Resonance

1 Introduction

In physics, **resonance** is the tendency of a system (usually a linear system) to oscillate with larger amplitude at some frequencies than at others. These are known as the system's **resonant frequencies**. At these frequencies, even small periodic driving forces can produce large amplitude oscillations.

Resonances occur when a system is able to store and easily transfer energy between two or more different storage modes (such as kinetic energy and potential energy in the case of a pendulum). However, there are some losses from cycle to cycle, called damping. When damping is small, the resonant frequency is approximately equal to a natural frequency of the system, which is a frequency of unforced vibrations. Some systems have multiple, distinct, resonant frequencies.

Example

Mechanical and acoustic resonance

- A pendulum, a playground swing
- The timekeeping mechanisms of all modern clocks and watches: the balance wheel in a mechanical watch and the quartz crystal in a quartz watch
- Acoustic resonances of musical instruments and human vocal cords

Electrical resonance

- Electrical resonance of tuned circuits in radios and TVs that allow individual stations to be picked up

Optical resonance

- Creation of coherent light by optical resonance in a laser cavity

Atomic, particle, and molecular resonance

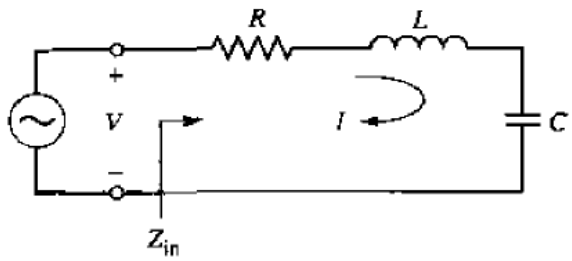
- Material resonances in atomic scale are the basis of several spectroscopic techniques that are used in condensed matter physics.
- Nuclear Magnetic Resonance (NMR), Magnetic Resonance Imaging (MRI)

2 Series RLC Resonant Circuit

In an RLC circuit, the resonance is the state at which the reactance of the inductor, X_L, and the reactance of the capacitor, X_C, are equal. It is used in various applications, including filters, oscillators, frequency meters and tuned amplifiers.

A series RLC resonant circuit is shown in the figure below. The governing equation is given by

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = v(t)$$



Using the voltage across the capacitor $v_C(t)$ as the unknown, the above differential equation can be rewritten as

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t) \text{ or}$$

$$\frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{v_C(t)}{LC} = \frac{d^2v_C(t)}{dt^2} + 2\alpha\frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = \frac{v_S(t)}{LC} = \omega_0^2 v_S(t)$$

where $\alpha = R/2L$; $\omega_0 = 1/\sqrt{LC}$ which are called *damping attenuation* and *natural* (*resonant*) *frequency*.

or $\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{i(t)}{LC} = 0$, which can be rewritten in the following form:

$\frac{d^2i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + \omega_r^2 i(t) = 0$. (Note that it is a second-order differential equation.)

α is called the *attenuation*, and is a measure of how fast the transient response of the circuit will die away after the stimulus has been removed. ω_r is the angular resonance (natural) frequency.
For the case of the series RLC circuit these two parameters are given by:
$\alpha = R/2L$; $\omega_r = 1/\sqrt{LC}$.
The ratio of the two parameters above, which is defined as the *damping factor*, ζ, i.e.,

$$\zeta = \frac{\alpha}{\omega_r} = \frac{R}{2}\sqrt{\frac{C}{L}}.$$

The differential equation above has the characteristic equation:
$s^2 + 2\alpha s + \omega_r^2 = 0$, which has the roots

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_r^2} = -\omega_r\left(\zeta - \sqrt{\zeta^2 - 1}\right);\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_r^2} = -\omega_r\left(\zeta + \sqrt{\zeta^2 - 1}\right).$$

The general solution is given by
$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, where A_1, A_2 are constants determined by enforcing the boundary conditions.
The *overdamped* response ($\zeta > 1$) is given by

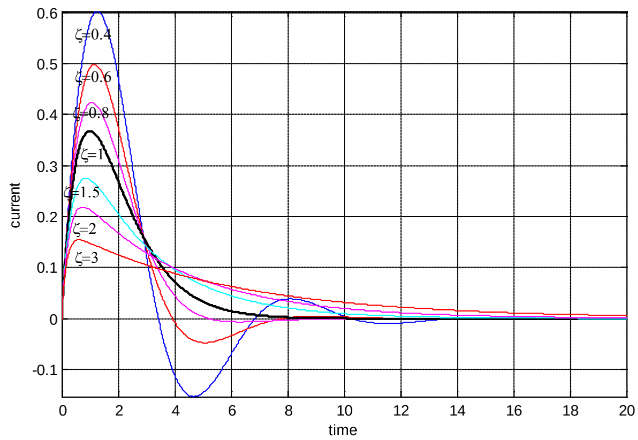
$$i(t) = A_1 \exp\left\{-\omega_r\left(\zeta - \sqrt{\zeta^2 - 1}\right)t\right\} + A_2 \exp\left\{-\omega_r\left(\zeta + \sqrt{\zeta^2 - 1}\right)t\right\},$$

and it is a decay of the transient current without oscillation.
The *underdamped* response ($\zeta < 1$) is given by

$$i(t) = A_1 e^{-\omega_r \zeta t} e^{j\omega_r \sqrt{1-\zeta^2}t} + A_2 e^{-\omega_r \zeta t} e^{-j\omega_r \sqrt{1-\zeta^2}t} = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t,$$

where $\omega_d = \omega_r\sqrt{1-\zeta^2}$, which is called the *damped resonant frequency*.

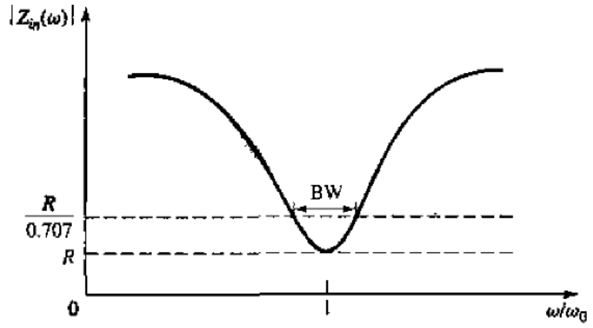
The *critically damped* response ($\zeta = 1$) is given by
$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$,
which represents the fastest decay without oscillation. The figure in the right shows the step response for numerous ζ assuming $L = C = 1$.



For the steady state assuming sinusoidal excitation, the input impedance is given by

$$Z_{in}(\omega) = R + j\omega L + \frac{1}{j\omega C}.$$

At the resonance, the reactance of the inductor and the reactance of the capacitor cancel each other resulting in the input impedance being the minimum, i.e., $Z_{in} = R$ as shown in the figure below.



Three frequency ranges can be defined as follows:
Low frequency range ($f < f_r$)

Resonant frequency ($f = f_r$)

High frequency range ($f > f_r$)

where f_r denotes the resonant frequency, given by $f_r = 1/(2\pi\sqrt{LC})$. Clearly, the current reaches the maximum at this frequency assuming a constant voltage source.
Question Sketch the phase of V_R, V_L, V_C and V_{total} with respective to that of I for all frequency ranges.

Quality factor (Q factor) Q factor is defined as

$$Q = \omega \frac{\text{average energy stored}}{\text{energy loss/second}} = \omega \frac{W_L + W_C}{P_{av}}$$

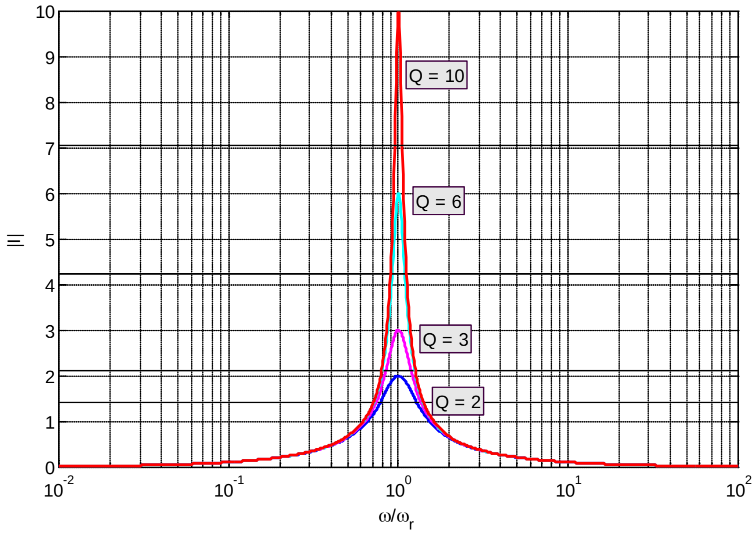
where $P_{av} = |I_R|^2 R$: dissipated power in the resistor, $W_L = |I_L|^2 L$: magnetic energy stored in the inductor, $W_C = |I_C|^2 / \omega^2 C$: electric energy stored in the capacitor (assuming the rms value). Thus, Q factor is a measure of the loss of a resonant circuit—lower loss implies higher Q. At the resonance frequency, the Q factor becomes

$$Q_r = \frac{\omega_r L}{R} = \frac{1}{\omega_r RC}; \omega_r = 2\pi f_r$$

For the low frequency range, $Q =$
For the high frequency range, $Q =$

Resonance response and bandwidth

Typically, bandwidth is defined as the frequency range where the average power delivered to the circuit is greater or equal to one-half that delivered at the resonance. The frequency response of a series resonant circuit is shown below. Here L, C are assumed to be constant, and R is calculated from Q.



It can be observed from the figure that

1. Bandwidth increases with Q decreases.
2. Resonant slope decreases with Q decreases.
3. Magnitude response at f_r decreases with Q decreases.

Now, consider the behavior of the input impedance of a series resonant circuit near its resonant frequency. Let $\omega = \omega_r + \Delta\omega$, where Δω is small.

$$Z_{in}(\omega) = R + j\omega L + \frac{1}{j\omega C} = R + j\omega L\left(1 - \frac{1}{\omega^2 LC}\right) = R + j\omega L \frac{\omega^2 - \omega_r^2}{\omega^2}.$$

Now, $\omega^2 - \omega_r^2 = (\omega - \omega_r)(\omega + \omega_r) = \Delta\omega(2\omega + \Delta\omega) \cong 2\omega\Delta\omega$ for small Δω. Therefore,

$$Z_{in}(\omega) \cong R + j\omega L \frac{2\omega\Delta\omega}{\omega^2} = R + j2L\Delta\omega = R + \frac{j2RQ_r\Delta\omega}{\omega_r}$$

where Q_r is the Q factor at the resonance frequency. Hence, the slope of reactance near the resonance frequency ($|dX/df|$) is $4\pi L$ while the slope in the low frequency range is $1/(2\pi C)$ and that in the high frequency range is $2\pi L$, since both inductor and capacitor affect the behavior near f_r.

Now consider the bandwidth, when the frequency is such that $|Z_{in}|^2=2R^2$ (or $X = R$), the power delivered to the circuit is one-half that at the resonance frequency. If BW = f_u-f_l is the bandwidth, then $\Delta\omega = 2\pi(f_u$-$f_r) = 2\pi$BW/2 at the upper band edge. Thus,

$$Z_{in}(\omega) \cong R + \frac{j2RQ_r\Delta\omega}{\omega_r} = R + jR \text{ or } \frac{2Q_r\Delta\omega}{\omega_r} = 1, \therefore 2\Delta\omega = \frac{\omega_r}{Q_r}$$

Hence, $BW = \dfrac{f_r}{Q_r}$.

<u>Question</u> Use Taylor's series expansion to show that the slope of reactance near the resonance frequency ($|dX/df|$) is $4\pi L$.

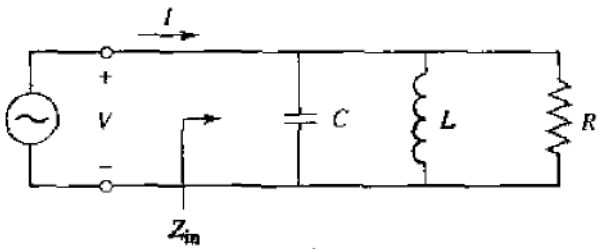
Add unloaded and loaded Q factors

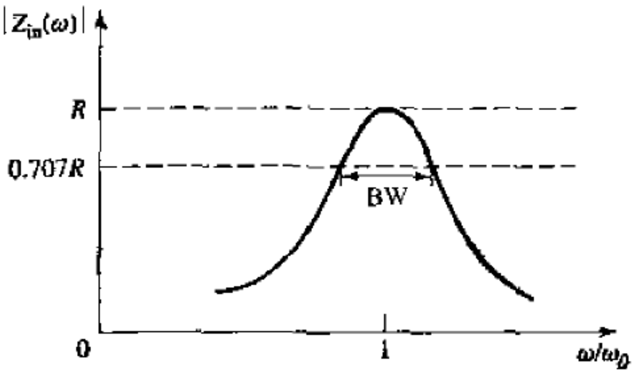
Add Tank circuits

3 Parallel RLC Resonant Circuit (Anti-resonant circuit)

A parallel RLC resonant circuit is shown in the figure below. The input impedance is given by

$$Z_{in}(\omega) = \left(\frac{1}{R} + j\omega C + \frac{1}{j\omega L}\right)^{-1}$$





At the resonance, the reactance of the inductor and the reactance of the capacitor cancel each other resulting in the input impedance being the maximum, i.e., Z_{in} = R as shown in the right figure. The resonant frequency is given by $f_r = \dfrac{1}{2\pi\sqrt{LC}}$ as in the case of series RLC circuits. Clearly, the voltage reaches the maximum at this frequency assuming a constant current source.

<u>Question</u> Sketch the phase of I_R, I_L, I_C and I_{total} with respective to that of V for all frequency ranges.

It is more convenient to use the input admittance:

$$Y_{in}(\omega) = G + j\omega C + \frac{1}{j\omega L},$$

where $G=1/R$, and the duality relationships ($V\leftrightarrow I$, $Z\leftrightarrow Y$, $R\leftrightarrow G$, $L\leftrightarrow C$, $C\leftrightarrow L$). Then, the approach used in the case of series RLC circuits can be applied.

<u>Quality factor (Q factor)</u> Since $P_{av} = |V_R|^2/R$, $W_C = |V_C|^2 C$ and $W_L = |V_L|^2/\omega^2 L$ (assuming the rms value), at the resonance frequency the Q factor becomes

$$Q_r = \omega_r RC = \frac{R}{\omega_r L}; \omega_r = 2\pi f_r$$

For the low frequency range, Q =
For the high frequency range, Q =

Resonance response and bandwidth

It can be shown that a parallel RLC circuit's behavior near the resonance frequency is similar to that of a series RLC circuit. Consider the behavior of the input admittance of a parallel resonant circuit near its resonant frequency. Let $\omega = \omega_r + \Delta\omega$, where Δω is small.

$$Y_{in}(\omega) = G + j\omega C + \frac{1}{j\omega L} = G + j\omega C\left(1 - \frac{1}{\omega^2 LC}\right) = G + j\omega C\frac{\omega^2 - \omega_r^2}{\omega^2}.$$

Now, $\omega^2 - \omega_r^2 = (\omega - \omega_r)(\omega + \omega_r) = \Delta\omega(2\omega + \Delta\omega) \cong 2\omega\Delta\omega$ for small Δω. Therefore,

$$Y_{in}(\omega) \cong G + j\omega C\frac{2\omega\Delta\omega}{\omega^2} = G + j2C\Delta\omega = G + \frac{j2GQ_r\Delta\omega}{\omega_r} = \frac{1}{R} + \frac{j2Q_r\Delta\omega}{\omega_r R}$$

where Q_r is the Q factor at the resonance frequency. Hence, the slope of susceptance near the resonance frequency ($|dB/df|$) is $4\pi C$ while the slope in the high frequency range is $2\pi C$.

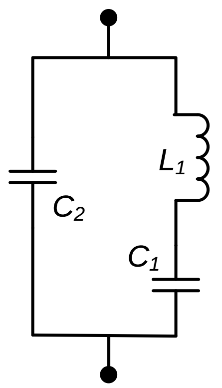
Now consider the bandwidth, when the frequency is such that $|Y_{in}|^2=2G^2$ (or $B = G$), the power delivered to the circuit is one-half that at the resonance frequency. If BW = f_u-f_l is the bandwidth, then $\Delta\omega = 2\pi(f_u - f_r) = 2\pi\,\mathrm{BW}/2$ at the upper band edge. Thus,

$$Y_{in}(\omega) \cong G + \frac{j2GQ_r\Delta\omega}{\omega_r} = G + jG \text{ or } \frac{2Q_r\Delta\omega}{\omega_r} = 1, \therefore 2\Delta\omega = \frac{\omega_r}{Q_r}$$

Hence, $BW = \dfrac{f_r}{Q_r}$.

4 Multiple Resonance

Multiple resonance is desirable when a device operates in various frequency bands. For example, consider the circuit in the right figure.



The impedance of the circuit is given by

$$Z(\omega) = -\frac{j}{\omega C_2}\frac{\omega^2 - 1/L_1C_1}{\omega^2 - \dfrac{C_1 + C_2}{L_1C_1C_2}}$$

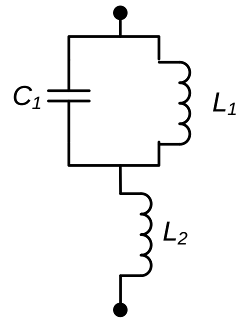
If we plot the reactance, two resonant frequencies can be observed. The frequency at which the reactance is zero is called zero frequency, while the frequency at which the reactance is infinity is called pole frequency. Two resonant frequencies are given by

$$f_{r1} = \frac{1}{2\pi\sqrt{L_1C_1}};\ f_{r2} = \frac{1}{2\pi\sqrt{L_1C_1C_2/(C_1 + C_2)}}$$

Therefore, the impedance can be rewritten as

$$Z(\omega) = -\frac{j}{\omega C_2}\frac{\omega^2 - \omega_{r1}^2}{\omega^2 - \omega_{r2}^2}$$

where f_{r1} is the zero frequency, and f_{r2} is the pole frequency.



Now consider the circuit in the right figure. The admittance of the circuit is given by

$$Y(\omega) = -\frac{j}{\omega L_2}\frac{\omega^2 - 1/L_1C_1}{\omega^2 - \frac{L_1+L_2}{C_1L_1L_2}},$$

thus the impedance can be written in the form:

$$Z(\omega) = j\omega L_2 \frac{\omega^2 - \omega_{r2}^2}{\omega^2 - \omega_{r1}^2}$$

Observe the two forms of impedance equations, one can conclude the general form of a lossless multiple resonant circuit's impedance equation as follows:

$$\boxed{Z(\omega) = xH\frac{\left(\omega^2-\omega_{z1}^2\right)\left(\omega^2-\omega_{z2}^2\right)\cdots\left(\omega^2-\omega_{zn}^2\right)}{\left(\omega^2-\omega_{p1}^2\right)\left(\omega^2-\omega_{p2}^2\right)\cdots\left(\omega^2-\omega_{pn}^2\right)}}$$

where x is either ω or $1/\omega$ and H is a constant.

5 Network Functions

To analyze more complicated networks, it is more convenient to use the Laplace transformation. It follows that the impedance (or admittance) is expressed in "transform" domain (or s domain). For one-port network, the input impedance can be given by the following function:

driving-point impedance function $\quad Z(s) = \dfrac{V(s)}{I(s)}.$

Likewise, the input admittance is given by

driving-point admittance function $\quad Y(s) = \dfrac{I(s)}{V(s)} = \dfrac{1}{Z(s)}.$

For two-port networks, four transfer functions are defined as follows:

Voltage transfer function $\quad G_{21}(s) = \dfrac{V_2(s)}{V_1(s)} \quad$ Current transfer function $\quad \alpha_{21}(s) = \dfrac{I_2(s)}{I_1(s)}$

Transfer admittance function $\quad Y_{21}(s) = \dfrac{I_2(s)}{V_1(s)} \quad$ Transfer impedance function $\quad Z_{21}(s) = \dfrac{V_2(s)}{I_1(s)}$

Note that, in general, $Z_{12} \neq 1/Y_{12}$.

Examples

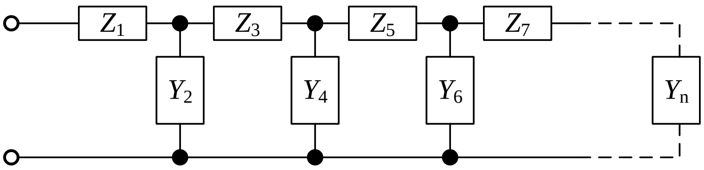
The general form of network functions is as follows:

$$N(s) = \frac{p(s)}{q(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0},$$

which is a rational function of s and m, n are integers.

Example The driving-point impedance function of the ladder network shown below is given by

$$Z = Z_1 + \cfrac{1}{Y_2 + \cfrac{1}{Z_3 + \cfrac{1}{Y_4 + \cfrac{1}{Z_5 + \cfrac{1}{Y_6 + \ddots + \cfrac{1}{Z_{2n-1} + \cfrac{1}{Y_{2n}}}}}}}}$$



Example

Poles and Zeroes

The network function is a ratio of two polynomials given by

$$N(s) = \frac{p(s)}{q(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0} = H\,\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)},$$

where $H = a_n/b_m$ is a constant known as the scale factor, $z_1,\ldots,z_n$, which are roots of $p(s)=0$, are called zeroes of $N(s)$ while $p_1,\ldots,p_m$, which are roots of $q(s)=0$, are called poles of $N(s)$. A dissipationless (lossless) network has only imaginary poles and zeroes. The necessary conditions for driving-point functions are

1. The coefficients in the polynomial $p(s)$ and $q(s)$ must be real and positive.
2. Complex and imaginary poles and zeroes must be conjugate.
3. (a) The real part of all poles and zeroes must not be positive.
   (b) If the real part is zero, then that pole or zero must be simple.
4. The polynomials $p(s)$ and $q(s)$ must not have missing terms between the highest and lowest degree, unless all even or all odd terms are missing.
5. The degree of $p(s)$ and $q(s)$ may differ by either zero or one only.
6. The terms of lowest degree in $p(s)$ and $q(s)$ may differ in degree by one at the most.

Assume that all poles are simple, then the function $N(s)$ can be rewritten as

$$N(s) = \frac{p(s)}{q(s)} = r(s) + \frac{m(s)}{q(s)} = r(s) + \sum_i \frac{k_i}{s-p_i},$$

where $r(s)$ is the quotient of $p(s)$ divided by $q(s)$ ($p(s) = r(s)q(s)+m(s)$) and k_i is the residue at the pole p_i given by

$$k_i = \left.\frac{p(s)}{q'(s)}\right|_{s=p_i} = (s-p_i)\left.\frac{p(s)}{q(s)}\right|_{s=p_i}.$$

If the degree of $q(s)$ is greater than that of $p(s)$, then $r(s) = 0$.

6 Foster's Reactance Theorem

For a positive real rational function $Z(s)=1/Y(s)$ to be realizable as the driving point impedance of a lossless one-port, the necessary and sufficient condition is that it should be expressible in the form

$$Z(s) \text{ or } Y(s) = \frac{a_n\left(s^2+\omega_{z1}^2\right)\left(s^2+\omega_{z2}^2\right)\cdots\left(s^2+\omega_{zn}^2\right)}{sb_m\left(s^2+\omega_{p1}^2\right)\left(s^2+\omega_{p2}^2\right)\cdots\left(s^2+\omega_{pm}^2\right)}$$

where a_n and b_m are constants. It follows that

1. $0 \le \omega_1 < \omega_2 < \omega_3$ (Interlacing poles and zeros, all on $j\omega$ axis)
2. Foster's Theorem further restricts the degrees of the numerator, n, and denominator, m, by requiring that they must differ by unity. In other words, if the numerator is an even degree, the denominator is odd, and vice versa.

From these conditions, the following properties can be deduced:

1. Unity degree difference between numerator and denominator implies that $Z(s)$ must have either a single pole or a single zero at both s=0 and s=∞. Therefore the function $Z(s)$ or $Y(s)$ will belong to one of the four types:
   a. Pole at s=0 and pole at s=∞
   b. Pole at s=0 and zero at s=∞
   c. Zero at s=0 and pole at s=∞
   d. Zero at s=0 and zero at s=∞
2. $Z(j\omega)$ is purely reactive. Therefore, it can be written as

$$Z(j\omega) = jX(\omega),$$

where $X(\omega)$ is the input reactance with

$$X(\omega) \text{ or } Y(\omega) = \frac{a_n\left(\omega^2-\omega_{z1}^2\right)\left(\omega^2-\omega_{z2}^2\right)\cdots\left(\omega^2-\omega_{zn}^2\right)}{\omega b_m\left(\omega^2-\omega_{p1}^2\right)\left(\omega^2-\omega_{p2}^2\right)\cdots\left(\omega^2-\omega_{pn}^2\right)}$$

Alternation of poles and zeros leads to the property

$$0 < \frac{|X(\omega)|}{\omega} < \frac{d}{d\omega}X(\omega)$$

In other words, the reactance $X(\omega)$ is always an increasing function of frequency. The rational functions satisfying these requirements are called Foster functions.

3. Since all poles of $Z(s)$ and $Y(s)$ are on the $s=j\omega$ axis, they can always be expanded as

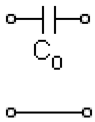
$$Z(s) = \frac{k_0}{s} + k_\infty s + 2\sum_i \frac{k_i s}{s^2+\omega_i^2} \text{ (Foster's I form)}$$

and

$$Y(s) = \frac{h_0}{s} + h_\infty s + 2\sum_i \frac{h_i s}{s^2+\omega_i^2} \text{ (Foster's II form)}$$

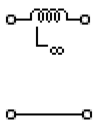
where the constants "k" and "h" are residues of the respective poles. Physically, they correspond to simple network elements, as follows:

- If $Z(s)$ has a pole at s=0, it can be extracted as a series capacitor:



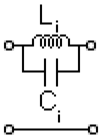
$$C_0 = \frac{1}{k_0} = \left.\frac{1}{sZ(s)}\right|_{s=0}$$

- If $Z(s)$ has a pole at s=∞, it can be extracted as a series inductor:



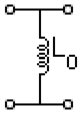
$$L_\infty = k_\infty = \left.\frac{Z(s)}{s}\right|_{s=\infty}$$

- If $Z(s)$ has a pole at $s=j\omega_i$, it can be extracted as a parallel resonator in series:



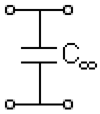
$$C_i = \frac{1}{2k_i} = \left.\frac{s}{\left(s^2 + \omega_i^2\right)Z(s)}\right|_{s=j\omega_i} ; L_i = \frac{1}{\omega_i^2 C_i}$$

- If $Y(s)$ has a pole at $s=0$, it can be extracted as a shunt inductor:



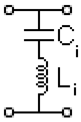
$$L_0 = \frac{1}{h_0} = \left.\frac{1}{sY(s)}\right|_{s=0}$$

- If $Y(s)$ has a pole at $s=\infty$, it can be extracted as a shunt capacitor:



$$C_\infty = h_\infty = \left.\frac{Y(s)}{s}\right|_{s=\infty}$$

- If $Y(s)$ has a pole at $s=j\omega_i$, it can be extracted as a series resonator to ground:



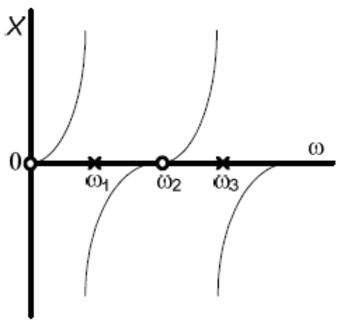
$$L_i = \frac{1}{2h_i} = \left.\frac{s}{\left(s^2 + \omega_i^2\right)Y(s)}\right|_{s=j\omega_i} ; C_i = \frac{1}{\omega_i^2 L_i}$$

Note that if a pole of $Z(s)$ at $s=0$ or $s=\infty$ is extracted, a zero appears at that frequency automatically in the remaining impedance function, which acts as a pole of the remaining admittance function. Hence, given $Z(s)$, one can synthesize a variety of circuits all having the same input impedance but with different structures by extracting elements in different orders from impedance or admittance functions. The resonance components shown above are called the canonical forms, which are used for synthesizing Foster's networks.

Since there are four possible reactance responses (given by a-d of 1) and each response can be synthesized by both series and parallel networks, hence 8 Foster's networks are valid.

<u>Example</u> Synthesize the network with the reactance response shown below.

$$Z(s) = \frac{sH\left(s^2 + \omega_2^2\right)}{\left(s^2 + \omega_1^2\right)\left(s^2 + \omega_3^2\right)} \left(Z(\omega) = \frac{-j\omega H\left(\omega^2 - \omega_2^2\right)}{\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_3^2\right)} \right)$$

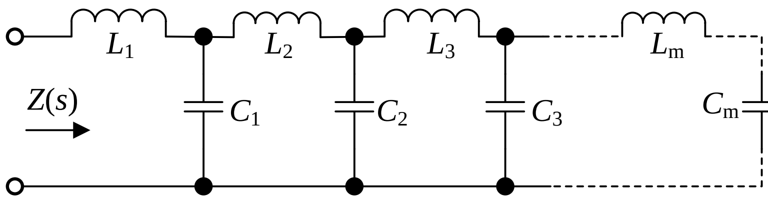


7 Cauer or Ladder Form

Recall the ladder network previously shown, using the previous result with $Z_k=a_ks$ and $Y_k=b_ks$, the driving-point impedance function is then given by

$$Z(s)=a_1s+\cfrac{1}{b_2s+\cfrac{1}{a_3s+\cfrac{1}{b_4s+\cfrac{1}{a_5s+\cfrac{1}{b_6s+\ddots}}}}}$$
$$+\cfrac{1}{a_{2n-1}s+\cfrac{1}{b_{2n}s}}\ .$$

As $s=j\omega$, obviously Z_k represents the reactance of a coil of inductance a_k [H], and Y_k the susceptance of a capacitor of capacitance b_k [F]. If we assign $L_k=a_k$ and $C_k=b_k$, we get the ladder network shown below, which is the realization of the Cauer first form for $Z(s)$.



A study of this network reveals that

a) $Z(s)$ possesses poles both at the origin and infinity.
b) $Z(s)$ will have a zero at the origin when C_m (the last element) is short-circuited.
c) $Z(s)$ will have a zero at infinity when L_1 (the first element) is short-circuited.
d) The impedance function will have zeroes both at the origin and at infinity when both L_1 and C_m (the end elements) are eliminated by short-circuiting.

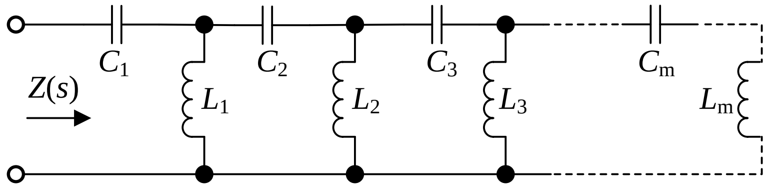
An alternative Cauer representation (Cauer second form) is obtained by writing the polynomials in $Z(s)$ in ascending powers of s:

$$Z(s)=\frac{a_0+a_2s^2+\cdots+a_{2m-2}s^{2m-2}+a_ms^{2m}}{b_1s+b_3s^3+\cdots+b_{2m-3}s^{2m-3}+b_{2m-1}s^{2m-1}}\ .$$

Carrying out the process of division and inversion, we can write

$$Z(s)=\frac{1}{C_1s}+\cfrac{1}{\frac{1}{L_1s}+\cfrac{1}{\frac{1}{C_2s}+\cfrac{1}{\frac{1}{L_2s}+\cfrac{1}{\frac{1}{C_3s}+\cfrac{1}{\frac{1}{L_3s}+\ddots}}}}}$$
$$+\cfrac{1}{\frac{1}{C_ms}+\cfrac{1}{\cfrac{1}{L_ms}}}\ .$$

The network corresponding to this continued fraction is shown in the figure below. This form is called the Cauer second form of the driving-point impedance.



This network exhibits poles both at the origin and at infinity. It will have a zero at the origin when C_1 is short-circuited. Zero of $Z(s)$ at infinity requires that L_m be deleted. Zeroes of $Z(s)$ at both the origin and at infinity will be obtained when both C_1 and L_m are removed by short-circuited.

Example Determine the Foster and Cauer forms of realization of the given driving-point impedance function

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)}.$$