

Likewise, for the network in Fig. 2(b), the matched condition is given by

$$\frac{1}{Z_0} = jB + \frac{1}{R_L + j(X + X_L)}.$$

Rearranging and separating into real and imaginary parts yield

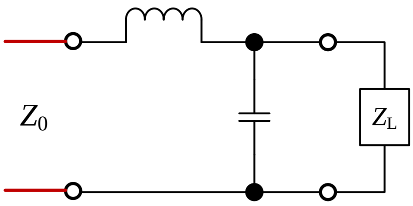
$$BZ_0(X+X_L)=Z_0-R_L\,;\quad X+X_L=BZ_0R_L\,.$$

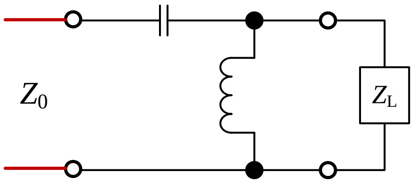
Solving for X and B gives

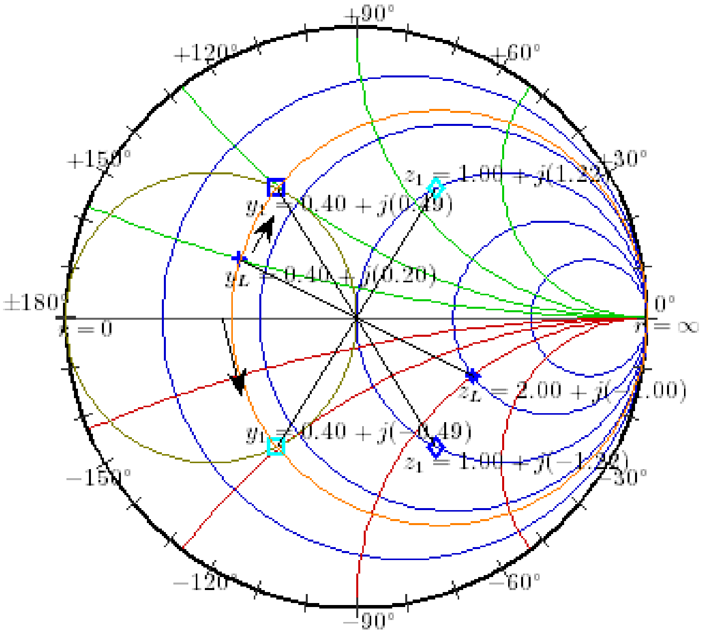
$$X=\pm\sqrt{R_L(Z_0-R_L)}-X_L\,;B=\frac{\pm\sqrt{(Z_0-R_L)/R_L}}{Z_0}\,.$$

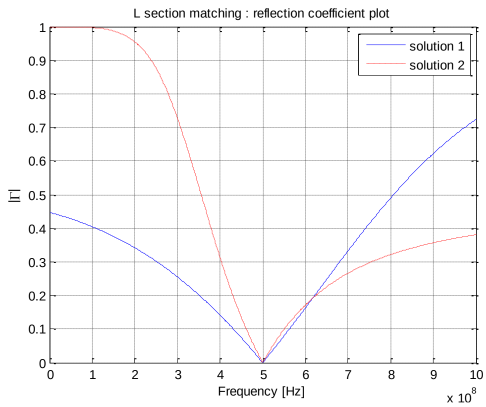
Note that $R_L < Z_0$ in this case, so the argument of the square root is always positive.

Example 1 Design an L-section matching network to match a series *RC* load with an impedance $Z_L = 200 - j100\ \Omega$, to a 100 Ω line, at a frequency of 500 MHz.









3 Single-Stub Tuning

The impedance matching using L-sections discussed previously requires lumped elements that might not be available, thus it is not practical in some cases. The single-stub tuning is the matching technique that uses a single open-circuited or short-circuited length of transmission-line (a "stub"), connected either in parallel or in series with the transmission feed line at a certain distance from the load. Note that there are two design parameters, namely the length of the stub and the distance from the load, which contribute degree of freedom 2, as in the matching with L-sections.

The choice of open-circuited stub or short-circuited stub depends on the type of transmission line media. For microstrip lines, open stubs are preferred due to ease of fabrication, while for coaxial lines or waveguides, short stubs are more desirable since such open-circuited stubs tend to radiate, resulting in reactance changes.

3.1 Shunt Stubs

The single-stub shunt tuning circuit configuration is shown in Fig. 3. Refer to the figure, to match the impedance, it is required that

$$Y_0 = Y_{in} = Y_1 + Y_{stub}.$$

Since Y_{stub} is purely susceptance (i.e., zero conductance), the real part of Y_1 must be equal to Z_0. Furthermore, the susceptance of Y_1 must cancel out the susceptance of Y_{stub}, resulting in Y_{in} becomes Y_0. Using the Smith chart makes the design process easier. The first step is to find the distance such that the normalized admittance is on the $1+jb$ circle. Then find the length such that the stub has susceptance $-b$.

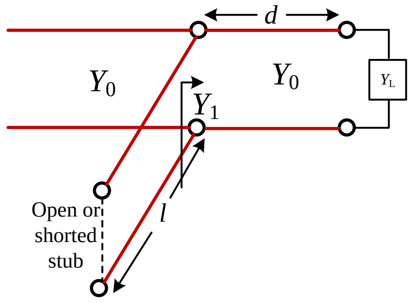
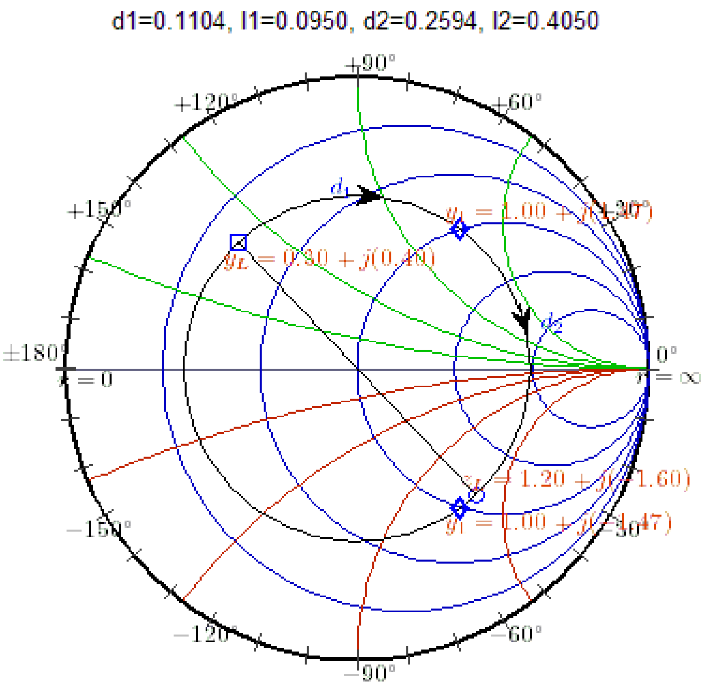
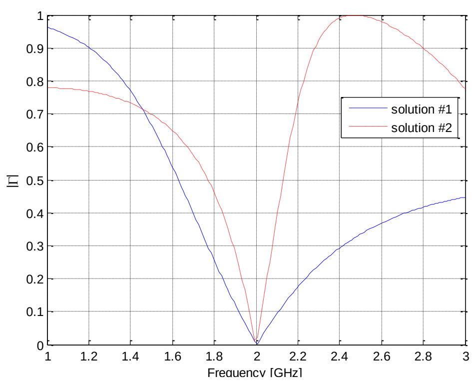


Fig. 3: Single-stub shunt tuning

Example 2 For a load impedance $Z_L = 60 - j80\ \Omega$, design two single-stub (short circuit) shunt tuning networks to match this load to a 50 Ω line. Assuming that the load is matched at 2 GHz and the load consists of a resistor and a capacitor in series.





3.2 Series Stubs

The single-stub series tuning circuit configuration is shown in Fig. 4. Refer to the figure, to match the impedance, it is required that

$$Z_0 = Z_{in} = Z_1 + Z_{stub}.$$

Since Z_{stub} is purely reactance (i.e., zero resistance), the real part of Z_1 must be equal to Z_0. Furthermore, the reactance of Z_1 must cancel out the reactance of Z_{stub}, resulting in Z_{in} becomes Z_0. As in the shunt tuning circuit design, using the Smith chart makes the design process easier. The first step is to find the distance such

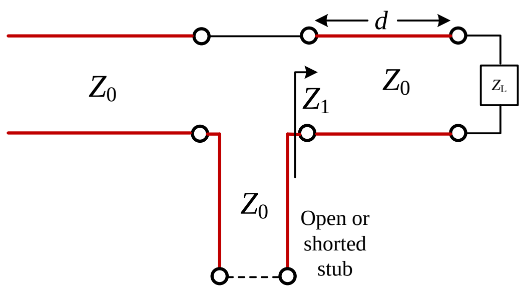
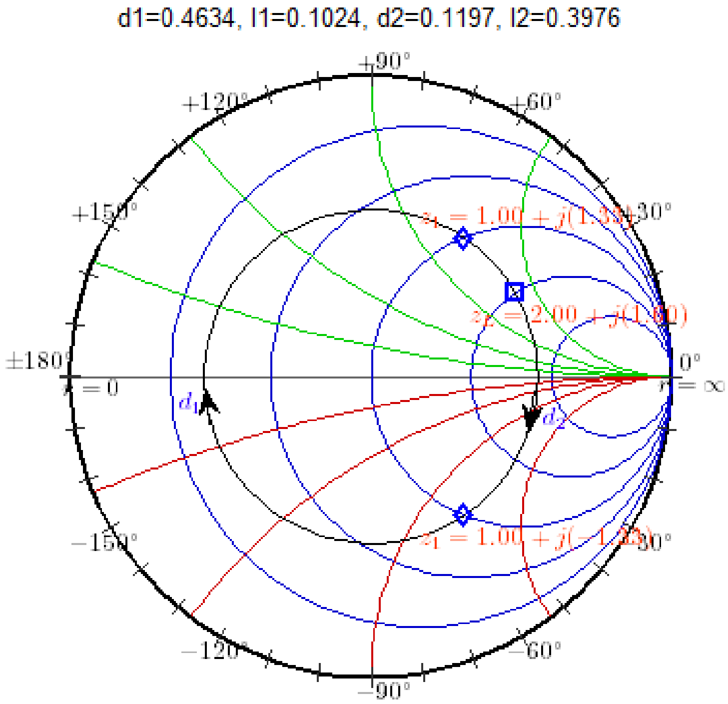
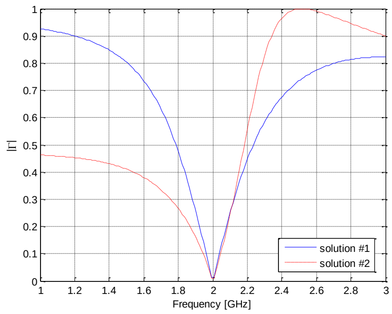


Fig. 4: Single-stub series tuning

that the normalized impedance is on the $1+jx$ circle. Then find the length such that the stub has reactance $-x$.

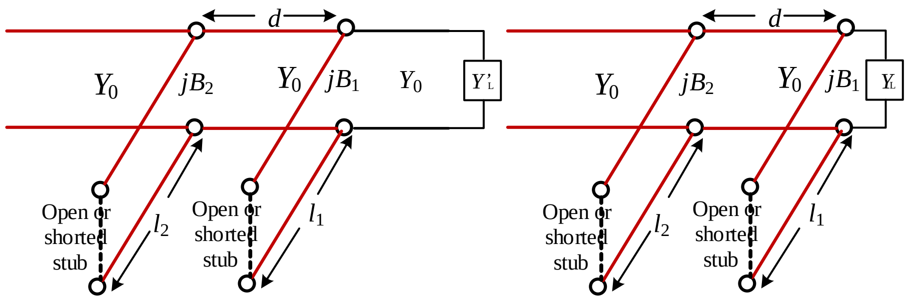
<u>Example 3</u> For a load impedance $Z_L = 100 + j80\ \Omega$, design two single-stub (open circuit) series tuning networks to match this load to a 50 Ω line. Assuming that the load is matched at 2 GHz and the load consists of a resistor and an inductor in series.





4 Double-Stub Tuning

The single-stub tuner requires a variable length of line between the load and the stub, thus it is difficult to make it "adjustable". The double-stub tuning shown in Fig. 5 uses 2 adjustable shunt stubs in fixed positions. However, the double-stub tuner cannot match **all** load impedances.



(a) (b)

Fig. 5: Double-stub tuning (a) Original circuit with the load an arbitrary distance from the first stub (b) Equivalent circuit with the load transformed to the first stub.

The Smith chart solution can be illustrated in Fig. 6. First, locate y_L and draw the rotated $1+jb$ circle with respect to the stub spacing d. Then move the load admittance onto the rotated $1+jb$ circle (points y_1, y'_1) using the susceptance b_1, b'_1 of the stub. Next, move the points y_1, y'_1 onto the $1+jb$ circle (points y_2, y'_2). Finally, add the susceptance b_2, b'_2 to match the load impedance. Note that there are two possible solutions as in the case of single-stub tuning.

Notice that if y_L is inside the shaded region in the figure, specified by g_0+jb circle, it is impossible to move this admittance onto the rotated circle, which means that it cannot be matched by a double-stub tuner (i.e., there is no solution). Therefore, this shaded region forms a forbidden range of load admittances that cannot be matched by this double-stub tuner. Reducing the space d can lead to the reduction in the size of this forbidden range, however, d must be kept sufficiently large for fabricating two separate stubs. In addition, spacings near 0 or $\lambda/2$ load to matching networks that are very frequency sensitive. In practice, stub spacings are usually chosen as $\lambda/8$ or $3\lambda/8$. Furthermore, if the length of line between the load and the first stub can be adjusted, then y_L can always be moved out of the forbidden region.

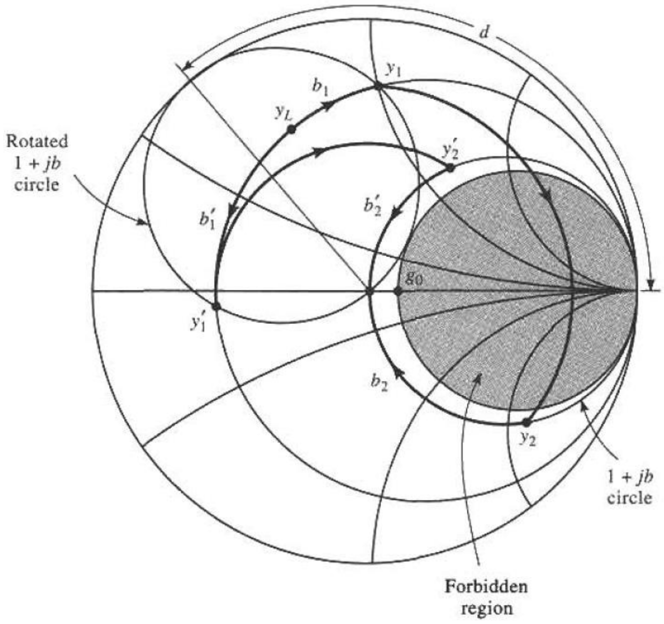
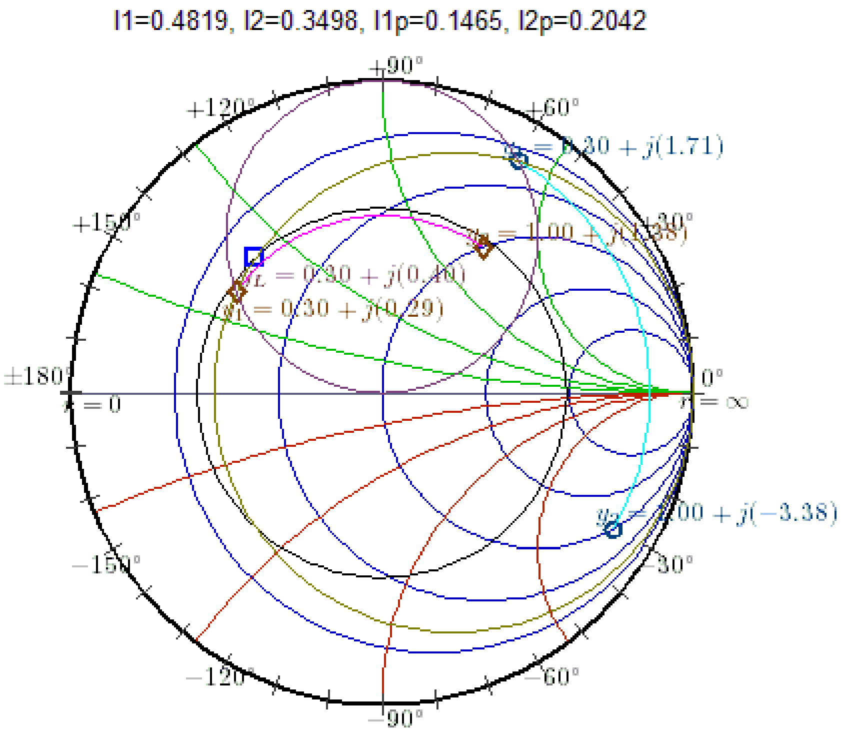
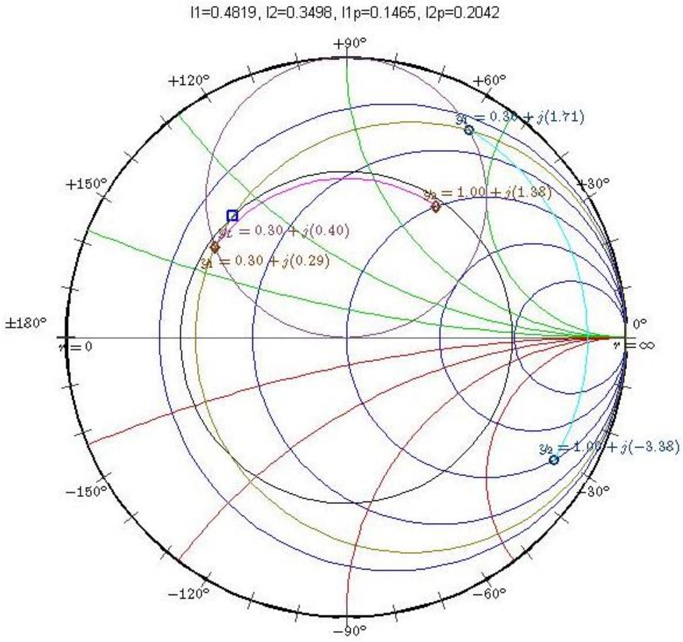
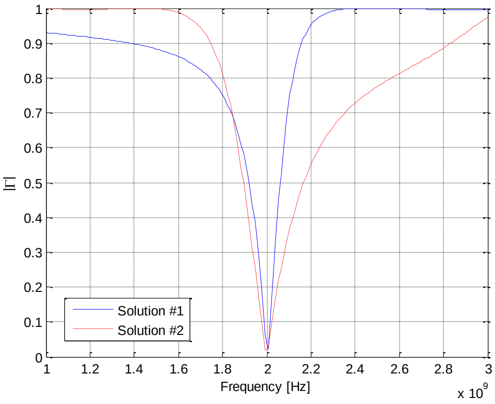


Fig. 6: Smith chart diagram for the operation of a double-stub tuner.

Example 4 For a load impedance Z_L = 60 - j80 Ω, design a shunt double-stub tuner to match this load to a 50 Ω line. The stubs are to be open-circuited and are spaced $\lambda/8$ apart. Also, the load is assumed to consist of a 60Ω-resistor and a 0.995pF-capacitor.







5 Quarter-Wave Transformer

Recall that, for a quarter-wavelength transmission line ($\ell = \lambda/4$), the input impedance becomes

$Z_{in} = Z_0^2 / Z_L$ or $Z_0^2 = Z_{in} Z_L$.

Therefore, a quarter-wavelength transmission line can be used to convert a resistive load to match a transmission line by choosing the proper characteristic impedance of the quarter-wavelength line. This is called a quarter-wave transformer. The general configuration of this quarter-wave transformer is shown in Fig. 6, where

$Z_1^2 = Z_0 R_L$.

To match an arbitrary Z_L using the quarter-wave transformer, one must somehow modify the load such that it becomes purely resistive. This may be done by adding certain lumped elements, transmission line of certain length, tuning circuits or stubs.

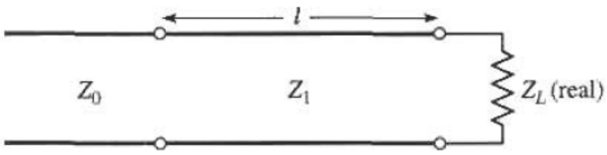


Fig. 6

Example 5 Repeat example 3 by using the quarter-wave transformer.

6 The Theory of Small Reflections

Quarter-wave transformers provide a simple mean of impedance matching, but cannot achieve broad bandwidth. To obtain more bandwidth, multisection transformers can be used.

Single-section Transformer

Consider the single-section transformer shown in Fig. 7, the partial reflection and transmission coefficients are given by

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}; \Gamma_2 = -\Gamma_1; \Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2};$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_2 + Z_1}; T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_2 + Z_1}.$$

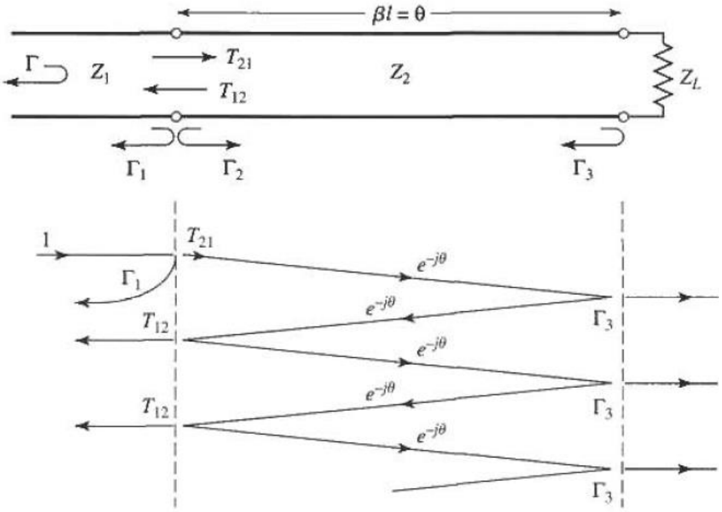


Fig. 7

The total reflection can then be given in terms of an infinite sum of partial reflections and transmissions as follows:

$$\Gamma = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-j4\theta} + \cdots$$

$$= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta}\sum_{n=0}^{\infty}\Gamma_2^n\Gamma_3^n e^{-j2n\theta}$$

Using the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x| < 1,$$

Γ can be rewritten as

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-j2\theta}}{1 - \Gamma_2\Gamma_3 e^{-j2\theta}}.$$

Using Γ_2=-Γ_1, T_{21}=1+Γ_1, T_{12}=1−Γ_1 yields

$$\Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1 + \Gamma_1\Gamma_3 e^{-j2\theta}}.$$

If the discontinuities between the impedances Z_1, Z_2 and Z_2, Z_L are small, then $|\Gamma_1\Gamma_3| \ll 1$, and

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-j2\theta}.$$

Multisection Transformer

Now consider the multisection transformer shown in Fig. 8. This transformer consists of N equal-length (*commensurate*) sections of transmission lines. Partial reflection coefficients can be defined at each junction as

$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}; \Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}; \Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}.$$

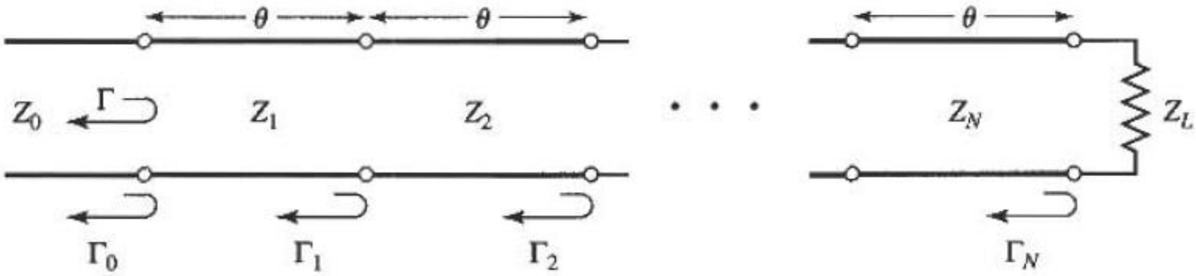


Fig. 8

We also assume that all Z_n increase or decrease monotonically across the transformer, and Z_L is real. This implies that Γ_n will be real and of the same sign. Then the total reflection coefficient Γ can be approximated as

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_N e^{-j2N\theta}.$$

Furthermore, assume that the transformer can be made symmetric, so that $\Gamma_0=\Gamma_N$, $\Gamma_1=\Gamma_{N-1}$, etc. (Note that this does not imply that the Z_n's are symmetrical.) Then,

$$\Gamma(\theta) = e^{-jN\theta}\left\{\Gamma_0(e^{-jN\theta} + e^{jN\theta}) + \Gamma_1(e^{-j(N-2)\theta} + e^{j(N-2)\theta}) + \cdots\right\}.$$

It follows that for N even,

$$\Gamma(\theta) = 2e^{-jN\theta}\left\{\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + \frac{1}{2}\Gamma_{N/2}\right\},$$

and for N odd,

$$\Gamma(\theta) = 2e^{-jN\theta}\left\{\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + \Gamma_{(N-1)/2} \cos\theta\right\}.$$

From these results, one can notice that any desired reflection response (as a function of θ) can be realized by choosing the proper Γ_n's and using enough sections. Recall the fact that a smooth function can be approximated by a Fourier series, if enough terms are used.

7 Binomial Multisection Matching Transformers

The passband response of a binomial transformer is optimum in the sense that, for a given number of sections, the response is flat as possible near the design frequency. Thus, such as response is also known as maximally flat. This type of response is designed, for an N-section transformer, by setting the first N-1 derivatives of $|\Gamma(\theta)|$ to zero, at the center frequency f_0. Such a response can be obtained if

$$\Gamma(\theta) = A(1+e^{-j2\theta})^N.$$

Then the magnitude $|\Gamma(\theta)|$ is

$$|\Gamma(\theta)| = |A||e^{-j\theta}||e^{j\theta} + e^{-j\theta}|^N = 2^N|A||\cos\theta|^N.$$

Note that $|\Gamma(\theta)|=0$ for $\theta=\pi/2$ and that $(d^n|\Gamma(\theta)|)/d\theta^n = 0$ at $\theta=\pi/2$ for $n = 1, 2, \ldots, N-1$. ($\theta=\pi/2$ corresponds to the center frequency f_0, for which $\ell=\lambda/4$ and $\theta=\beta\ell=\pi/2$.)

Let $f\rightarrow 0$, then $\theta=\beta\ell = 0$, and

$$\Gamma(\theta=0) = A(1+1)^N = A2^N = \frac{Z_L - Z_0}{Z_L + Z_0},$$

since for $f = 0$ all sections are of zero electrical length. Thus,

$$A = 2^{-N}\frac{Z_L - Z_0}{Z_L + Z_0}.$$

Now expanding $\Gamma(\theta)$ according to the binomial expansion yields

$$\Gamma(\theta) = A(1+e^{-j2\theta})^N = A\sum_{n=0}^{N} C_n^N e^{-j2n\theta}$$,where $C_n^N = \dfrac{N!}{(N-n)!n!}$. Since,

$$\Gamma(\theta) = A\sum_{n=0}^{N} C_n^N e^{-j2n\theta} = \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_N e^{-j2N\theta},\ \Gamma_n = AC_n^N.$$

If Γ_n's are assumed to be small, the following approximation can be applied:

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \cong \frac{1}{2}\ln\frac{Z_{n+1}}{Z_n}$$, since $\ln x \cong 2\dfrac{x-1}{x+1}$. Therefore,

$$\ln\frac{Z_{n+1}}{Z_n} \cong 2\Gamma_n = 2AC_n^N = 2(2^{-N})\frac{Z_L - Z_0}{Z_L + Z_0}C_n^N \cong 2^{-N}C_n^N \ln\frac{Z_L}{Z_0}.$$

To calculate the bandwidth, let Γ_m denote the maximum value of reflection coefficient that can be tolerated over the passband. Then,

$\Gamma_m = 2^N |A| \cos^N \theta_m$, where $\theta_m < \pi/2$ is the lower edge of the passband. Thus,

$\theta_m = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$, and the fractional bandwidth is given by

$$\frac{\Delta f}{f} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right].$$

Example 6 Design a three-section transformer to match a 50 Ω load to a 100 Ω line, and calculate the bandwidth for $\Gamma_m = 0.05$.

8 Chebyshev Multisection Matching Transformers

In contrast with the binomial matching transformer, the Chebyshev transformer optimizes bandwidth at the expense of passband ripple. The Chebyshev transformer is designed by equating $\Gamma(\theta)$ to a Chebyshev polynomial, which has the optimum characteristics needed for this type of transformer.

Chebyshev Polynomial

The n^{th} order Chebyshev polynomial is a polynomial of degree n, and is denoted by $T_n(x)$. The first four Chebyshev polynomials are

$$T_1(x) = x; T_2(x) = 2x^2 - 1; T_3(x) = 4x^3 - 3x; T_4(x) = 8x^4 - 8x^2 + 1.$$

Higher-order polynomials can be found using the following recurrence formula:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Some important properties of Chebyshev polynomials are listed here:

1. For $-1 \le x \le 1$, $|T_n(x)| \le 1$. In this range, the Chebyshev polynomials oscillate between ±1. This is the equal ripple property, and this region will be mapped to the passband of the matching transformer.
2. For $|x|>1$, $|T_n(x)|>1$. This region will be mapped to the frequency range outside the passband.
3. For $|x|>1$, $|T_n(x)|$ increases faster with x as n increases.

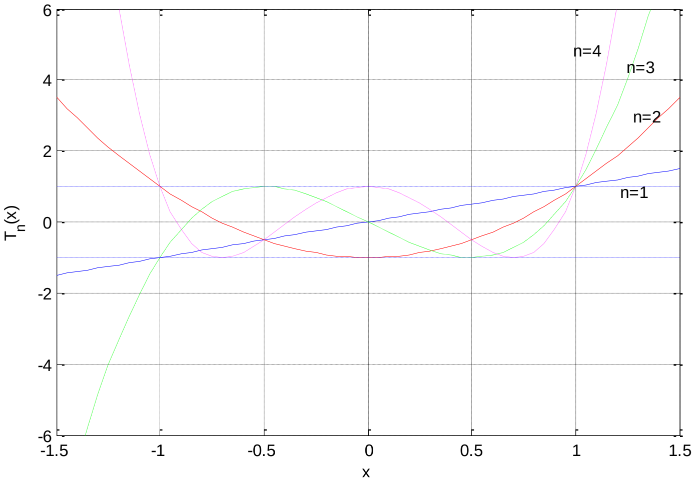


Fig. 8: First four Chebyshev polynomials

Now, let $x = \cos\theta$ for $|x| < 1$. Then it can be shown that the Chebyshev polynomials can be expressed as

$T_n(\cos\theta) = \cos n\theta$, or more generally as

$$T_n(x) = \begin{cases} \cos(n\cos^{-1} x) & \text{for } |x|<1 \\ \cosh(n\cosh^{-1} x) & \text{for } |x|>1 \end{cases}$$

Since equal ripple is desirable in the passband, it is necessary to map θ_m to $x = 1$ and $\pi-\theta_m$ to $x = -1$, where θ_m and $\pi-\theta_m$ are the lower and upper edges of the passband. This can be accomplished by replacing $\cos\theta$ in the above equation with $\cos\theta/\cos\theta_m$:

$$T_n\left(\frac{\cos\theta}{\cos\theta_m}\right) = T_n(\sec\theta_m\cos\theta) = \cos n\left[\cos^{-1}\left(\frac{\cos\theta}{\cos\theta_m}\right)\right].$$

Then $|\sec\theta_m\cos\theta| \le 1$ for $\theta_m < \theta < \pi-\theta_m$, so $|T_n(\sec\theta_m\cos\theta)| \le 1$ over this same range.
It follows that the first four terms of the Chebyshev polynomials can be written as

$$T_1(\sec\theta_m\cos\theta) = \sec\theta_m\cos\theta;\ T_2(\sec\theta_m\cos\theta) = \sec^2\theta_m(\cos 2\theta + 1) - 1;$$

$$T_3(\sec\theta_m\cos\theta) = \sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3\sec\theta_m\cos\theta;$$

$$T_4(\sec\theta_m\cos\theta) = \sec^4\theta_m(\cos 4\theta + 4\cos 2\theta + 3) - 4\sec^2\theta_m(\cos 2\theta + 1) + 1.$$

The above results can be used to design matching transformers with up to four sections.

Design of Chebyshev Transformers

A Chebyshev equal-ripple passband can be synthesized by making $\Gamma(\theta)$ proportional to $T_N(\sec\theta_m\cos\theta)$, where N denotes the number of sections. Thus,

$$\Gamma(\theta) = 2e^{-jN\theta}\{\Gamma_0\cos N\theta + \Gamma_1\cos(N-2)\theta + \cdots + \Gamma_n\cos(N-2n)\theta + \cdots\} = Ae^{-jN\theta}T_N(\sec\theta_m\cos\theta)$$

where the last term in the series is $(1/2)\Gamma_{N/2}$ for N even and $\Gamma_{(N-1)/2}\cos\theta$ for N odd. The constant A can be found from letting $\theta = 0$:

$$\Gamma(\theta = 0) = \frac{Z_L - Z_0}{Z_L + Z_0} = AT_N(\sec\theta_m),\ \text{or}\ A = \frac{Z_L - Z_0}{Z_L + Z_0}\frac{1}{T_N(\sec\theta_m)}.$$

Now if the maximum allowable reflection coefficient magnitude in the passband is Γ_m (i.e., the *ripple*), then $\Gamma_m = |A|$, since the maximum value of $T_n(\sec\theta_m\cos\theta)$ in the passband is unity. Using the approximation introduced in the previous section yields

$$T_N(\sec\theta_m) = \frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right| \cong \frac{1}{2\Gamma_m}\left|\ln\frac{Z_L}{Z_0}\right|.$$ It follows that

$$\sec\theta_m = \cosh\left\{\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right\} \cong \cosh\left\{\frac{1}{N}\cosh^{-1}\left(\left|\frac{\ln(Z_L/Z_0)}{2\Gamma_m}\right|\right)\right\}.$$

Once θ_m is known, the fractional bandwidth can be calculated from

$$\frac{\Delta f}{f} = 2 - \frac{4\theta_m}{\pi}.$$

Each Γ_n can be determined by expanding $T_N(\sec\theta_m\cos\theta)$ and equating similar terms of the form $\cos(N-2n)\theta$. The following approximation can be applied to improve the accuracy:

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \cong \frac{1}{2}\ln\frac{Z_{n+1}}{Z_n}.$$

Example 7 Design a three-section Chebyshev transformer to match a 100 Ω load to a 50 Ω line, with $\Gamma_m = 0.05$.