Transmission Lines

1 Introduction

For efficient point-to-point transmission of power and information, the source energy must be directed or guided. Here, the transverse electromagnetic (TEM) waves guided by transmission lines will be discussed. The characteristics of TEM waves here are the same as those for a uniform plane wave propagating in an unbounded dielectric medium. Basically, these TEM waves are like "guided" waves on a line in contrast with "unguided" waves in free space for radiated waves.

There are three common types of guiding structures that support TEM waves, namely, *parallel-plate transmission line*, *two-wire transmission line*, and *coaxial transmission line*. The following points should be first pointed out here:

1. Only perfect conductor supports TEM mode whose electric field has only component perpendicular to the line. For good conductors, longitudinal component can exist due to the line currents passing through the "imperfect" conductors. This mode is referred to as the "quasi-TEM" mode.
2. In addition, if the surrounding medium is lossy (either through Ohmic loss or dielectric loss of the medium), this additional loss should also be taken into account.

2 General Transmission-Line Equations

In general, transmission lines are used when the physical dimensions of electric networks are usually a considerable fraction of a wavelength (typically, larger than quarter-wavelength) and may even be many wavelengths long, whereas ordinary circuit theory is applied to networks considerably much smaller than the operating wavelength. A transmission line is a distributed-parameter network, or "distributed" circuit, and must be described by circuit parameters that are distributed throughout its length, while ordinary electric networks are considered "lumped" circuits, whose elements are discrete and currents flowing in lumped-circuit elements do not vary spatially over the elements.

Now, consider a "lossy" transmission line consisting of two wires in xz-plane shown in Figure 1. Applying Faraday's law

$$\oint_C \mathcal{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathcal{B}}{\partial t}\cdot d\mathbf{s} \tag{1.1}$$

to the contour C and surface S shown in (a) yields

$$\int_{x_1}^{x_2}\left[\mathcal{E}_x(x,z+\Delta z;t)-\mathcal{E}_x(x,z;t)\right]dx+\int_z^{z+\Delta z}\left[\mathcal{E}_z(x_1,z;t)-\mathcal{E}_z(x_2,z;t)\right]dz=-\int_z^{z+\Delta z}\int_{x_1}^{x_2}\frac{\partial \mathcal{B}_y(x,z;t)}{\partial t}dxdz \tag{1.2}$$

Define the voltage between the wires

$$v(z;t)=-\int_{x_1}^{x_2}\mathcal{E}_x(x,z)dx, \tag{1.3}$$

then, $v(z+\Delta z;t)-v(z;t)=-\int_{x_1}^{x_2}\left[\mathcal{E}_x(x,z+\Delta z;t)-\mathcal{E}_x(x,z;t)\right]dx$. (1.4)

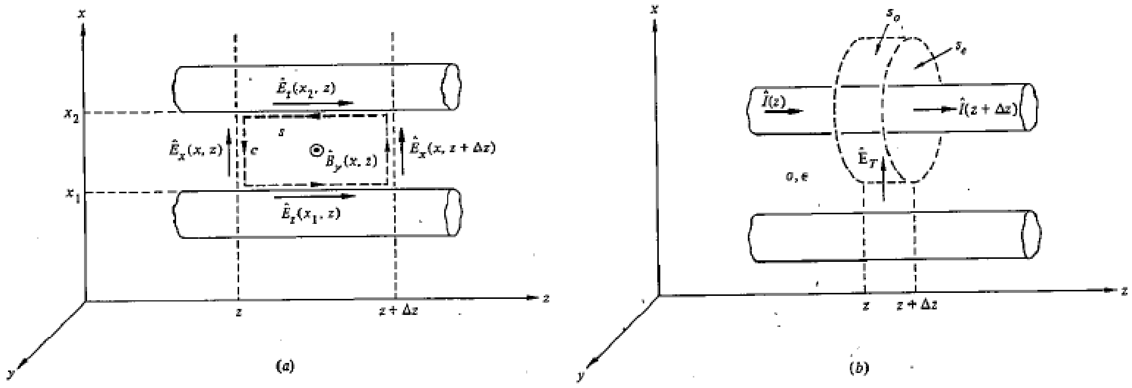


Fig. 1: Contours and surfaces for the derivation of the transmission-line equations

Suppose a current $i(z)$ exists in the upper wire in the positive z direction and returning in the lower wire. Assume that the losses in the wires can be lumped as an impedance through which $i(z)$ passes. The lossy nature of the conductors will result in the resistance per unit length, R. There is also the internal inductance per unit length L_i due to the current $i(z)$ partially penetrating the wires, which are not perfect conductors. Thus,

$$\int_z^{z+\Delta z}\left[\mathscr{E}_z(x_1,z)-\mathscr{E}_z(x_2,z)\right]dz=-R\Delta z i(z;t)-L_i\Delta z\frac{\partial i(z;t)}{\partial t}. \quad (1.5)$$

Furthermore, the RHS of (1.2) is related to the magnetic flux external to the wires produced by the current, i.e.,

$$i(z;t)\Delta z L_e=-\int_z^{z+\Delta z}\int_{x_1}^{x_2}\mathscr{B}_y(x,z;t)dxdz \quad (1.6)$$

where L_e denotes the external inductance per unit length.
Substituting (1.4),(1.5),(1.6) into (1.2) yields

$$v(z+\Delta z;t)-v(z;t)=-\left[(Ri(z;t)+(L_i+L_e)\frac{\partial i(z;t)}{\partial t}\right]\Delta z \quad (1.7)$$

Dividing both sides by Δz and taking the limit as $\Delta z\to 0$, we obtain

$$-\frac{\partial v(z;t)}{\partial z}=Ri(z;t)+L\frac{\partial i(z;t)}{\partial t}, \quad (1.8)$$

where $L = L_i + L_e$. Now, consider Figure 1(b). Applying the continuity equation to the surface S yields

$$\oint_S \mathscr{J}\cdot d\mathbf{s}=-\frac{dQ}{dt} \quad (1.9)$$

where Q is the total charge enclosed by S. Over the ends of the cylinder

$$\oint_{S_e}\mathscr{J}\cdot d\mathbf{s}=i(z+\Delta z;t)-i(z;t). \quad (1.10)$$

Over the sides of the cylinder, the conductivity of the medium results in a transverse conduction current through S_o so that

$$\oint_{S_o}\mathscr{J}\cdot d\mathbf{s}=G\Delta z v(z;t) \quad (1.11)$$

where G is the per-unit length conductance between the wires produced by the lossy medium. In addition, let C denote the per-unit length capacitance, then over the length Δz

$$Q=C\Delta z v(z;t) \quad (1.12)$$

Inserting (1.10),(1.11),(1.12) into (1.9) yields

$$i(z+\Delta z;t)-i(z;t)+G\Delta z v(z;t)=-C\Delta z\frac{\partial v(z;t)}{\partial t} \quad (1.13)$$

Rewriting (1.13), then dividing both sides by Δz and taking the limit as $\Delta z\to 0$, we obtain

$$-\frac{\partial i(z;t)}{\partial z}=Gv(z;t)+C\frac{\partial v(z;t)}{\partial t} \quad (1.14)$$

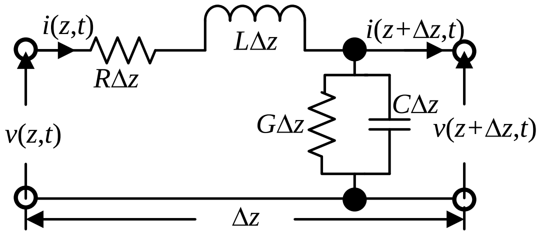


Fig. 2: The per-unit length model of a lossy transmission line

Therefore, the per-unit length model can be shown in Figure 2. Equations (1.8) and (1.14) are a pair of first-order partial differential equations in $v(z;t)$ and $i(z;t)$, which are called the *general transmission line equations* or *telegrapher's equations*.

For the *time-harmonic* electromagnetic field, the use of phasor representation helps simplify the analysis. Introducing the following quantities:

$$\mathcal{E}(x,z;t)=\mathrm{Re}\left[\mathbf{E}(x,z)e^{j\omega t}\right];\mathcal{H}(x,z;t)=\mathrm{Re}\left[\mathbf{H}(x,z)e^{j\omega t}\right];\mathcal{B}(x,z;t)=\mathrm{Re}\left[\mathbf{B}(x,z)e^{j\omega t}\right];$$

$$\mathcal{J}(x,z;t)=\mathrm{Re}\left[\mathbf{J}(x,z)e^{j\omega t}\right];\hat{Q}(x,z;t)=\mathrm{Re}\left[Q(x,z)e^{j\omega t}\right];$$

$$v(z;t)=\mathrm{Re}\left[V(z)e^{j\omega t}\right];i(z;t)=\mathrm{Re}\left[I(z)e^{j\omega t}\right],$$

where **E**, **H**, **B**, **J** are vector phasors, and *V*, *I* are (scalar) phasors, then equations (1.8) and (1.14) become

$$-\frac{dV(z)}{dz}=\hat{Z}I(z) \quad (1.15a) \qquad -\frac{dI(z)}{dz}=\hat{Y}V(z) \quad (1.15b),$$

where $\hat{Z}=R+j\omega L$ and $\hat{Y}=G+j\omega C$. Equations (1.15a) and (1.15b) are called *time-harmonic transmission-line equations*, which is useful for steady state analysis with sinusoidal waveform. Note that phasors are *complex* quantities in general.

From (1.15a) and (1.15b), one obtains the wave equations for the lossy case as follows:

$$\frac{d^2V(z)}{dz^2}=\hat{Z}\hat{Y}V(z)=\gamma^2V(z) \quad (1.16a) \qquad \frac{d^2I(z)}{dz^2}=\hat{Z}\hat{Y}I(z)=\gamma^2I(z) \quad (1.16b),$$

where γ is the propagation constant given by

$$\gamma=\alpha+j\beta=\sqrt{\hat{Z}\hat{Y}}=\sqrt{(R+j\omega L)(G+j\omega C)} \quad (1.17)$$

(α is called <u>attenuation constant</u> [Np/m] and β is called <u>phase constant</u> [rad/m]). These equations are similar to wave equations in lossy media which can be derived from Maxwell's equations. They would reduce to those for the lossless case when $R = G = 0$, which are quite similar to those for the waves in lossless media. The solutions to (1.16) become

$$V(z)=V_0^+e^{-\gamma z}+V_0^-e^{\gamma z}=V_0^+e^{-\alpha z}e^{-j\beta z}+V_0^-e^{\alpha z}e^{j\beta z} \quad (1.18a)$$

$$I(z)=I_0^+e^{-\gamma z}+I_0^-e^{\gamma z}=I_0^+e^{-\alpha z}e^{-j\beta z}+I_0^-e^{\alpha z}e^{j\beta z} \quad (1.18b),$$

where the plus and minus superscripts denote waves traveling in the +z and –*z* directions, respectively, and $V_0^{\pm}$, $I_0^{\pm}$ denote wave amplitudes. Substituting (1.18) into (1.15) yields the relationships between voltage and current wave amplitudes, which are defined as the *characteristic impedance*:

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{\hat{Z}}{\hat{Y}}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}. \qquad (1.19)$$

Clearly, using Z_0, (1.18b) can be rewritten as

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} = \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{\alpha z} e^{j\beta z}. \qquad (1.20)$$

Note that $V(z)$ and $I(z)$ are similar to the plane wave (electric field, magnetic field) in a lossy medium. The wave decays in z direction due to the $e^{-\alpha z}$ and $e^{\alpha z}$ terms. The velocity of propagation or the phase velocity is given by

$$u_p = \frac{\omega}{\beta} \text{ with } \beta = \mathrm{Im}(\gamma) \qquad (1.21)$$

Attenuation constant has unit Np/m where 1 Np/m means attenuation factor $1/e$ after 1-m propagation. Typically represented in dB unit where $\alpha_{dB} = 20\log_{10}e^{\alpha} = 8.686\alpha$ and $\alpha = 1$ Np/m -> 8.686 dB/m.

Note that γ and Z_0 are characteristic properties of a transmission line, which depend on R, L, G, C and ω.

Special Cases For the following special cases, the expressions are simplified.

1. *Lossless Line* ($R = 0$, $G = 0$)
   a) Propagation constant:
   $\gamma = \alpha + j\beta = j\omega\sqrt{LC}$; $\alpha = 0$; $\beta = \omega\sqrt{LC}$ (linear function of ω)
   b) Phase velocity
   $u_p = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}}$ (constant)
   c) Characteristic impedance
   $Z_0 = R_0 + jX_0 = \sqrt{\dfrac{L}{C}}$; $R_0 = \sqrt{\dfrac{L}{C}}$; $X_0 = 0$

For the lossless lines, $\gamma = \alpha + j\beta = j\beta; u = 1/\sqrt{LC}; \beta = \omega/u = 2\pi f/u$. Thus, the propagation constant (the phase constant) is linearly dependent on frequency, i.e., linear phase, but the phase velocity is independent of frequency.

2. *Low-loss Line* or *slightly lossy* ($R << \omega L$, $G << \omega C$) low-loss conditions are more easily satisfied at very high frequencies.
   a) Propagation constant:

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)^{1/2}\left(1 + \frac{G}{j\omega C}\right)^{1/2} \cong j\omega\sqrt{LC}\left(1 + \frac{R}{j2\omega L}\right)\left(1 + \frac{G}{j2\omega C}\right)$$

$$\cong j\omega\sqrt{LC}\left[1 + \frac{1}{j2\omega}\left(\frac{R}{L} + \frac{G}{C}\right)\right]$$

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right); \; \beta = \omega\sqrt{LC} \text{ (linear function of } \omega)$$

   b) Phase velocity

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{LC}} \text{ (constant)}$$

c) Characteristic impedance

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{-1/2} \cong \sqrt{\frac{L}{C}}\left[1+\frac{1}{j2\omega}\left(\frac{R}{L}-\frac{G}{C}\right)\right];$$

$$R_0 \cong \sqrt{\frac{L}{C}}\,;\; X_0 \cong -\sqrt{\frac{L}{C}}\frac{1}{2\omega}\left(\frac{R}{L}-\frac{G}{C}\right) \cong 0\,.$$

3. *Distortionless Line* (*R*/*L* = *G*/*C*)

For the lossy lines, if the condition $\boxed{\frac{R}{L}=\frac{G}{C}}$ is satisfied, then

a) Propagation constant:

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{R(1+j\omega\frac{L}{R})G(1+j\omega\frac{C}{G})} = \sqrt{RG}\left(1+j\omega\frac{C}{G}\right) = \alpha + j\beta$$

or $\alpha = \sqrt{RG}; \beta = \omega\sqrt{RG}\frac{C}{G} = \omega C\sqrt{\frac{R}{G}} = \omega C\sqrt{\frac{L}{C}} = \omega\sqrt{LC}$

b) Phase velocity: $u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$ (constant)

c) Characteristic impedance

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{R(1+j\omega L/R)}{G(1+j\omega C/G)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}\,;\; R_0 = \sqrt{\frac{L}{C}}\,;\; X_0 = 0.$$

Thus, α does not depend on frequency, β is linearly dependent on frequency, which is similar to the lossless lines, and there is no distortion. Thus, it is called “distortionless” lines. If two frequency components have different phase velocity, then the signal will distort. For example, let $y = \cos(\omega_1 t - \beta_1 z)\cos(\omega_2 t - \beta_2 z)$, and $\omega_2 = 5\omega_1$, $\beta_1\ell = 2\pi$. Figure 3 shows the signal at z=0 and $z=\ell$ for different u_1, u_2.

In general, the phase constant is not a linear function of ω, thus it will lead to a u_p, which depends on frequency. As the different components of a signal propagate along the line with different velocities, the signal suffer *dispersion*. A general, lossy transmission line is therefore *dispersive*, as is a lossy dielectric.

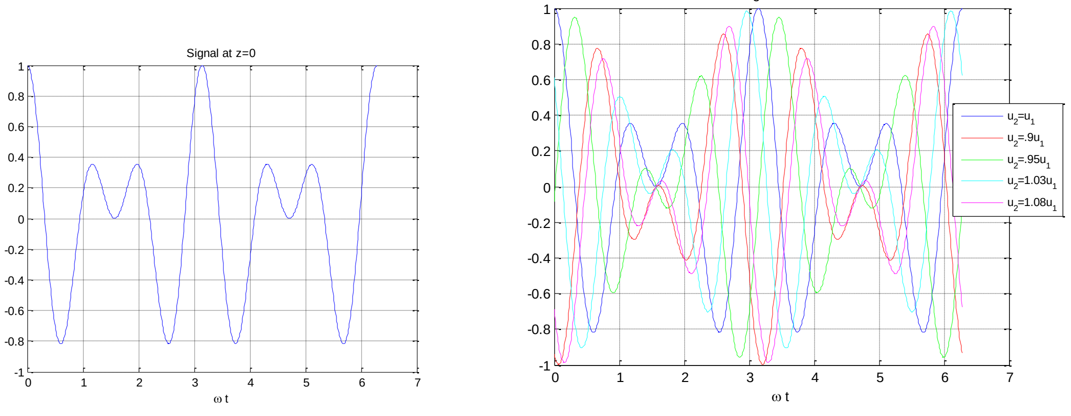


Fig. 3 : Signal distortion

Example 2.1 A distortionless line has $Z_0 = 60\ \Omega$, $\alpha = 20$ mNp/m, $u = 0.6c$, where c is the speed of light in a vacuum. Find R, L, G, C and λ at 100 MHz.

3 Transmission-Line Parameters

Capacitance Recall that the total charge is related to the voltage by $Q = CV$, the capacitance can be found from

$$C = \frac{\oint_S \mathbf{D}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot d\mathbf{l}} = \frac{\oint_S \varepsilon\mathbf{E}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot d\mathbf{l}}.$$

Conductance (the shunt resistance) When the dielectric medium is lossy (having a small but nonzero conductivity), a current will flow from the positive to the negative conductor, and a current density field will be established in the medium. Using the Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, yields

$$G = \frac{I}{V} = \frac{\oint_S \mathbf{J}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot d\mathbf{l}} = \frac{\oint_S \sigma\mathbf{E}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot d\mathbf{l}}.$$

For homogeneous media (or when σ and ε have the same spatial dependence) the following relationship holds:

$$\frac{C}{G} = \frac{\varepsilon}{\sigma},$$

which is derived from two above equations.

Inductance The inductance can be directly calculated from

$$L = \frac{\Phi}{I} = \frac{\oint_S \mathbf{B}\cdot d\mathbf{s}}{\oint_S \mathbf{J}\cdot d\mathbf{s}} = \frac{\oint_S \mu\mathbf{H}\cdot d\mathbf{s}}{\oint_S \sigma\mathbf{E}\cdot d\mathbf{s}}.$$

However, a comparison of the propagation constant for the TEM wave on a transmission line with $R = 0$ and that for the wave in a medium with constitutive parameters (μ, ε, σ)

$$\gamma = j\omega\sqrt{LC}\left(1 + \frac{G}{j\omega C}\right)^{1/2};\ \gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

together with $C/G = \varepsilon/\sigma$ yields

$LC = \mu\varepsilon.$

Resistance Series resistance R is determined by introducing a small E_z as a slight perturbation of the TEM wave and by finding the Ohmic power dissipated in a unit length of the line. Typically, it is calculated by the resistance due to the *skin depth*.

i) *Parallel plate transmission line* Let w, d denote the width, separation and assume the medium between plates has constitutive parameters (μ, ε, σ), then

$$C=\varepsilon\frac{w}{d};L=\mu\frac{d}{w};G=\sigma\frac{w}{d}.$$

The per-unit-length resistance can be calculated from $R = 1/\sigma_c S$, where σ_c denotes the conductivity of the conducting plate and S denotes the cross section which equals the product of width w and skin depth δ. Thus,

$$R=\frac{2}{\sigma_c w\delta}=\frac{2}{w}\sqrt{\frac{\pi f\mu_c}{\sigma_c}},$$

where μ_c denotes the permeability of the plate and f is the frequency. Also, the factor 2 comes from the fact that the transmission line consists of two plates. Note also that a good conductor is assumed here in the calculation of skin depth, i.e.,

$$\delta=\frac{1}{\sqrt{\pi f\mu_c\sigma_c}}.$$

ii) *Two-wire transmission line* Assume the two conducting wires of radius a, separated by D in the medium with constitutive parameters (μ, ε, σ), then

$$C=\frac{\pi\varepsilon}{\cosh^{-1}(D/2a)};L=\frac{\mu}{\pi}\cosh^{-1}(D/2a);G=\frac{\pi\sigma}{\cosh^{-1}(D/2a)},\text{ and}$$

$$R=\frac{2}{\sigma_c S}=\frac{2}{\sigma_c 2\pi a\delta}=\frac{1}{\sigma_c \pi a\delta}=\frac{1}{\pi a}\sqrt{\frac{\pi f\mu_c}{\sigma_c}}.$$

iii) *Coaxial transmission line* Assume the (inner, outer) radii be (a, b) and the medium with constitutive parameters (μ, ε, σ), then

$$C=\frac{2\pi\varepsilon}{\ln(b/a)};L=\frac{\mu}{2\pi}\ln\frac{b}{a};G=\frac{2\pi\sigma}{\ln(b/a)},\text{ and}$$

$$R=\frac{1}{\sigma_c S_i}+\frac{1}{\sigma_c S_o}=\frac{1}{\sigma_c 2\pi\delta}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{1}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{\frac{\pi f\mu_c}{\sigma_c}}.$$

Note that (S_i, S_o) in the equation above denote the cross sections of the (inner,outer) conductors, respectively.

4 Wave Characteristics on Finite Transmission Lines (Steady State Analysis)

Consider the finite transmission line shown in Fig. 4. The length of the line is ℓ. Then,

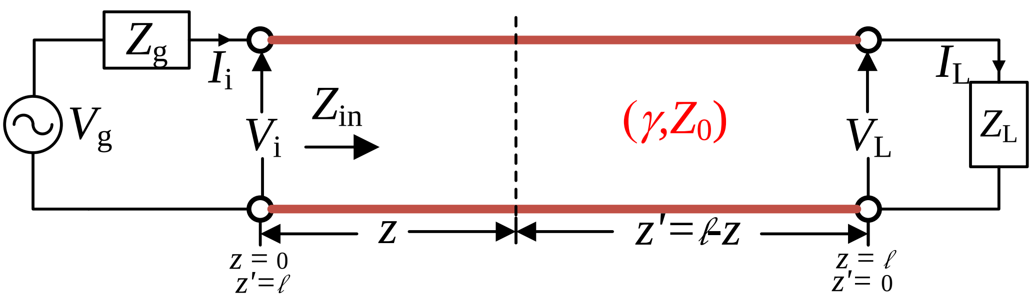


Fig. 4: Finite transmission line terminated with load impedance Z_L.

$$Z_L=\left(\frac{V}{I}\right)_{z=\ell}=\frac{V_L}{I_L}=\frac{V_0^+e^{-\gamma\ell}+V_0^-e^{\gamma\ell}}{I_0^+e^{-\gamma\ell}+I_0^-e^{\gamma\ell}}=\frac{V_0^+e^{-\gamma\ell}+V_0^-e^{\gamma\ell}}{V_0^+e^{-\gamma\ell}/Z_0-V_0^-e^{\gamma\ell}/Z_0} \quad (4.1).$$

Recall that $V(z)=V_0^+e^{-\gamma z}+V_0^-e^{\gamma z}$ (4.2a); $I(z)=\frac{V_0^+}{Z_0}e^{-\gamma z}-\frac{V_0^-}{Z_0}e^{\gamma z}$ (4.2b), thus,

$V_L=V(\ell)=V_0^+e^{-\gamma\ell}+V_0^-e^{\gamma\ell}$ (4.3a); $I_L=I(\ell)=\frac{V_0^+}{Z_0}e^{-\gamma\ell}-\frac{V_0^-}{Z_0}e^{\gamma\ell}$ (4.3b).

Solving for V_0^+, V_0^- from the above equations yields

$V_0^+=\frac{1}{2}(V_L+I_LZ_0)e^{\gamma\ell}=\frac{I_L}{2}(Z_L+Z_0)e^{\gamma\ell}$ (4.4a); $V_0^-=\frac{1}{2}(V_L-I_LZ_0)e^{-\gamma\ell}=\frac{I_L}{2}(Z_L-Z_0)e^{-\gamma\ell}$ (4.4b).

Substituting (4.4a),(4.4b) into (4.2a),(4.2b) yields

$V(z)=\frac{I_L}{2}\left[(Z_L+Z_0)e^{\gamma(\ell-z)}+(Z_L-Z_0)e^{-\gamma(\ell-z)}\right]$ (4.5a);

$I(z)=\frac{I_L}{2Z_0}\left[(Z_L+Z_0)e^{\gamma(\ell-z)}-(Z_L-Z_0)e^{-\gamma(\ell-z)}\right]$ (4.5b).

Introducing a new variable *z'*=ℓ-*z*, then (4.5) can be rewritten as

$V(z')=\frac{I_L}{2}\left[(Z_L+Z_0)e^{\gamma z'}+(Z_L-Z_0)e^{-\gamma z'}\right]$(4.6a) $I(z')=\frac{I_L}{2Z_0}\left[(Z_L+Z_0)e^{\gamma z'}-(Z_L-Z_0)e^{-\gamma z'}\right]$ (4.6b).

The use of hyperbolic functions simplifies the equations above to be

$V(z')=I_L[Z_L\cosh\gamma z'+Z_0\sinh\gamma z']$ (4.7a) $I(z')=\frac{I_L}{Z_0}[Z_L\sinh\gamma z'+Z_0\cosh\gamma z']$ (4.7b).

The ratio of (*V*(*z'*)/*I*(*z'*) is the impedance when one looks toward the load end of the line at a distance *z'* from the load, which is given by

$$Z(z')=\frac{V(z')}{I(z')}=Z_0\frac{Z_L\cosh\gamma z'+Z_0\sinh\gamma z'}{Z_L\sinh\gamma z'+Z_0\cosh\gamma z'}=Z_0\frac{Z_L+Z_0\tanh\gamma z'}{Z_0+Z_L\tanh\gamma z'} \quad (4.8).$$

At the source end of the line *z'*=ℓ, the generator looking into the line sees an ***input impedance*** Z_{in}, which is given by

$$Z_{in}=Z(z'=\ell)=Z_0\frac{Z_L+Z_0\tanh\gamma\ell}{Z_0+Z_L\tanh\gamma\ell} \quad (4.9).$$

Then the load impedance and the transmission line can be replaced by the input impedance Z_{in} as depicted in Fig. 5. The input voltage V_i and input current I_i in Fig. 5 are found from the equivalent circuit as follows:

$$V_i=\frac{Z_{in}}{Z_g+Z_{in}}V_g \quad (4.10)$$

$$I_i=\frac{V_g}{Z_g+Z_{in}} \quad (4.11)$$

The average power delivered by the generator to the input terminals of the line is

$$(P_{av})_i=\frac{1}{2}\mathrm{Re}[V_iI_i^*], \quad (4.12)$$

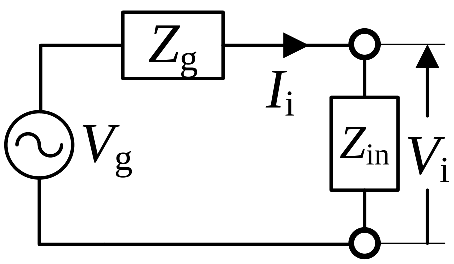


Fig. 5: Equivalent circuit for finite transmission line at generator end

and the average power delivered to the load is

$$\left(P_{av}\right)_L = \frac{1}{2}\mathrm{Re}[V_L I_L{*}] = \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2}\left|I_L\right|^2 R_L \qquad (4.13).$$

For a lossless transmission line, conservation of power requires that $(P_{av})_i = (P_{av})_L$.
Note that once V_i, I_i are obtained, $V_0^{\pm}$ can be calculated from (4.2a),(4.2b) as

$$V(z=0) = V_i = V_0^+ + V_0^- ; I(z=0) = I_i = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0},$$

then the voltage and current along the transmission line can be found.

Voltage Reflection Coefficient and Voltage Standing Wave Ratio (VSWR)
Define the "complex" ***voltage reflection coefficient*** as

$$\Gamma(z) = \frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}} = \frac{V_0^-}{V_0^+} e^{2\gamma z}, \qquad (4.14)$$

then (4.2a), (4.2b) can be rewritten as

$$V(z) = V_0^+ e^{-\gamma z}\left[1+\Gamma(z)\right] \quad (4.15a) \qquad I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z}\left[1-\Gamma(z)\right] \quad (4.15b).$$

It follows that

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)} \qquad (4.16)$$

and

$$Z(z=\ell) = \frac{V(\ell)}{I(\ell)} = Z_L = Z_0 \frac{1+\Gamma(\ell)}{1-\Gamma(\ell)} = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}. \qquad (4.17)$$

Therefore, the voltage reflection coefficient at the load is given by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| e^{j\theta_\Gamma}. \qquad (4.18)$$

Then from (4.14),

$$\frac{V_0^-}{V_0^+} = \Gamma(\ell) e^{-2\gamma\ell} = \Gamma_L e^{-2\gamma\ell}, \qquad (4.19)$$

thus, $\Gamma(z) = \Gamma_L e^{2\gamma(z-\ell)}$. (4.20)

The magnitude ratio of the maximum to the minimum voltages along a finite, terminated line is defined as the ***voltage standing wave ratio* (VSWR)**, i.e.,

$$\mathrm{VSWR} = S = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1+|\Gamma|}{1-|\Gamma|}, \qquad (4.21)$$

which measures "degree of mismatch". S=1 denotes the matched-load condition.

Power Flow Due to losses, the power is a function of the position on the line.

$$P_{av}(z) = \frac{1}{2}\mathrm{Re}\left[V(z)I^*(z)\right] = \frac{1}{2}\mathrm{Re}\left[\left(V_0^+ e^{-\alpha z}e^{-j\beta z} + V_0^- e^{\alpha z}e^{j\beta z}\right)\left(\frac{V_0^+}{Z_0}e^{-\alpha z}e^{-j\beta z} - \frac{V_0^-}{Z_0}e^{\alpha z}e^{j\beta z}\right)^*\right] \quad (4.22)$$

Recall that

$$V(z) = V_0^+ e^{-\gamma z}\left[1+\Gamma(z)\right] \quad (4.15a) \qquad I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z}\left[1-\Gamma(z)\right] \quad (4.15b),$$

then,

$$P_{av}(z) = \frac{1}{2}\text{Re}\left[V_0^+ e^{-\gamma z}\left[1+\Gamma(z)\right]\left(\frac{V_0^+}{Z_0}e^{-\gamma z}\right)^* \left[1-\Gamma(z)\right]^*\right]$$
$$= \frac{1}{2}\left|V_0^+\right|^2 e^{-2\alpha z}\,\text{Re}\left[\frac{1-\left|\Gamma(z)\right|^2 + j2\,\text{Im}\,\Gamma(z)}{Z_0^*}\right] \quad (4.23)$$

The law of energy conservation requires that the rate of decrease of $P_{av}(z)$ with distance along the line equals the time-average power loss P_L per unit length. Thus,

$$-\frac{\partial P_{av}(z)}{\partial z} = P_L(z) = 2\alpha P_{av}(z)\,, \quad (4.24)$$

from which we obtain the following formula:

$$\alpha = \frac{P_L(z)}{2P_{av}(z)}\ \text{(Np/m)} \quad (4.25)$$

Example 4.1 A certain transmission line operating at $\omega = 10^6$ rad/s has $\alpha = 8$ dB/m, $\beta = 1$ rad/m, and $Z_0 = 60 + j40\ \Omega$, and is 2 m long. If the line is connected to a generator of $10\angle 0$ V, $Z_g = 40\ \Omega$ and terminated by a load of $20 + j50\ \Omega$, determine

a) The input impedance
b) The sending end current
c) The power at the sending end and the load

5 Steady State Analysis of Lossless Transmission Lines

In most practical applications, low-loss transmission lines are used and thus they can be approximated as lossless to simplify the analysis. As previously mentioned, for lossless lines, $\gamma = \alpha + j\beta = j\omega\sqrt{LC}$ (5.1) ; $\alpha = 0$ (5.2); $\beta = \omega\sqrt{LC}$ (5.3).

Hence, the input impedance becomes

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell}. \qquad (5.4)$$

where Z_0 is real. Note that $\beta\ell = \omega\ell/u_p = 2\pi\ell/\lambda$, thus it is more convenient to express ℓ in terms of wavelength, i.e., $\lambda = u_p/f$. Also, the voltage reflection coefficient becomes

$$\Gamma(z) = \frac{V_0^- e^{j\beta z}}{V_0^+ e^{-j\beta z}} = \frac{V_0^-}{V_0^+} e^{j2\beta z} = \Gamma_L e^{j2\beta(z-\ell)} = |\Gamma_L| e^{j\theta_\Gamma}. \qquad (5.5)$$

and $V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} = V_0^+ e^{-j\beta z}[1 + \Gamma(z)]$. (5.6)

Let $V_0^+ = |V_0^+| e^{j\theta^+}, V_0^- = |V_0^-| e^{j\theta^-}$, then

$$|\Gamma(z)| = \left|\frac{V_0^-}{V_0^+}\right| = |\Gamma_L|, \text{ and } \theta_\Gamma = 2\beta z - \theta^+ + \theta^- = 4\pi z/\lambda - \theta^+ + \theta^-.$$

1+Γ(z) can be illustrated in the complex plane as shown in Fig. 6, which is often referred to as the *crank diagram*.

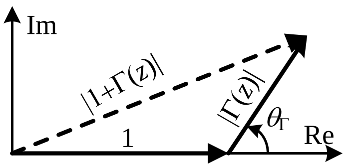


Fig. 6: The crank diagram

It follows that

$$|V(z)| = |V_0^+ e^{-j\beta z}[1 + \Gamma(z)]| = |V_0^+||1 + \Gamma(z)|, \text{ thus}$$

$$|V(z)|_{\max} = |V_0^+|(1 + |\Gamma(z)|) \qquad (5.7) \qquad |V(z)|_{\min} = |V_0^+|(1 - |\Gamma(z)|) \qquad (5.8)$$

Clearly, for a matched load $|\Gamma_L|=0$ and $|V(z)|_{\max} = |V(z)|_{\min}$.

The VSWR then becomes

$$\text{VSWR} = S = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1 + |\Gamma(z)|}{1 - |\Gamma(z)|} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}. \qquad (5.9)$$

Alternatively, $|\Gamma(z)| = |\Gamma_L| = \frac{S-1}{S+1}$. (5.10)

<u>Power Flow</u> The average power is given by

$$P_{av}(z) = \frac{1}{2}\text{Re}[V(z)I^*(z)] = \frac{1}{2}\text{Re}\left[\left(V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}\right)\left(\frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}\right)^*\right]$$

$$= \frac{1}{2}\text{Re}\left[V_0^+ e^{-j\beta z}(1 + \Gamma(z))\left(\frac{V_0^+ e^{-j\beta z}}{Z_0}\right)^* (1 - \Gamma^*(z))\right] = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}\left(1 - |\Gamma_L|^2\right) \qquad (5.11)$$

(5.11) can be found from summing power traveling in +*z* and –*z* directions as follows:

$$P_{av}^+(z) = \frac{1}{2}\text{Re}[V^+(z)I^{+*}(z)] = \frac{1}{2}\text{Re}\left[V_0^+ e^{-j\beta z} \frac{V_0^{+*}}{Z_0} e^{j\beta z}\right] = \frac{1}{2}\frac{|V_0^+|^2}{Z_0} \qquad (5.12)$$

$$P_{av}^{-}(z)=\frac{1}{2}\mathrm{Re}\left[V^{-}(z)I^{-}*(z)\right]=\frac{1}{2}\mathrm{Re}\left[V_0^{-}e^{j\beta z}\frac{V_0^{-*}}{Z_0}e^{-j\beta z}\right]=\frac{1}{2}\frac{\left|V_0^{-}\right|^2}{Z_0}=\frac{1}{2}\frac{\left|V_0^{+}\right|^2\left|\Gamma_L\right|^2}{Z_0} \qquad (5.13)$$

$$P_{av}(z)=P_{av}^{+}(z)+P_{av}^{-}(z)=\frac{1}{2}\frac{\left|V_0^{+}\right|^2}{Z_0}\left(1-\left|\Gamma_L\right|^2\right). \qquad (5.14)$$

Note that the power is uniform along the transmission line, i.e., independent of the position on the line, thus the power delivered to the line is the same as the power delivered to the load.

Clearly, $\frac{P_{av,\mathrm{reflected}}}{P_{av,\mathrm{incident}}}=\frac{P_{av}^{-}}{P_{av}^{+}}=\left|\Gamma_L\right|^2.$ (5.15)

Also, the power delivered to the line and to the load can be found from

$$P_{av,\mathrm{toline}}=\frac{1}{2}\mathrm{Re}\left[V(0)I*(0)\right]=\frac{1}{2}\mathrm{Re}\left[\frac{V(0)V*(0)}{Z_{in}^{*}}\right]=\frac{1}{2}\frac{\left|V_i\right|^2}{\left|Z_{in}\right|}\cos\angle Z_{in}, V(0)=\frac{Z_{in}}{Z_{in}+Z_g}V_g \quad (5.16)$$

$$P_{av,\mathrm{toload}}=\frac{1}{2}\mathrm{Re}\left[V(\ell)I*(\ell)\right]=\frac{1}{2}\mathrm{Re}\left[\frac{V(\ell)V*(\ell)}{Z_{L}^{*}}\right]=\frac{1}{2}\frac{\left|V_L\right|^2}{\left|Z_L\right|}\cos\angle Z_L. \qquad (5.17)$$

<u>Special Cases</u>

a) Short-circuit load (Z_L = 0)

$Z_{in}=jZ_0\tan\beta\ell=jZ_0\tan(2\pi\ell/\lambda)$, which implies an "inductive" reactance.

$|\Gamma_L|=1\rightarrow$ No power delivered to load (i.e., the load does not consume any power).

b) Open-circuit load ($Z_L=\infty$)

$Z_{in}=Z_0/(j\tan\beta\ell)=-jZ_0/\tan(2\pi\ell/\lambda)$, which implies a "capacitive" reactance.

$|\Gamma_L|=1\rightarrow$ No power delivered to load (i.e., the load does not consume any power).

c) Matched load ($Z_L=Z_0$)

$Z_{in}=Z_0$

$|\Gamma_L|=0\rightarrow P_{av}=P_{av}^{+}\rightarrow$ all power is delivered to the load.

d) Quarter-wavelength transmission line ($\ell=\lambda/4$)

Since $\beta\ell=\pi/2$, $\tan\beta\ell\rightarrow\infty$, thus

$$Z_{in}=\frac{Z_0^2}{Z_L}.$$

It follows that $\Gamma(0)=\Gamma_L e^{-j2\beta\ell}=-\Gamma_L$, and

for $Z_L=0$, $\Gamma_L=-1$, $\Gamma(0)=1$, $Z_{in}=\infty$: short-circuit $\rightarrow$ open-circuit

$Z_L=\infty$, $\Gamma_L=1$, $\Gamma(0)=-1$, $Z_{in}=0$: open-circuit $\rightarrow$ short-circuit

e) Transmission lines' length equal multiples of half-wavelength ($\ell=n\lambda/2$)

Since $\beta\ell=n\pi$, $\tan\beta\ell=0$, thus $Z_{in}=Z_L$, i.e., input impedance is equal to load impedance.
Note also that since function tan(x) is a "periodic" function with period π,

$$\tan\beta\ell=\tan(\beta\ell\pm n\pi)=\tan\beta(\ell\pm n\pi/\beta)=\tan\beta(\ell\pm n\lambda/2),\ n=1,2,3,\ldots,$$

thus adding multiples of half-wavelength does not change the input impedance.

Example 5.1 A 10-m section of lossless transmission line having $Z_0 = 50\ \Omega$ and $u = 200$ m/μs is driven by a 26-MHz generator having an open-circuit voltage of $V_g = 100$ V and generator impedance $Z_g = 50\ \Omega$. The line terminated in a load impedance of $Z_L = 100 + j50\ \Omega$. Determine the input impedance to the line and the instantaneous voltage at the input to the line and the at the load, i.e., $V(0, t)$ and $V(\ell, t)$.

Example 5.2 A 1-m section of lossless transmission line having $C = 200$ pF/m and $L = 0.5$ μH/m is driven by a 30-MHz generator having an open-circuit voltage of $V_g = 1$ V and generator impedance $Z_g = 10\ \Omega$. The line terminated in a load impedance of $Z_L = 100 + j50\ \Omega$. Determine the load voltage and the average power delivered to the line and to the load.

6 Transient Analysis

In this section, the transient behavior of lossless transmission lines will be discussed. Such practical situations include cases where non-time-harmonic signals are used or where the conditions are not steady-state. Examples are digital (pulse) signals in computer networks and sudden surges in power and telephone lines.

Recall that the general transmission line equations previously derived are given for lossless transmission lines by

$$-\frac{\partial v(z;t)}{\partial z} = L\frac{\partial i(z;t)}{\partial t} \quad (6.1a) \quad \text{and} \quad -\frac{\partial i(z;t)}{\partial z} = C\frac{\partial v(z;t)}{\partial t} \quad (6.1b)$$

and the wave equations derived from (6.1a), (6.1b) are

$$\frac{\partial^2 v(z;t)}{\partial z^2} = LC\frac{\partial^2 v(z)}{\partial t^2} \quad (6.2a) \quad \text{and} \quad \frac{\partial^2 i(z;t)}{\partial z^2} = LC\frac{\partial^2 i(z)}{\partial t^2} \quad (6.2b),$$

The general solutions to (6.2a), (6.2b) are

$$v(z;t) = v^+(t - z/u) + v^-(t + z/u) \quad (6.3a) \quad i(z;t) = i^+(t - z/u) + i^-(t + z/u) \quad (6.3b)$$

where +,- signs denote the waves traveling in +*z*, -*z*, respectively. v^+, v^-, i^+, i^- depend on *t*, *z* and *u*.

<u>Proof</u>

As before, $i^{\pm}$ can be related to $v^{\pm}$ via the characteristic impedance as

$$i^+(t - z/u) = v^+(t - z/u)/Z_0; \quad (6.4a) \qquad i^-(t + z/u) = -v^-(t + z/u)/Z_0 \quad (6.4b)$$

where $Z_0 = \sqrt{L/C}$. Hence,

$$i(z;t) = \left[v^+(t - z/u) - v^-(t + z/u)\right]/Z_0 \quad (6.5)$$

Now, consider a terminated transmission line where a generator with generator resistance R_g is applied at $t = 0$, as depicted in Fig. 7.

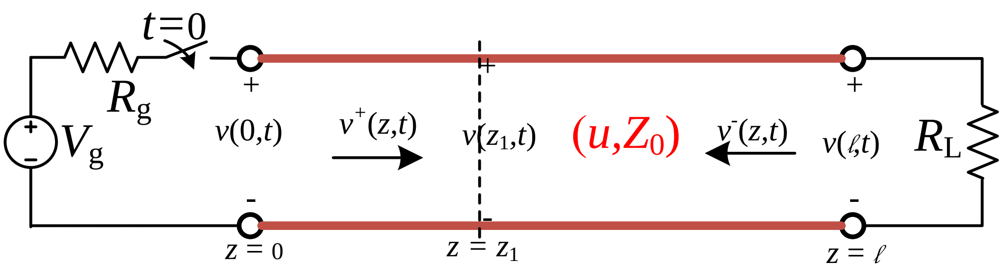


Fig. 7: A generator applied to a terminated lossless line at $t = 0$.

At the load end $(z = \ell)$, $v(\ell;t) = R_L i(\ell;t)$ (6.6)

The discontinuity at the load results in the reflection. Recall the voltage reflection coefficient mentioned previously, which is given by

$$\Gamma_L = \frac{v^-(t + \ell/u)}{v^+(t - \ell/u)}. \quad (6.7)$$

Using Γ_L to rewrite *v*, *i* at the load as

$$v(\ell;t) = v^+(t-\ell/u) + v^-(t+\ell/u) = v^+(t-\ell/u)(1+\Gamma_L) \qquad (6.8a)$$

$$i(\ell;t) = \{v^+(t-\ell/u) - v^-(t+\ell/u)\}/Z_0 = v^+(t-\ell/u)(1-\Gamma_L)/Z_0 \qquad (6.8b)$$

Likewise, the current reflection coefficient can be defined as

$$-\Gamma_L = \frac{i^-(t+\ell/u)}{i^+(t-\ell/u)}. \qquad (6.9)$$

Substituting (6.8a),(6.8b) into (6.6) yields

$v^+(t-\ell/u)(1+\Gamma_L) = R_L v^+(t-\ell/u)(1-\Gamma_L)/Z_0$ or

$$R_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}. \qquad (6.10)$$

Solving for Γ_L yields

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}, \qquad (6.11)$$

which is the same as that for the steady-state analysis, except this Γ_L is real. Note that at the load, reflection is like a mirror; reflected wave v^-, a replica of v^+ which is flipped around and is multiplied by Γ_L, is a “mirror image” of v^+.

Now, consider the portion of the line at the generator end $z = 0$. Let $T = \ell/u$ be the time delay and the generator is turned on at $t = 0$.

During $0 \le t \le 2T$, no backward-traveling waves will appear at $z = 0$. Thus,

$$v(0;t) = v^+(t-0/u), 0 \le t \le 2T \quad (6.12a) \qquad i(0;t) = v^+(t-0/u)/Z_0, 0 \le t \le 2T \quad (6.12b)$$

Hence, the ratio of the voltage and current on the line is Z_0 for $0 \le t \le 2T$, i.e., the input impedance seen by the generator is Z_0 during $0 \le t \le 2T$. Thus,

$$v(0;t) = V_g(t) - R_g i(0;t) = V_g(t) - R_g v(0;t)/Z_0, \qquad (6.13)$$

so that $v(0;t) = \dfrac{Z_0}{Z_0 + R_g} V_g(t), 0 \le t \le 2T$. (6.14)

Therefore, the initial v^+ has the same shape as $V_g(t)$ and is multiplied by $Z_0/(Z_0+R_g)$. Likewise,

$$i(0;t) = \frac{V_g(t)}{Z_0 + R_g}, 0 \le t \le 2T. \qquad (6.15)$$

After $2T$, there exist backward-traveling waves at the generator end. The discontinuity at this end will cause the reflection in the same manner as that at the load end. Define the voltage reflection coefficient at the generator as

$\Gamma_g = \dfrac{v^-(t+0/u)}{v^+(t-0/u)}$, (6.16) then

$$v(0;t) = v^+(t-0/u) + v^-(t+0/u) = v^+(t-0/u)(1+\Gamma_g) \qquad (6.17a)$$

$$i(0;t) = [v^+(t-0/u) - v^-(t+0/u)]/Z_0 = v^+(t-0/u)(1-\Gamma_g)/Z_0 \qquad (6.17b)$$

Since $v(0;t) = R_g i(0;t)$,

$v^+(t-0/u)(1+\Gamma_g) = R_g v^+(t-0/u)(1-\Gamma_g)/Z_0$ or

$$R_g = Z_0 \frac{1+\Gamma_g}{1-\Gamma_g}. \qquad (6.18)$$

Solving for Γ_g yields

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0}, \qquad (6.19)$$

which is similar to the voltage reflection coefficient at the load. The reflections at both ends will continue until the system reaches the steady-state.

<u>Reflection Diagram or Bounce Diagram</u> To calculate the voltage and current at a particular time and location on a transmission line with arbitrary resistive load tends to be tedious and difficult to visualize when there are many reflected waves. In such cases, the graphical construction of a *reflection diagram* (or *a bounce diagram*) is helpful. Fig. 8 shows a typical reflection diagram. V_1^+ in the figure denotes the initial voltage, i.e., $v(0;t)$ in (6.14). It will take the time T for this wave to reach the load end and then the reflection will take place creating a reflected wave, which will reach the generator end at the time $2T$. These processes will continue until the steady-state.

If one is interested in finding the voltage at a specific location on the line, say z_1, first find the time when each wave reaches that location, as denoted here by t_1, t_2 and vice versa in the figure. These are time instants when the voltage discontinuities occur. Then the voltage at these instants can be obtained by simply adding all components together.

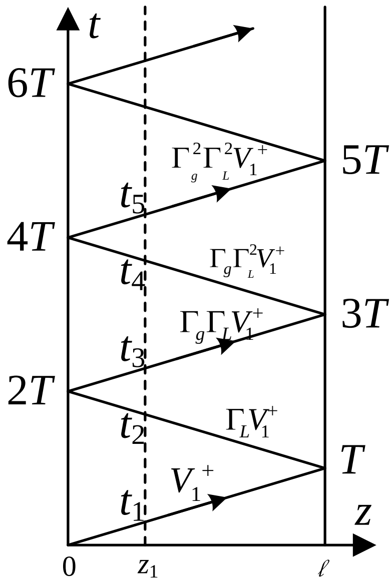
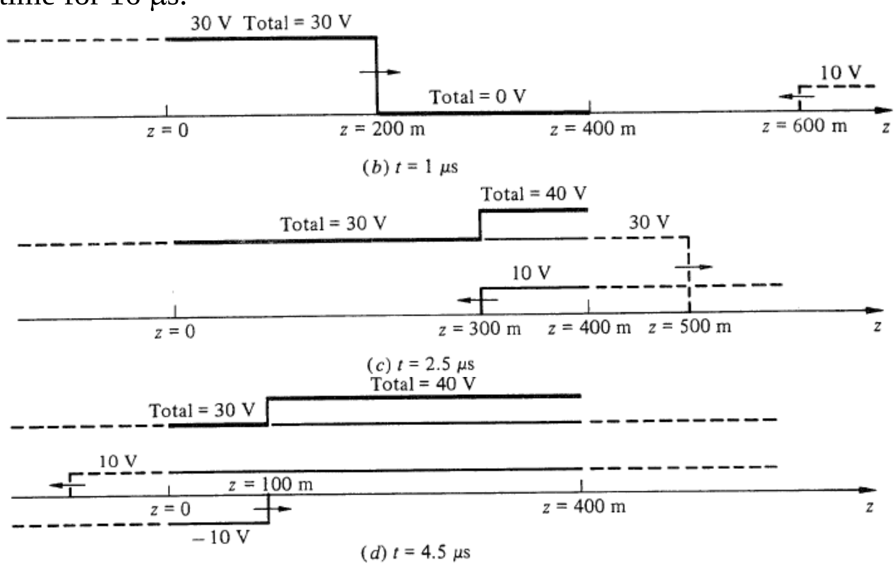
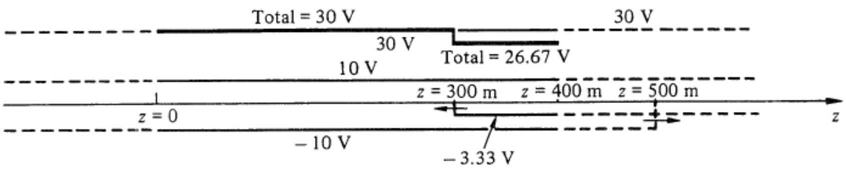


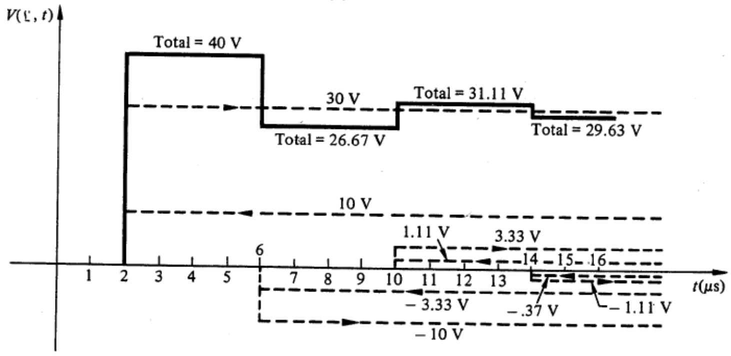
Fig. 8: A reflection diagram

<u>Example 6.1</u> Consider a 400-m section of lossless transmission line having Z_0 = 50 Ω and u = 200 m/μs connected to a load Z_L = 100 Ω. At t = 0 a 30-V battery with zero generator resistance is connected to the line. Sketch the distribution of voltage along the line for several instants of time. Then sketch the voltage at the load to the line, $v(\ell;t)$, as a function of time for 16 μs.

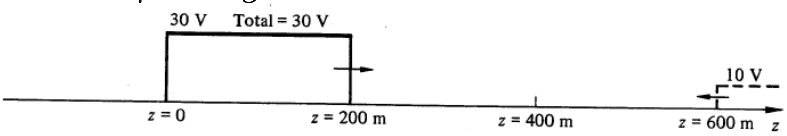




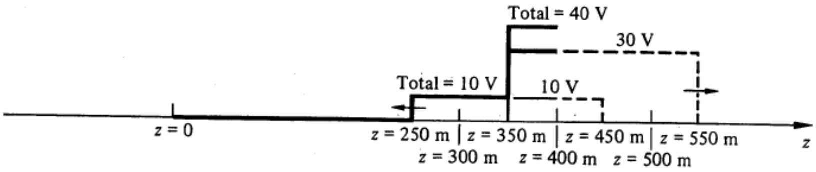
(e) t = 6.5 μs



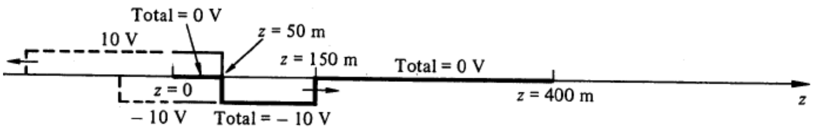
Example 6.2 Repeat the previous example where the voltage source is a pulse of 30 V but duration 1 μs. The generator resistance remains zero.



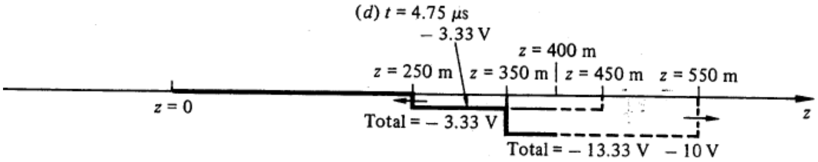
(b) t = 1 μs



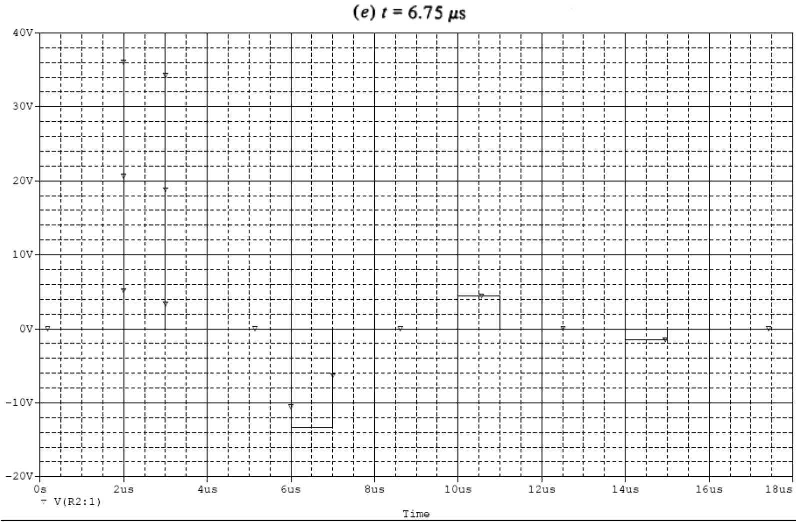
(c) t = 2.75 μs



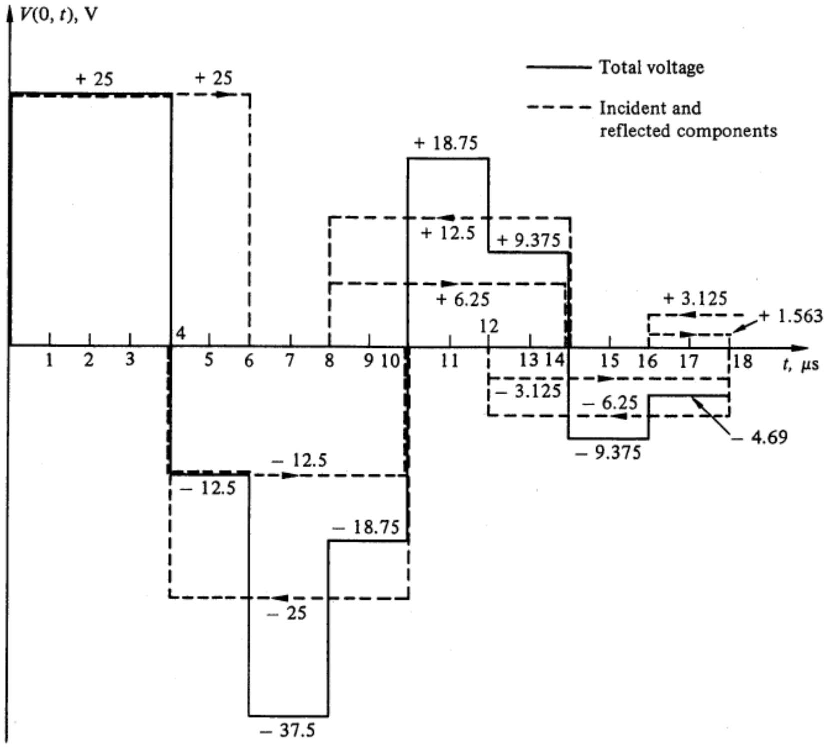
(d) t = 4.75 μs

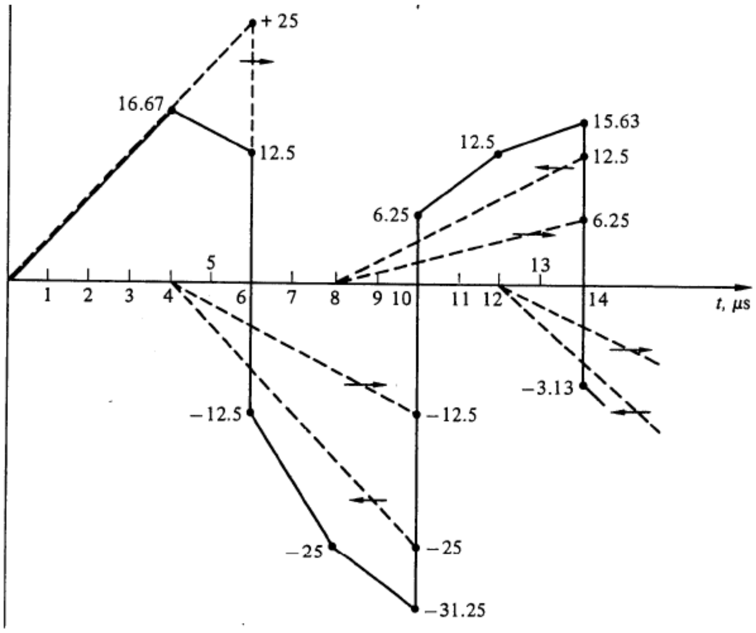


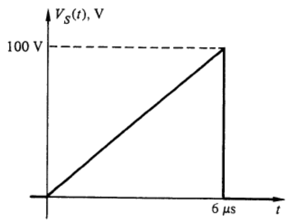
(e) t = 6.75 μs



Example 6.3 Consider a 400-m length of coaxial cable with C = 100 pF/m and L = 0.25 μH/m. The cable is terminated in a short circuit and is driven by a pulse source with internal resistance of 150 Ω. The pulse has a magnitude of 100 V and duration of 6 μs. Sketch the voltage at the input to the line, $v(0;t)$, as a function of time for 18 μs.







Capacitive load termination

Assume that a lossless transmission line is terminated with a capacitive load C_L, and a DC voltage source V_g with internal resistance R_g is applied. When the switch is closed at t = 0, a voltage wave of an amplitude

$$V_1^+ = v(0;t) = \frac{Z_0}{Z_0 + R_g} V_g \,, 0 \le t \le 2T \quad (6.20).$$

travels toward the load. Upon reaching the load at $t = \ell/u = T$, a reflected wave $V_1^-(t)$ is produced because of mismatch. At $z = \ell$, for all $t \ge T$,

$$v(\ell;t) = v_L(t) = V_1^+ + V_1^-(t) \text{ (6.21)}; \; i_L(t) = \frac{1}{Z_0}\left[V_1^+ - V_1^-(t)\right] \text{ (6.22)}; i_L(t) = C_L \frac{dv_L(t)}{dt} \text{ (6.23)}.$$

From (6.21), (6.22), one obtains

$v_L(t) = 2V_1^+ - Z_0 i_L(t)$. (6.24)

Substituting (6.24) into (6.23) yields

$$C_L \frac{dv_L(t)}{dt} + \frac{1}{Z_0} v_L(t) = \frac{2V_1^+}{Z_0}, \; t \ge T. \qquad (6.25)$$

The solution of (6.25) is given by

$$v_L(t) = 2V_1^+ \left\{1 - e^{-(t-T)/Z_0 C_L}\right\}, \; t \ge T, \qquad (6.26)$$

which can be easily obtained via Laplace's transform.

It follows that

$$i_L(t) = \frac{2V_1^+}{Z_0} e^{-(t-T)/Z_0 C_L}, \; t \ge T \quad (6.27)\; ; \;\; V_1^-(t) = 2V_1^+\left[1/2 - e^{-(t-T)/Z_0 C_L}\right], \; t \ge T. (6.28)$$

7 The Smith Chart

The Smith chart is a graphical tool for calculating the characteristics of transmission lines; it is constructed with a unit circle (radius 1), i.e., $|\Gamma| \le 1$. Here, assume that Z_0 is real[1], and recall that $\Gamma = \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$ (7.1) or $\Gamma = \Gamma_L = |\Gamma_L| e^{j\theta_L} = \Gamma_r + j\Gamma_i$ (7.2).

To generalize for use with different Z_0, the chart is "normalized" by Z_0. For the load impedance Z_L, normalized impedance z_L is given by $z_L = \frac{Z_L}{Z_0} = r + jx$ (7.3)

Using (7.3) in (7.1), (7.2) yields

$$\Gamma = \Gamma_L = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1} \qquad (7.4) \;\text{ or }\; z_L = r + jx = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i}. \; (7.5)$$

Since $\frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i} = \frac{\left[(1+\Gamma_r) + j\Gamma_i\right]\left[(1-\Gamma_r) + j\Gamma_i\right]}{(1-\Gamma_r)^2 + \Gamma_i^2}$,

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad (7.6a) \;\text{ and }\; x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad (7.6b)$$

Rearranging terms in (7.6) yields

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \quad (7.7a) \text{ and } (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \qquad (7.7b)$$

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[1] Recall that most practical transmission lines are assumed lossless or slightly lossy, where the characteristic impedance is real.

These represent circles with (7.7a) centered at $(r/(1+r),0)$ with radius $1/(1+r)$ and (7.7b) centered at $(1,1/x)$ with radius $1/x$. Note also that VSWR = $(1+|\Gamma|)/(1-|\Gamma|)$ is also a circle, thus if Γ is known, so is VSWR.

Figure 9 shows some of constant *r* circles and constant *x* circles given by (7.7).

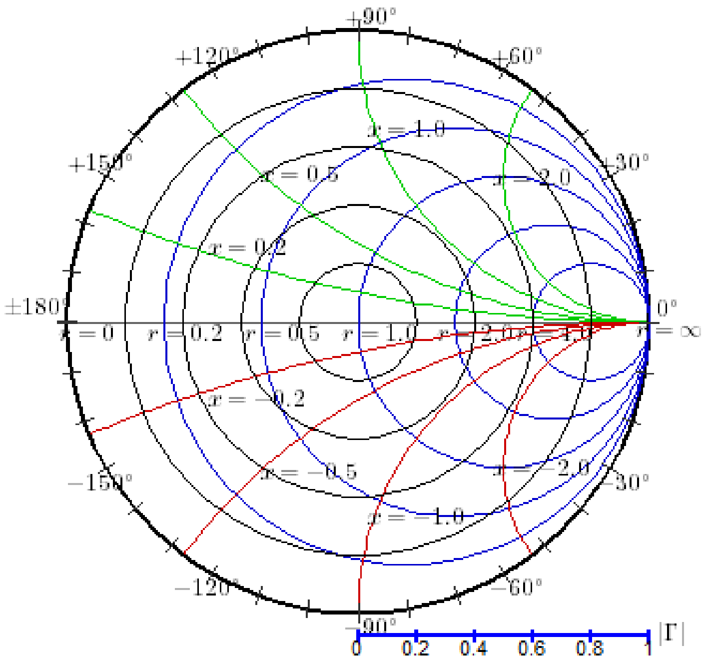


Fig. 9 : Constant *r* circles and constant *x* circles on the Smith chart

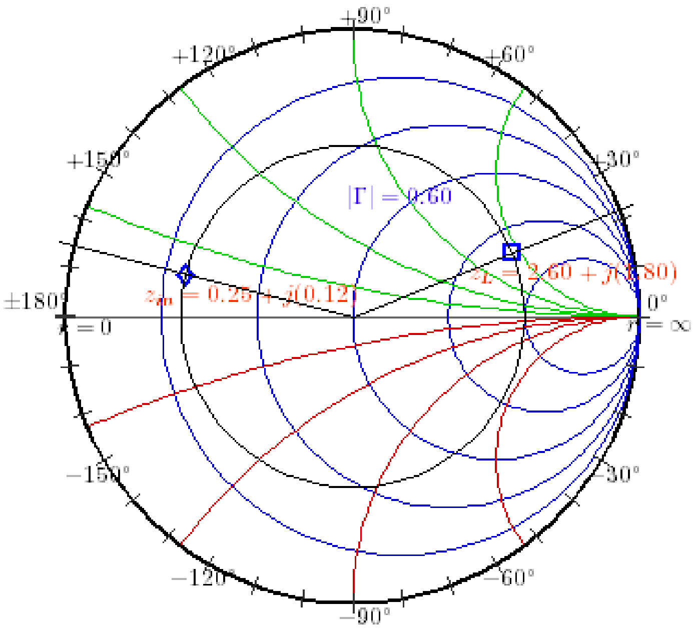
<u>Observations</u>

1. Short circuit ($r = x = 0$) corresponds to $Z_L = 0$ and open circuit ($r = x = \infty$) corresponds to $Z_L = \infty + j\infty$, but short circuit is transformed into open circuit with $\ell = \lambda/4$ (quarter-wavelength transformer), i.e., $Z_{in} = Z_0^2/Z_L$, and vice versa.
2. Complete revolution (2π) around the Smith chart represents a distance of $\lambda/2$ on the line with clockwise denoting "toward the generator (G)" and counterclockwise denoting "toward the load (L)".
3. Three scales around the periphery of the Smith chart : λ toward G, λ toward L, λ in degree.
4. V_{max} at $Z_{in,max}$ when V^+, V^- in phase and V_{min} at $Z_{in,min}$ when V^+, V^- out of phase, which are $\lambda/4$ apart.
5. Can also be used as admittance chart with normalized admittance, $y = g + jb$.

<u>Application of Smith Chart</u>

<u>Example 7.1: Input impedance calculation</u> A load impedance of 130+j90 Ω terminates a 50-Ω transmission line that is .3λ long. Find Γ_L, $\Gamma(z=0)$, Z_{in}, SWR.

<u>Solution</u> The normalized load impedance $z_L = (130+j90)/50 = 2.6+j1.8$. Plot this value on the Smith chart, one can find that $\Gamma_L = 0.60\angle 21.8°$ and SWR = 3.98. Since the line is .3λ long, by moving the distance $.3\lambda \times 4\pi / \lambda = 1.2\pi = 216°$ toward the generator along the $|\Gamma|=0.60$ circle, one can find the input impedance and reflection coefficient at the generator to be : $Z_{in} = Z_0 z_{in} = 50(0.255+j0.117) = 12.7+j5.8$ Ω and $\Gamma(z=0) = \Gamma_{in} = 0.60\angle 165.8°$.



<u>Quiz 1</u> Repeat example 1 for a 100-Ω transmission line.

Example 7.2: Admittance calculation Let y_L denote the normalized load admittance, then y_L = $1/z_L$ and $\frac{y_L-1}{y_L+1}=\frac{1/z_L-1}{1/z_L+1}=\frac{1-z_L}{z_L+1}$. Recall that $\Gamma=\Gamma_L=\frac{z_L-1}{z_L+1}$, it follows that $\frac{y_L-1}{y_L+1}=-\Gamma=\Gamma e^{j\pi}$. Thus, the normalized admittance can be found by rotating the normalized impedance by 180°.

Use the previous example as an example. y_L = 0.26-j0.18, thus Y_L = 5.2-j3.6 mS. Then by moving 1.2π toward the generator, one can find that y_{in} = 3.24-j1.48, thus Y_{in} = 64.8-29.6 mS, which is the reciprocal of 12.7+j5.8 Ω.

Quiz 2 Suppose the input impedance is found to be 25 + *j*20 Ω and the transmission line has Z_0 = 50 Ω and is 0.3625λ long. Find the load impedance.

Example 7.5: Quarter-wavelength transformer Recall that when $\ell = \lambda/4$, $Z_{in}=Z_0^2/Z_L$, and thus on the Smith chart, it becomes $z_{in} = 1/z_L$ or $y_{in} = z_L$. Therefore, the input impedance of the quarter-wavelength transformer can be found by rotating z_L by π. For example, the short circuit is transformed to the open circuit and vice versa.

The quarter-wavelength transformer has one significant application in matching load impedance Z_L to a transmission line with characteristic impedance Z_0. By inserting a quarter-wavelength transformer with characteristic impedance $Z_1=\sqrt{Z_L Z_0}$ between the transmission line and the load, the input impedance becomes $Z_{in} = Z_0$, thus the line is now matched.

The Slotted Line A slotted line is a transmission line configuration (usually waveguide or coax) that allows the sampling of the electric field amplitude of a standing wave on a terminated line. With the device, the VSWR and the distance of the first voltage minimum from the load can be measured, and from this data the load impedance can be determined. Note that the load impedance is generally a complex number, two distinct quantities must be measured to determine the impedance.

Although the slotted line used to be the principal way to measure an unknown impedance at microwave frequencies, it has been superseded by the modern vector network analyzer in terms of accuracy, versatility and convenience. The slotted line is still of some use in certain applications such as high-millimeter wave frequencies or where it is desired to avoid connector mismatches by connecting the unknown load directly to the slotted line, thus avoiding the use of imperfect transitions. Another reason for studying the slotted line is that it provides an excellent tool for learning basic concepts of standing waves and mismatched transmission lines.

Assume that, for a certain terminated line, the VSWR on the line and ℓ_{min}, the distance from the load to the first voltage minimum on the line, are measured. Recall that $|\Gamma|$ = (VSWR - 1)/(VSWR + 1) and a voltage minimum occurs when $e^{j(\theta-2\beta\ell)}=-1$, where θ is the phase angle of the reflection coefficient, $\Gamma=|\Gamma|e^{j\theta}$. The phase angle of the reflection coefficient is then $\theta=\pi+2\beta\ell_{\min}$. Actually, since the voltage minima repeat every $\lambda/2$, any multiple of $\lambda/2$ can be added to ℓ_{min} without changing the reflection coefficient, i.e., $\theta=\pi+2\beta(\ell_{\min}+n\lambda/2)$.

Thus, VSWR and ℓ_{min} can be used to determine the reflection coefficient and the load impedance can be determined from the reflection coefficient.

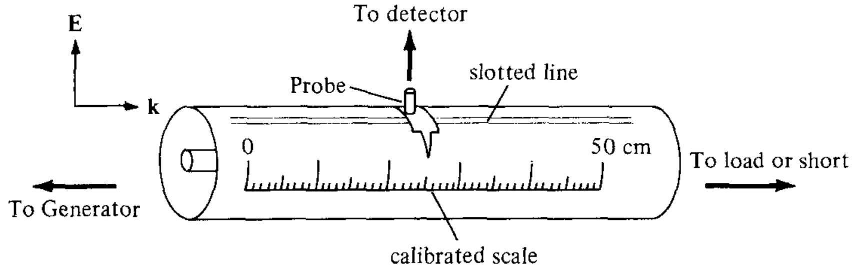
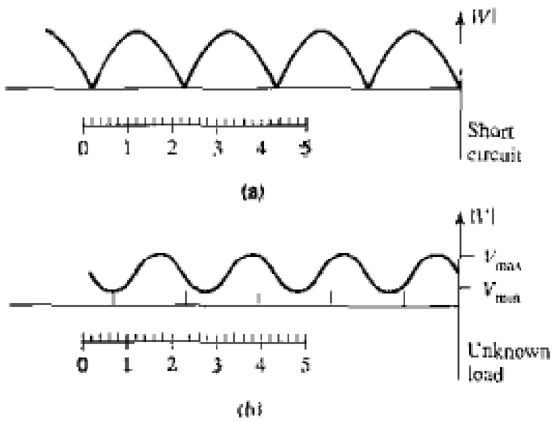


Figure 10: A slotted line

<u>Example 7.6 Impedance measurement with a slotted line</u> The following two step procedure has been carried out with a 50-Ω coaxial slotted line to determine an unknown impedance:

1. A short circuit is placed at the load plane, resulting in a standing wave on the line with infinite VSWR and sharply defined voltage minima as shown in figure (a). Voltage minima are recorded at z = 0.2, 2.2, 4.2 cm.
2. The short circuit is replaced by the unknown load. VSWR=1.5 is measured and voltage minima are recorded at z = 0.72, 2.72, 4.72 cm.



Find the unknown impedance.

<u>Solution</u> Since the voltage minima repeat every $\lambda/2$, λ = 4.0 cm. In addition, because the reflection coefficient and the input impedance also repeat every $\lambda/2$, we can consider the load terminals to be effectively located at any of the voltage minima locations listed in step 1. Thus, if we say the load is at 4.2 cm, then the next voltage minimum away from the load occurs at 2.72 cm, giving ℓ_{min} = 4.2 – 2.72 = 1.48 cm = 0.37λ. Therefore,

$$|\Gamma| = \frac{VSWR-1}{VSWR+1} = 0.2;$$

$$\theta = \pi + 2\beta\ell_{min} = \pi + 2(2\pi)0.37 = 0.48\pi$$

$$\rightarrow \Gamma = 0.2e^{j0.48\pi} = 0.0126 + j0.1996$$

The load impedance is then

$$Z_L = Z_0 \frac{1+\Gamma}{1-\Gamma} = 47.3 + j19.7\,\Omega.$$

To use the Smith chart, first noticing that the voltage minimum is located on the horizontal axis to the left of the origin. Thus, beginning with this point and then moving 0.37λ toward the load, we can get z_L = 0.95 + j0.39 and thus Z_L = 47.5 + j19.5 Ω.