Filters

0 Overview

A filter is a two-port device used to control the frequency response at a certain point in a system by providing transmission at frequencies within the *passband* of the filter and attenuation in its *stopband*. It can be classified by magnitude response as low-pass filter (LPF), high-pass filter (HPF), band-pass filter (BPF), and band-stop filter (BSF). It has wide range of applications including:

- Desired frequency band selection and unwanted band rejection (i.e., SNR improvement and Interference reduction)
- Noise reduction
- Channel selection in mobile and satellite communications

1 Image Impedance

In a two-port network, if two impedances Z_{1i} and Z_{2i} are such that Z_{1i} is the driving point impedance at port 1 with impedance Z_{2i} is connected across port 2 and Z_{2i} is the driving point impedance at port 2 with impedance Z_{1i} is connected across port 1, then the impedances Z_{1i} and Z_{2i} are called *the image impedances* of the network. For symmetrical network, image impedances are equal to each other, i.e., $Z_{1i} = Z_{2i}$, and is called the characteristic or iterative impedance Z_0.

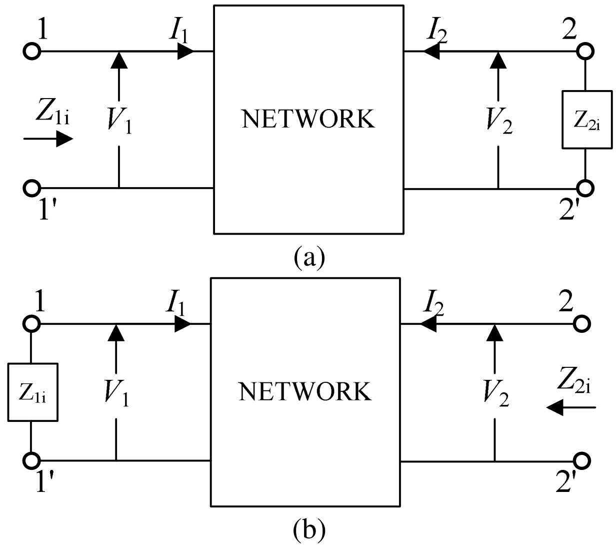


Fig. 1: Image impedance

Consider networks shown in Fig. 1, the driving point impedance at port 1 is given by

$$Z_{1i} = \frac{V_1}{I_1} = \frac{AZ_{2i} + B}{CZ_{2i} + D}.$$

Likewise, the driving point impedance at port 2 is given by

$$Z_{2i} = \frac{V_2}{I_2} = \frac{DZ_{1i} + B}{CZ_{1i} + A}.$$

Here, *ABCD* denote the transmission parameters. Solving the two equations above yields

$$Z_{1i} = \sqrt{\frac{AB}{CD}};\ Z_{2i} = \sqrt{\frac{BD}{AC}}.$$

Since the open-circuit input impedance Z_{ioc} and the short–circuit input impedance Z_{isc} are given by

$$Z_{ioc} = \frac{A}{C};\ Z_{isc} = \frac{B}{D},$$

the image impedance at port 1 can be rewritten as

$$Z_{1i} = \sqrt{Z_{ioc} Z_{isc}}.$$

Likewise, since the open-circuit output impedance Z_{ooc} and the short–circuit input impedance Z_{osc} are given by

$$Z_{ooc} = \frac{D}{C};\ Z_{osc} = \frac{B}{A},$$

the image impedance at port 2 can be rewritten as

$$Z_{2i} = \sqrt{Z_{ooc} Z_{osc}}.$$

2 Symmetric T and π networks

Consider a T network interposed between a generator with internal impedance Z_{1i} and a load impedance of Z_{2i}, as shown in Fig. 2. It is desired that the maximum power transfer occurs, i.e., the impedance at 1,1' terminals into which the generator supplies power be equal to Z_{1i}, and the impedance at 2,2' terminals be equal to Z_{2i}. Hence,

$$Z_{1in} = Z_1 + \frac{Z_3(Z_2 + Z_{2i})}{Z_2 + Z_3 + Z_{2i}} = Z_{1i} \quad \text{and}$$

$$Z_2 + \frac{Z_3(Z_1 + Z_{1i})}{Z_1 + Z_3 + Z_{1i}} = Z_{2i}.$$

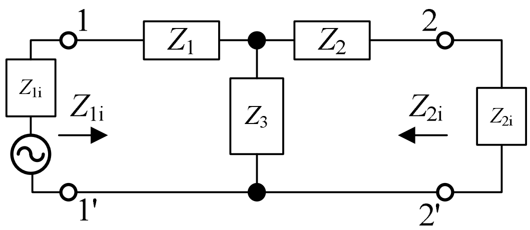


Fig. 2: A T-network interposed between load and source

Solving both equations yields

$$Z_{1i} = \sqrt{\frac{(Z_1 + Z_3)(Z_1Z_2 + Z_2Z_3 + Z_1Z_3)}{Z_2 + Z_3}};$$

$$Z_{2i} = \sqrt{\frac{(Z_2 + Z_3)(Z_1Z_2 + Z_2Z_3 + Z_1Z_3)}{Z_1 + Z_3}}.$$

Note also that

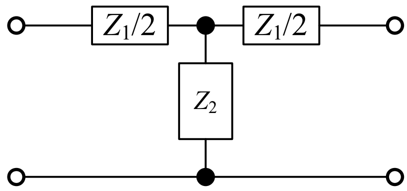
$Z_{1i} = \sqrt{Z_{1oc}Z_{1sc}}$; $Z_{2i} = \sqrt{Z_{2oc}Z_{2sc}}$, as before.

When Z_1=Z_2, i.e., two series arms of a T-network are equal, the network is said to be symmetric. For symmetric networks, $Z_{1i} = Z_{2i} = Z_0$ (characteristic impedance). Filter networks are usually set up as symmetrical sections of T or π types (Fig. 3(a), Fig. 4(a)). T section can be considered as built up of unsymmetrical L-half sections (Fig. 3(b)). For the T network shown in Fig. 3(a) terminated by its characteristic impedance Z_0, the input impedance is given by

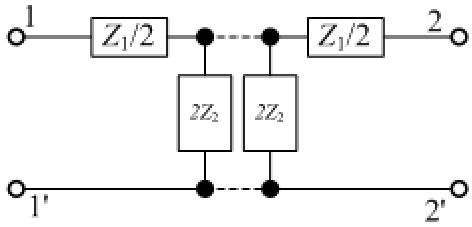
$$Z_{1in} = \frac{Z_1}{2} + \frac{Z_2(Z_1/2 + Z_0)}{Z_1/2 + Z_2 + Z_0}.$$

With proper choice of Z_0, it is possible to make Z_{1in}=Z_0,

$$Z_{1in} = Z_0 = \frac{Z_1}{2} + \frac{Z_2(Z_1/2 + Z_0)}{Z_1/2 + Z_2 + Z_0} = \sqrt{Z_1^2/4 + Z_1Z_2}.$$



(a) T-section



(b) two L sections

Fig. 3: A symmetrical T-section

Hence, for symmetrical T-section, Z_0 is given by

$$Z_{0T} = \sqrt{Z_1^2/4 + Z_1Z_2} = \sqrt{Z_1Z_2(1 + Z_1/4Z_2)}.$$

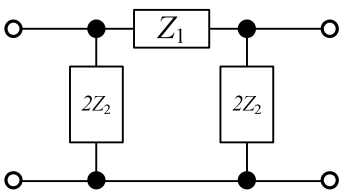
Again, from open and short-circuit measurements for the symmetrical T section,

$$Z_{1oc} = Z_{2oc} = Z_{oc} = Z_1/2 + Z_2;\ Z_{1sc} = Z_{2sc} = Z_{sc} = Z_1/2 + \frac{Z_1Z_2/2}{Z_1/2 + Z_2} \text{ and } Z_{1oc}Z_{1sc} = Z_1^2/4 + Z_1Z_2 = Z_{0T}^2$$

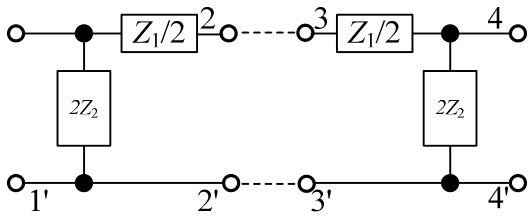
Thus, $Z_{0T} = \sqrt{Z_{oc}Z_{sc}}$.

Likewise, π section can be considered as built up of unsymmetrical L-half sections (Fig. 4(b)). For the π network shown in Fig. 4(a) terminated by its characteristic impedance Z_0, the input impedance is given by

$$Z_{1in} = \frac{\left(Z_1 + \dfrac{2Z_2Z_0}{2Z_2 + Z_0}\right)2Z_2}{Z_1 + \dfrac{2Z_2Z_0}{2Z_2 + Z_0} + 2Z_2}.$$



(a) π -section



(b) two L sections

Fig. 4: A symmetrical π -section

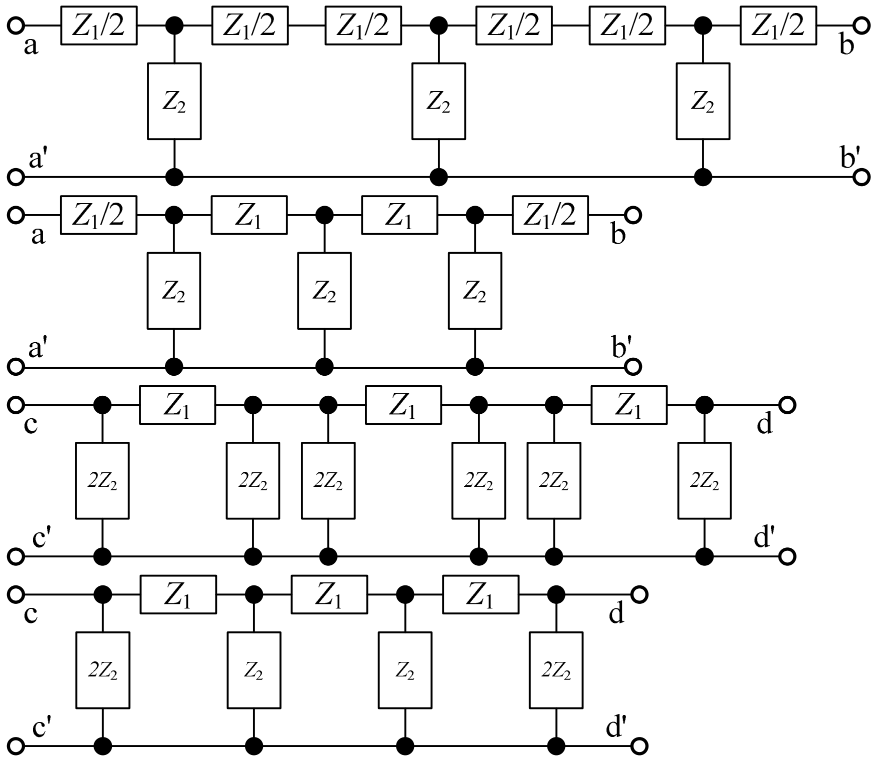
Requiring Z_{1in}=Z_0 leads to

$$Z_{0\pi} = \sqrt{\frac{Z_1Z_2}{1 + Z_1/4Z_2}} = \frac{Z_1Z_2}{Z_{0T}}.$$

It can also be shown that

$$Z_{0\pi} = \sqrt{Z_{oc}Z_{sc}}.$$

A series connection of several T or π networks leads to so-called ladder networks, as shown in Fig. 5 (a)-(d). Terminal half-section matching is obtained by connecting the ends of the T-network with the half sections of the π-network (Fig. 4 (b)), i.e., connect terminals 2,2’ of Fig. 4(b) with terminals a,a’ of Fig. 5(a) and 3,3’ with b,b’. Similarly, for the π-network of Fig. 5(c), terminal matching is to be done by the half-sections of the T-network (Fig. 3(b)), i.e., connecting terminals 2,2’ to c,c’ and 1,1’ to d,d’.



(a) Ladder networks of T sections

(b) Equivalent network of (a)

(c) Ladder networks of π sections

(d) Equivalent network of (c)

Fig. 5: Ladder networks made of T-sections and π-sections.

3 Propagation Constant

Under Z_0 termination, input and output impedances are equal, i.e.,

$$Z_0 = V_1 / I_1 = V_2 /(-I_2),$$

then $V_1 / V_2 = I_1 /(-I_2) = e^{\gamma}$,

where γ is a complex number and is defined as

$$\gamma = \alpha + j\beta,$$

where γ, α, β are propagation constant, attenuation constant, and phase constant, respectively. Furthermore,

$$V_1 / V_2 = I_1 /(-I_2) = A\angle\beta = \left| I_1 / I_2 \right| e^{j\beta} = e^{\alpha + j\beta}.$$

For n number of sections cascaded, with all of them having the same Z_0 value, the ratio of currents can be written as

$$\frac{I_1}{-I_2} \times \frac{-I_2}{-I_3} \times \cdots \times \frac{-I_{n-1}}{-I_n} = \frac{I_2}{-I_n} = e^{\gamma_1} \times e^{\gamma_2} \times \cdots \times e^{\gamma_n} = e^{\gamma}.$$

The overall propagation constant γ can be expressed as

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

4 Properties of Symmetrical Network

For a symmetrical T-section terminated with a load Z_0 and fed with a generator E_0, as shown in Fig. 6,

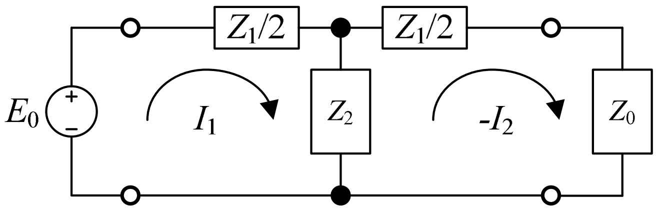


Fig. 6 Symmetrical network terminated by Z_0

$E_0 = (Z_1/2 + Z_2)I_1 + Z_2 I_2$

$0 = Z_2 I_1 + (Z_1/2 + Z_2 + Z_0)I_2$.

Thus, $\dfrac{I_1}{-I_2} = \dfrac{Z_1/2 + Z_2 + Z_0}{Z_2} = e^{\gamma}$, or

$Z_0 = Z_2(e^{\gamma} - 1) - Z_1/2$.

Applying the previous result $Z_0 = \sqrt{Z_1^2/4 + Z_1 Z_2} = \sqrt{Z_1 Z_2(1 + Z_1/4Z_2)}$ yields

$$Z_0^2 = \left[Z_2(e^{\gamma} - 1) - Z_1/2\right]^2 = Z_2^2(e^{\gamma} - 1)^2 - Z_1 Z_2(e^{\gamma} - 1) + Z_1^2/4 = Z_1^2/4 + Z_1 Z_2.$$

After simplification,

$Z_2(e^{\gamma} - 1)^2 - Z_1 e^{\gamma} = 0$ or $e^{2\gamma} - 2e^{\gamma} + 1 = (Z_1/Z_2)e^{\gamma}$. Hence,

$$\frac{e^{\gamma} + e^{-\gamma}}{2} = \cosh\gamma = 1 + Z_1/2Z_2.$$

Since, $\cosh^2\gamma - \sinh^2\gamma = 1$,

$$\sinh^2\gamma = \cosh^2\gamma - 1 = (1 + Z_1/2Z_2)^2 - 1 = Z_1^2/4Z_2^2 + Z_1/Z_2 = Z_0^2/Z_2^2 \rightarrow \sinh\gamma = Z_0/Z_2,$$

and $\tanh\gamma = \dfrac{Z_0}{Z_1/2 + Z_2}$.

Using the half-angle identity,

$$\sinh(\frac{\gamma}{2}) = \sqrt{\frac{1}{2}\left[\cosh(\gamma) - 1\right]} = \sqrt{\frac{1}{2}\left(1 + \frac{Z_1}{2Z_2} - 1\right)} = \sqrt{\frac{Z_1}{4Z_2}}.$$

Again, $\dfrac{I_1}{-I_2} = e^{\gamma} = \dfrac{Z_1/2 + Z_2 + Z_0}{Z_2} = 1 + \dfrac{Z_1}{2Z_2} + \sqrt{\left(\dfrac{Z_1}{2Z_2}\right)^2 + \dfrac{Z_1}{Z_2}}$,

So, $\gamma = \ln\left[1 + \dfrac{Z_1}{2Z_2} + \sqrt{\left(\dfrac{Z_1}{2Z_2}\right)^2 + \dfrac{Z_1}{Z_2}}\right]$.

5 Filter Fundamentals

The purpose of a filter network is to pass a desired frequency band without loss and stop or completely attenuate all undesired frequency bands. Since $\gamma = \alpha + j\beta$, $\alpha = 0$ means there is no attenuation in transmission with only a phase shift, i.e., $|I_1| = |I_2|$ and the operation is in the pass band. If $\alpha > 0$, then $|I_1| > |I_2|$, i.e., the attenuation occurs and the operation is in the stop band.

Recall that

$$\sinh(\frac{\gamma}{2}) = \sqrt{\frac{Z_1}{4Z_2}} = \sinh\left(\frac{\alpha + j\beta}{2}\right) = \sinh\frac{\alpha}{2}\cos\frac{\beta}{2} + j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

<u>Case I</u> When Z_1 and Z_2 are of the same type of reactances, then $Z_1/4\,Z_2 > 0$ and $\sinh(\gamma/2)$ is real, i.e.,

(i) $\cosh(\alpha/2)\sin(\beta/2) = 0$ or $\sin(\beta/2) = 0; \beta = n\pi, n = 0,2,4,\ldots$

(ii) $\sinh(\alpha/2)\cos(\beta/2) = \sqrt{Z_1/4Z_2}$.

Therefore, $\cos(\beta/2) = 1$ as $\sin(\beta/2) = 0$. Hence,

$$\sinh(\alpha/2) = \sqrt{Z_1/4Z_2}\,;\ \alpha = 2\sinh^{-1}\sqrt{Z_1/4Z_2}\ .$$

<u>Case II</u> If Z_1 and Z_2 are of the opposite type of reactances, then $Z_1/4\,Z_2$ is negative, i.e., $Z_1/4\,Z_2 < 0$ and obviously, $\sqrt{Z_1/4Z_2}$ is imaginary. Therefore, the following conditions must be satisfied:

(i) $j\cosh(\alpha/2)\sin(\beta/2) = \sqrt{Z_1/4Z_2}$

(ii) $\sinh(\alpha/2)\cos(\beta/2) = 0$

Two conditions may arise

(a) $\sinh(\alpha/2) = 0$, i.e., $\alpha = 0$ when $\beta \neq 0$ and

$$j\sin(\beta/2) = \sqrt{Z_1/4Z_2}\ \because \cosh(\alpha/2) = 1.$$

This signifies the region of zero attenuation or pass band which is limited by the upper limit of the sine term, i.e., $\sin(\beta/2) = |1|$, or it is required that

$$-1 < Z_1/4Z_2 < 0.$$

The phase angle in the pass band is given by

$$\beta = 2\sin^{-1}\sqrt{-Z_1/4Z_2}\ .$$

(b) $\cos(\beta/2) = 0$; therefore $\sin(\beta/2) = \pm 1$; $\beta = (2n\text{-}1)\pi$ when $\alpha \neq 0$ and

$$j\cosh(\alpha/2) = \sqrt{Z_1/4Z_2} \rightarrow \cosh(\alpha/2) = \sqrt{-Z_1/4Z_2}\,;\ \alpha = 2\cosh^{-1}\sqrt{-Z_1/4Z_2}\ .$$

Since hyperbolic cosine has no value below 1, the condition for stop band is $Z_1/4\,Z_2 < -1$. The frequencies at which the network changes from pass band to stop band and vice versa are called the cut-off frequencies. These frequencies occur when

$Z_1/4Z_2 = 0, \text{or } Z_1 = 0$ and $Z_1/4Z_2 = -1, \text{or } Z_1 = -4Z_2$,

where Z_1 and Z_2 are of the opposite type of reactances.

For symmetrical T- and π-network made up entirely of pure reactances, Z_0 is given by

$$Z_{0T} = \sqrt{-X_1X_2(1 + X_1/4X_2)}\,;\ Z_{0\pi} = -X_1X_2/Z_{0T}\,.$$

Table 1 summarizes the two bands, namely the pass band and the stop band with respect to the different values of $X_1/4X_2$.

Table 1

$X_1/4X_2$	0 to -1	-1 to -∞
Band	Pass	Stop
α	0	$2\cosh^{-1}\sqrt{X_1/4X_2}$
β	$2\sin^{-1}\sqrt{X_1/4X_2}$	π
Z_{0T}	positively real	purely reactive

In a pass band, Z_0 is real and positive. If the network is terminated with a resistive $Z_0 = R_0$, then the input impedance is R_0 and the network will accept and transmit power to the resistive load without loss. If the network is fed by a generator having an internal impedance R_0, then the system will be matched and the maximum power transfer occurs. In a stop band, Z_0 is reactive. If the network is terminated in its reactive Z_0, it may transmit voltage or current with 90° phase difference between input and output with considerable attenuation.

6 The constant-*k* Filters

In constant-*k* filters, Z_1 and Z_2 are of opposite reactances. Then

$$Z_1Z_2 = k^2,$$

where *k* is a constant.

<u>6.1 Low-Pass Filters</u>

For low-pass filters, $Z_1 = j\omega L$, $Z_2 = 1/j\omega C$, then

$$Z_1 Z_2 = L/C = R_k^2 = k^2 .$$

The cut-off frequency can be found from

$$-\frac{Z_1}{4Z_2} = -\frac{j\omega L}{4} j\omega C = \frac{\omega^2 LC}{4} = 1 \rightarrow \omega_c = \frac{2}{\sqrt{LC}} .$$

Fig. 7 shows the low-pass T-section filter.

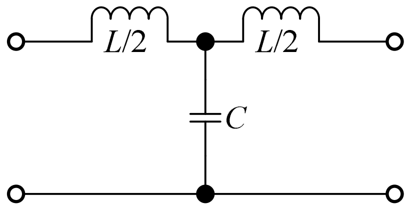


Fig. 7 Low-pass T-section

The characteristic impedance of the T-section and π-section are given by

$$Z_{0T} = \sqrt{(L/C)(1-\omega^2 LC/4)} = R_k\sqrt{1-(\omega/\omega_c)^2} = R_k\sqrt{1-(f/f_c)^2} \text{ and}$$

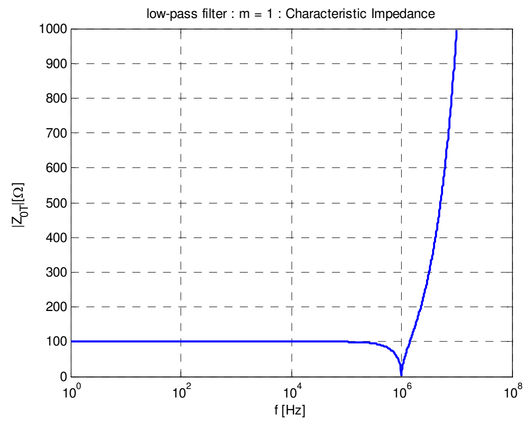
$$Z_{0\pi} = R_k/\sqrt{1-(f/f_c)^2} .$$

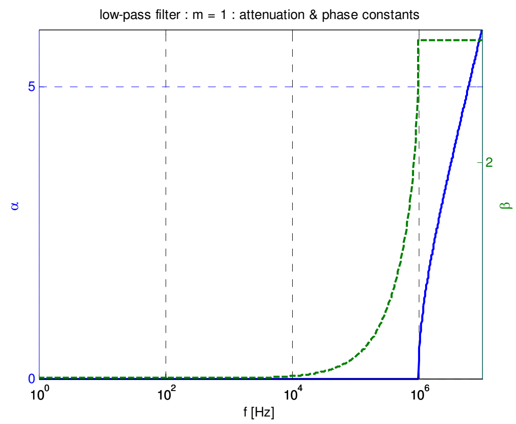
<u>Design Procedure</u> To determine the values of L, C, the value of R_k, i.e., the characteristic impedance at zero frequency, and the cut-off frequency are required. Then, from

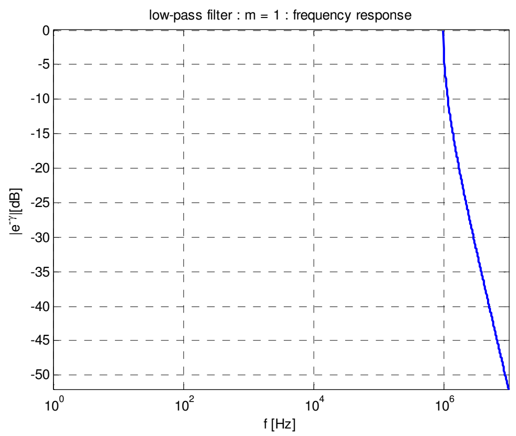
$$L/C = R_k^2 \text{ and } \sqrt{LC} = 1/\pi f_c ,$$

L, C can be calculated.

<u>Low-pass Filter Example</u> Design a low-pass filter with cut-off frequency of 1 MHz, and the characteristic impedance of 100 Ω.







6.2 High-Pass Filters

For high-pass filters, $Z_1 = 1/j\omega C$, $Z_2 = j\omega L$, then

$$Z_1 Z_2 = L/C = R_k^2 = k^2 .$$

The cut-off frequency can be found from

$$-\frac{Z_1}{4Z_2} = -\frac{1}{j\omega L}\frac{1}{4j\omega C} = \frac{1}{4\omega^2 LC} = 1 \rightarrow \omega_c = \frac{1}{2\sqrt{LC}} .$$

Fig. 8 shows the high-pass T-section filter.

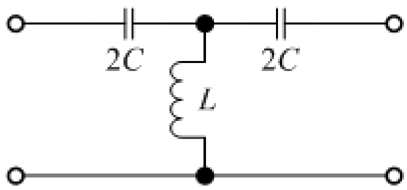


Fig. 8 High-pass T-section

The characteristic impedance of the T-section and π-section are given by

$$Z_{0T} = \sqrt{(L/C)(1-1/4\omega^2 LC)} = R_k\sqrt{1-(\omega_c/\omega)^2} = R_k\sqrt{1-(f_c/f)^2} \text{ and}$$

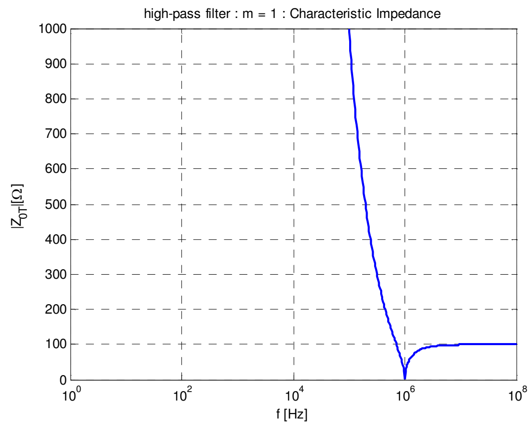
$$Z_{0\pi} = R_k/\sqrt{1-(f_c/f)^2}\ .$$

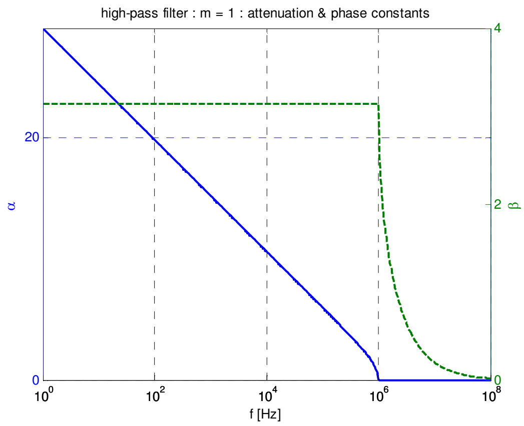
Design Procedure To determine the values of L, C, the value of R_k, i.e., the characteristic impedance at infinite frequency, and the cut-off frequency are required. Then, from

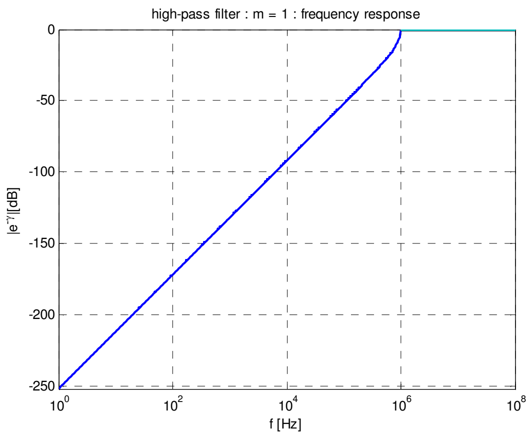
$$L/C = R_k^2 \text{ and } \sqrt{LC} = 1/4\pi f_c,$$

L, C can be calculated.

High-pass Filter Example Design a high-pass filter with cut-off frequency of 1 MHz, and the characteristic impedance of 100 Ω.







6.3 Band-Pass Filters

For band-pass filters, Z_1 is a series LC circuit, i.e., $Z_1 = j(\omega L_1 - 1/\omega C_1)$, and Z_2 is a parallel LC circuit, i.e., $Z_2 = j\omega L_2 \,//\, 1/j\omega C_2$, as shown in Fig. 9. The condition for the band-pass filter is that both series and parallel LC circuits have equal resonant frequencies, i.e.,

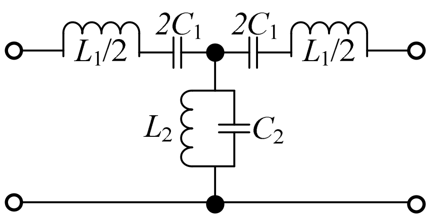


Fig. 9 Band-pass T-section

$\omega_0^2 L_1 C_1 = 1 = \omega_0^2 L_2 C_2$ or $L_1 C_1 = L_2 C_2$. Then,

$$Z_1 Z_2 = \frac{L_2(1-\omega^2 L_1 C_1)}{C_1(1-\omega^2 L_2 C_2)} = \frac{L_2}{C_1} = R_k^2 = k^2.$$

The cut-off frequency can be found from

$$Z_1 = -4Z_2 \rightarrow Z_1^2 = -4Z_1 Z_2 = -4R_k^2 \rightarrow Z_1 = \pm j2R_k.$$

Hence, Z_1 at lower cut-off frequency f_L is equal to $-Z_1$ at upper cut-off frequency f_H, i.e.,

$$1/\omega_L C_1 - \omega_L L_1 = \omega_H L_1 - 1/\omega_H C_1, \text{ or } 1-\omega_L^2 L_1 C_1 = (\omega_L/\omega_H)(\omega_H^2 L_1 C_1 - 1).$$

Using $\omega_0^2 = 1/L_1 C_1$ yields

$$1-\omega_L^2/\omega_0^2 = (\omega_L/\omega_H)(\omega_H^2/\omega_0^2 - 1) \text{ or } \omega_0 = \sqrt{\omega_L \omega_H}\ .$$

Also, from

$$Z_1 = \pm j2R_k \rightarrow Z_1\big|_{\omega=\omega_H} - Z_1\big|_{\omega=\omega_L} = 4jR_k \text{, or } \omega_H L_1 - 1/\omega_H C_1 - (\omega_L L_1 - 1/\omega_L C_1) = 4R_k \text{,}$$

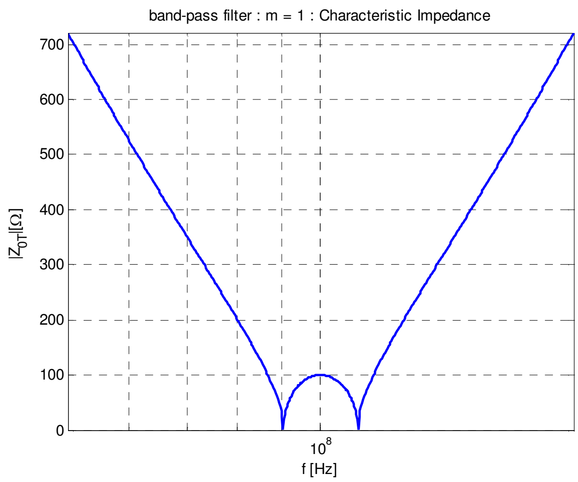
one can derive the condition

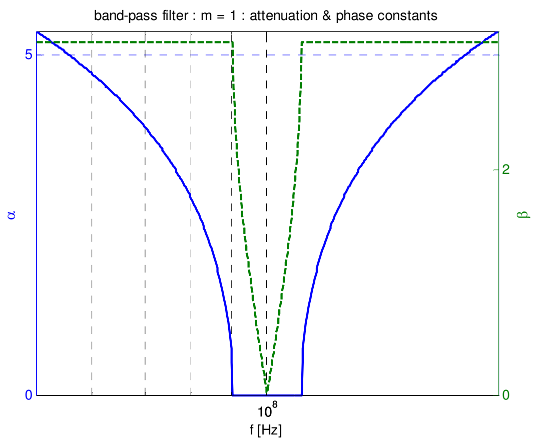
$$\omega_H - \omega_L = 2R_k C_1 \omega_0^2 \text{.}$$

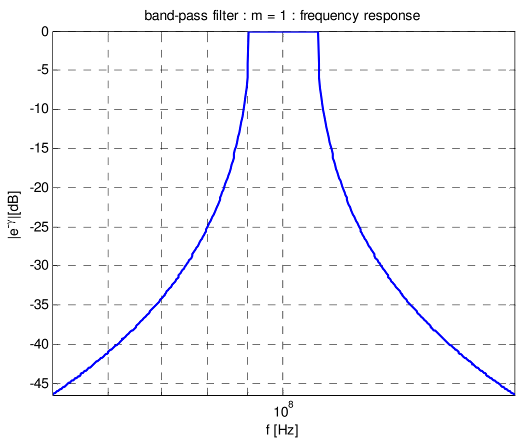
Design Procedure To determine the values of L_1, C_1, L_2, and C_2, one needs to specify the center frequency, the bandwidth and the desired characteristic impedance, then using the following procedures:

1. Determine C_1 from $\omega_H - \omega_L = 2R_k C_1 \omega_0^2$.
2. Determine L_1 from $L_1 = 1/\omega_0^2 C_1$.
3. Determine L_2 from $L_2 = k^2 C_1$, since $L_2/C_1 = k^2$.
4. Determine C_2 from $C_2 = 1/\omega_0^2 L_2$.

Band-pass Filter Example Design a band-pass filter with center frequency of 100 MHz, the bandwidth of 20 MHz, and the characteristic impedance of 100 Ω.







6.4 Band-Stop Filters (or Band-elimination filters, Band-rejection filters)

For band-stop filters, Z_1 is a parallel LC circuit, i.e., $Z_1 = j\omega L_1 \,//\, 1/j\omega C_1$, and Z_2 is a series LC circuit, i.e., $Z_2 = j(\omega L_2 - 1/\omega C_2)$, as shown in Fig. 10. The condition for the band-stop filter is that both series and parallel LC circuits have equal resonant frequencies, i.e.,

$$\omega_0^2 L_1 C_1 = 1 = \omega_0^2 L_2 C_2 \text{ or } L_1 C_1 = L_2 C_2 \text{. Then,}$$

$$Z_1 Z_2 = \frac{L_1(1-\omega^2 L_1 C_1)}{C_2(1-\omega^2 L_2 C_2)} = \frac{L_1}{C_2} = R_k^2 = k^2 \text{.}$$

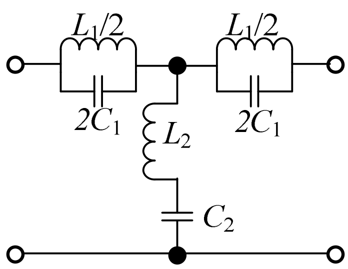


Fig. 10 Band-stop T-section

The cut-off frequency can be found from

$Z_1 = -4Z_2 \to -4Z_2^2 = Z_1 Z_2 = R_k^2 \to Z_2 = \pm jR_k / 2$.

Hence, Z_2 at lower cut-off frequency f_L is equal to $-Z_2$ at upper cut-off frequency f_H, i.e.,

$1/\omega_L C_2 - \omega_L L_2 = \omega_H L_2 - 1/\omega_H C_2$, or $1 - \omega_L^2 L_2 C_2 = (\omega_L / \omega_H)(\omega_H^2 L_2 C_2 - 1)$.

Using $\omega_0^2 = 1/L_2 C_2$ yields

$1 - \omega_L^2 / \omega_0^2 = (\omega_L / \omega_H)(\omega_H^2 / \omega_0^2 - 1)$ or $\omega_0 = \sqrt{\omega_L \omega_H}$.

Also, from

$Z_2 = \pm jR_k / 2 \to Z_2\big|_{\omega=\omega_H} - Z_2\big|_{\omega=\omega_L} = jR_k$, or $\omega_H L_2 - 1/\omega_H C_2 - (\omega_L L_2 - 1/\omega_L C_2) = R_k$,

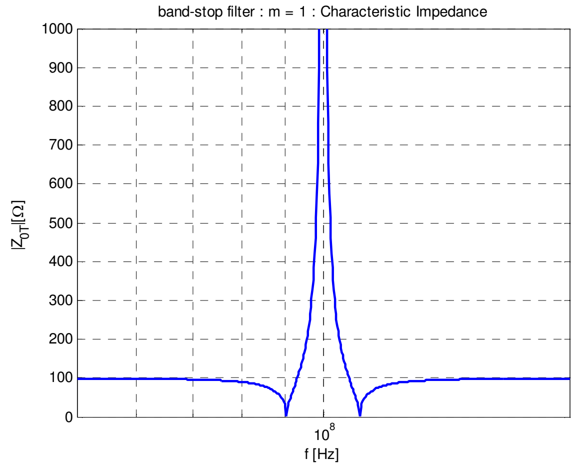
one can derive the condition

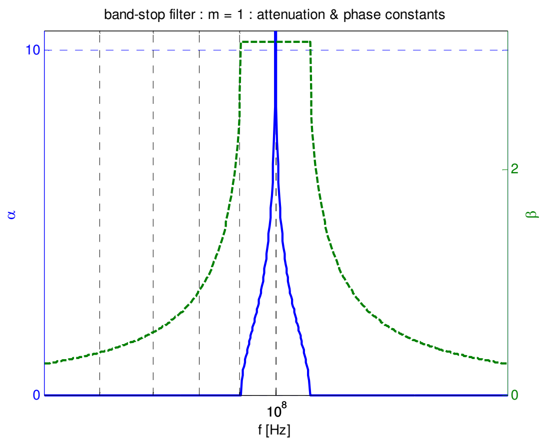
$\omega_H - \omega_L = R_k C_2 \omega_0^2 / 2$.

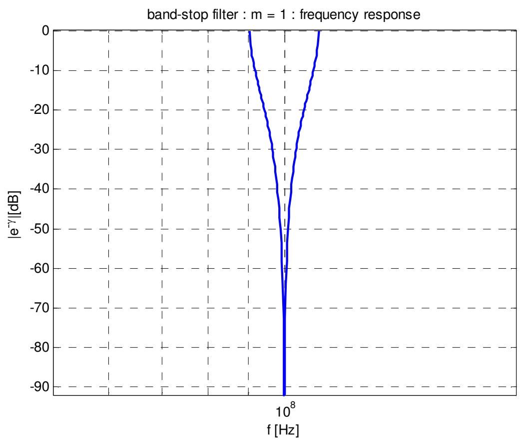
Design Procedure To determine the values of L_1, C_1, L_2, and C_2, one needs to specify the center frequency, the bandwidth and the desired characteristic impedance, then using the following procedures:

1. Determine C_2 from $\omega_H - \omega_L = R_k C_2 \omega_0^2 / 2$.
2. Determine L_2 from $L_2 = 1/\omega_0^2 C_2$.
3. Determine L_1 from $L_1 = k^2 C_2$, since $L_1/C_2 = k^2$.
4. Determine C_1 from $C_1 = 1/\omega_0^2 L_1$.

Band-stop Filter Example Design a band-stop filter with center frequency of 100 MHz, the bandwidth of 20 MHz, and the characteristic impedance of 100 Ω.







7 The *m*-derived T-section

The constant-k prototype filter section, though simple, has two major disadvantages, namely (i) the characteristic impedance varies widely over the pass band so that impedance matching is not possible, (ii) the cut-off rate is not appreciably high, i.e., the drop-off rate is not sufficiently fast. The cut-off rate may be raised by cascading a number of constant-k sections in series, but this is not economical. The *m*-derived filters are designed to achieve this objective.

The approach used here is to introduce a zero frequency into the impedance of the shunt arm. At this frequency, denoted by f_∞, the shunt arm becomes a short circuit and the attenuation becomes infinity. If f_∞ is chosen to be close to the cut-off frequency, then the cut-off rate can be raised. The attenuation may be kept at high value throughout the stop band by cascading the constant-k prototype section with the *m*-derived section. Now, consider the *m*-derived T-section, let us assume

$Z_1' = mZ_1$,

where $0 < m < 1$. Then, solving for Z_2' that achieves the same value of Z_{0T} yields

$$Z_{0T} = \sqrt{Z_1^2/4 + Z_1 Z_2} = \sqrt{Z_1'^2/4 + Z_1' Z_2'} = \sqrt{m^2 Z_1^2/4 + mZ_1 Z_2'} \text{ or } Z_2' = \frac{Z_2}{m} + \frac{1-m^2}{4m} Z_1 .$$

For a low-pass filter section, $Z_1 = j\omega L$, $Z_2 = 1/j\omega C$, then $Z_1' = jm\omega L$, $Z_2 = 1/jm\omega C + (1-m^2)\, jm\omega L/4m$, as shown in Fig. 11. The resonant frequency of the shunt arm becomes

$$\frac{1-m^2}{4}\omega_\infty^2 LC = 1 \rightarrow \omega_\infty^2 = \frac{4}{(1-m^2)LC} = \frac{\omega_c^2}{1-m^2} \text{ or}$$

$\omega_\infty = \omega_c / \sqrt{1-m^2}$, where ω_c is the cut-off frequency.

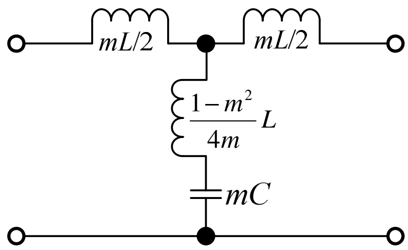


Fig. 11 *m*-derived low-pass T-section

Therefore, the smaller the value of *m*, the sharper the cut-off. Notice that

$$Z_2' = \frac{Z_2}{m} + \frac{1-m^2}{4m} Z_1 = \frac{1}{jm\omega C} + \frac{1-m^2}{4m} j\omega L = \frac{1}{4m}\frac{4-(1-m^2)\omega^2 LC}{j\omega C} = \frac{1}{m}\frac{1-(1-m^2)(\omega^2/\omega_c^2)}{j\omega C},$$

the pass and stop bands can be characterized as follows:

(a) <u>Pass band</u> $-1 < Z_1/4Z_2 < 0$ and $\alpha = 0$.

$$\beta = 2\sin^{-1}\sqrt{-Z_1'/4Z_2'} = 2\sin^{-1}\sqrt{\frac{m^2\omega^2 LC}{1-(1-m^2)(\omega^2/\omega_c^2)}} = 2\sin^{-1}\frac{m\omega/\omega_c}{\sqrt{1-(1-m^2)(\omega^2/\omega_c^2)}} .$$

(b) <u>Stop band</u> $-\infty < Z_1'/4Z_2' < -1$ and $\beta = (2n-1)\pi$.

For $f_c < f < f_\infty$,

$$\alpha = 2\cosh^{-1}\sqrt{-Z_1/4Z_2} = 2\cosh^{-1}\frac{m\omega/\omega_c}{\sqrt{1-(1-m^2)(\omega^2/\omega_c^2)}} = 2\cosh^{-1}\frac{m\omega/\omega_c}{\sqrt{1-(\omega^2/\omega_\infty^2)}} .$$

For $f > f_\infty$

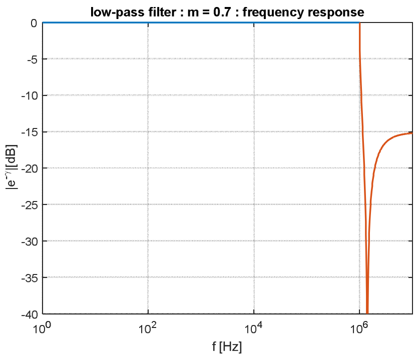
$$\alpha = 2\cosh^{-1}\sqrt{Z_1/4Z_2} = 2\cosh^{-1}\frac{m\omega/\omega_c}{\sqrt{(1-m^2)(\omega^2/\omega_c^2)-1}} = 2\cosh^{-1}\frac{m\omega/\omega_c}{\sqrt{(\omega^2/\omega_\infty^2)-1}} .$$

Similar analysis procedure can be applied to the *m*-derived high-pass T-section, as shown in Fig. 12. Here, $\omega_\infty = \omega_c\sqrt{1-m^2}$.

Likewise, the *m*-derived band-pass T-section is shown in Fig. 13.

<u>Question</u> Find ω_∞ for the T-section in Fig. 13.

<u>*m*-derived Low-pass Filter Example</u> Design a low-pass filter with cut-off frequency of 1 MHz, and the characteristic impedance of 100 Ω. Here, use $m = 0.7$.



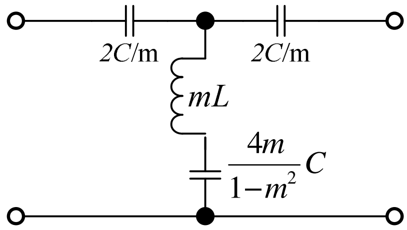


Fig. 12 *m*-derived high-pass T-section

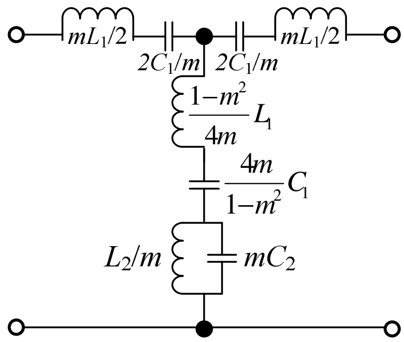


Fig. 13 *m*-derived band-pass T-section

8 Termination with *m*-derived half sections

The *m*-derived T- or π-sections can be formed by the splitted *m*-derived half sections or L-sections, as shown in Fig. 14. These *m*-derived half sections, having $m = 0.6$, are called terminating half sections.

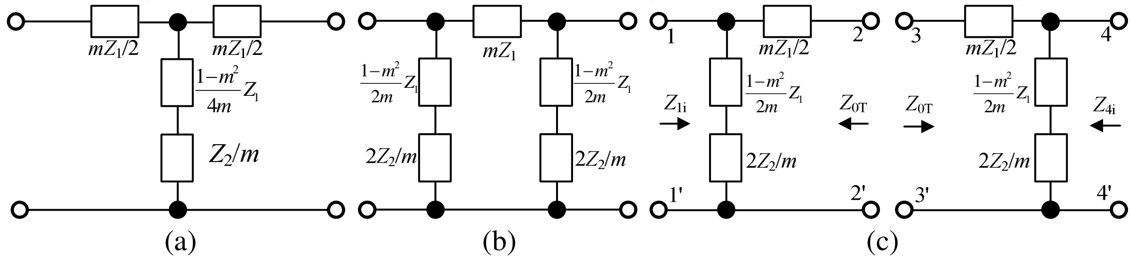


Fig. 14 (a) *m*-derived T-section (b) *m*-derived π-section (c) *m*-derived half sections

Zobel discovered that an *m*-derived half section could be made to change its characteristics with frequency in such a way that the filter is approximately matched to its load at all frequencies over most of the pass band.

Now, the image impedance of the left half section at the 1,1' terminals is given by

$$Z_{1i} = \sqrt{Z_{1oc}Z_{1sc}} = \sqrt{\frac{\left[(1-m^2)Z_1/2m + 2Z_2/m\right]^2 (mZ_1/2)}{(1-m^2)Z_1/2m + 2Z_2/m + mZ_1/2}} = \left[1+(1-m^2)Z_1/4Z_2\right]Z_{0\pi},$$

where $Z_{0\pi} = \sqrt{Z_1 Z_2/(1+Z_1/4Z_2)}$. The impedance of the left half section at terminal 2,2' is

$$Z_{2i} = \sqrt{Z_{2oc}Z_{2sc}} = \sqrt{\left(mZ_1/2 + (1-m^2)Z_1/2m + 2Z_2/m\right)mZ_1/2} = \sqrt{Z_1Z_2(1+Z_1/4Z_2)} = Z_{0T}.$$

The image impedance at 3,3' terminals is equal to Z_{0T}, and at terminals 4,4' is equal to Z_{1i}. For low-pass filters, using

$Z_{0\pi} = R_k/\sqrt{1-(f/f_c)^2}$ yields

$$Z_{1i} = \frac{R_k\left[1-(1-m^2)(f/f_c)^2\right]}{\sqrt{1-(f/f_c)^2}}.$$

The variation of image impedance as a function of f/f_c is plotted in Fig. 15. It is seen that $m = 0.6$ half section has a nearly constant value of Z_{1i} can be obtained over 85% of the pass band. Following the same procedure, the image impedance for high-pass filters can be given by

$$Z_{1i} = \frac{R_k\left[1-(1-m^2)(f_c/f)^2\right]}{\sqrt{1-(f_c/f)^2}}$$

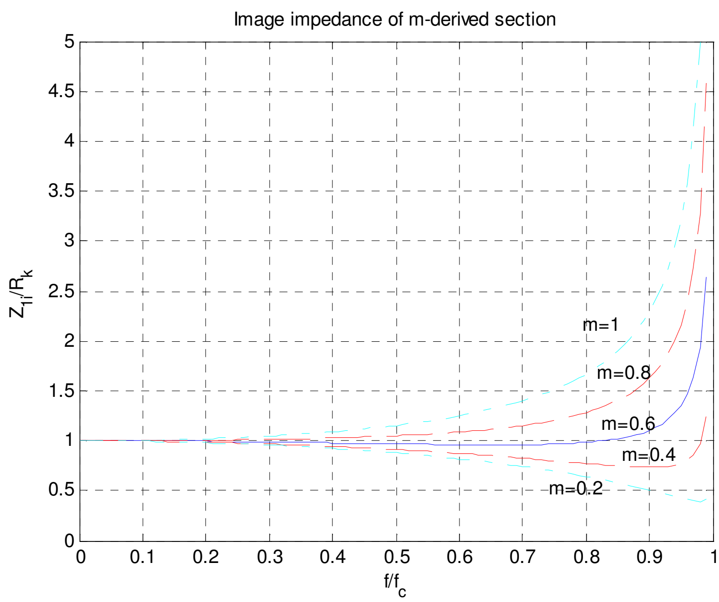


Fig. 15: Variation of image impedance of m-derived section

Fig. 16 summarizes the T- and π-sections used for low-pass and high-pass T-section filter designs.

8 Composite Filter Design

By combining in cascade the constant-k, *m*-derived sharp cut-off, and the *m*-derived matching sections, one can realize a filter with the desired attenuation and matching properties. This type of design is called *a composite filter*. Fig. 17 shows an example of composite filter design. The constant-k sections, the *m*-derived section as well as the matching half π-sections are shown in Fig. 16.

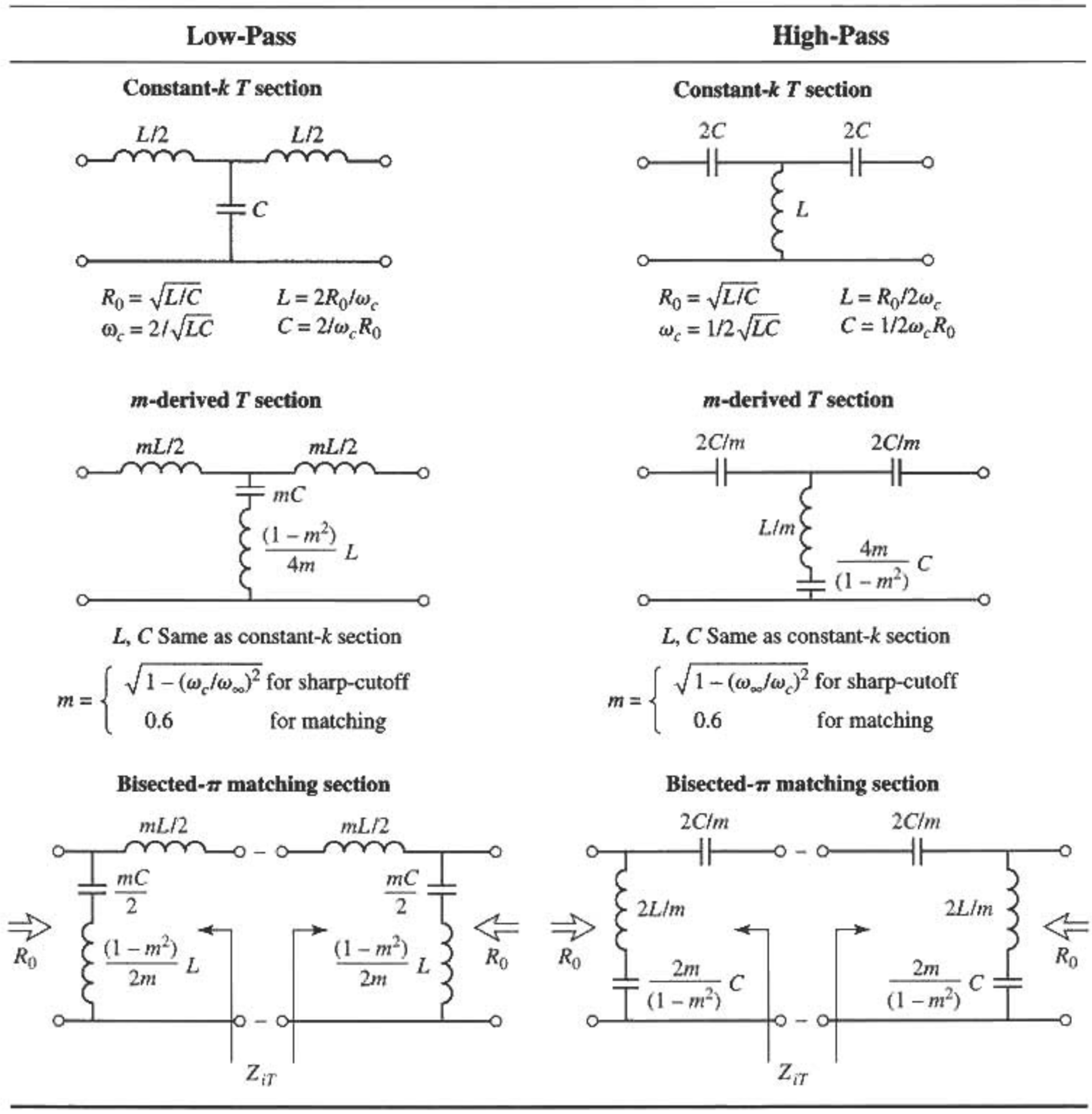


Fig. 16 Summary of composite filter design

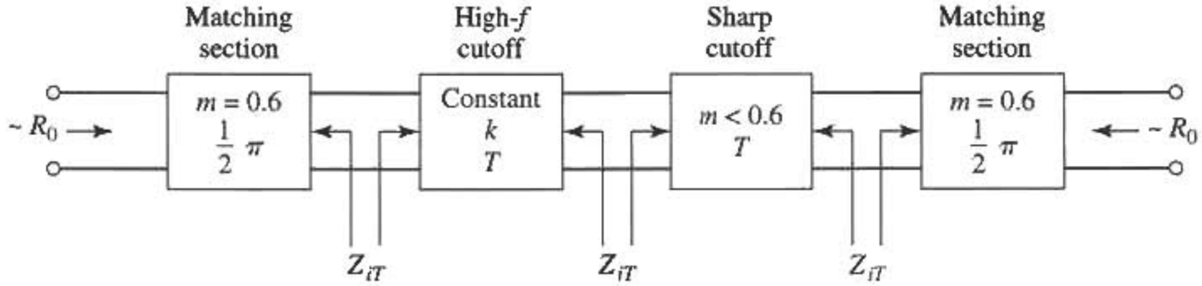


Fig. 17: A four-stage composite filter

9 Reactance Plot

Fig. 18 shows a typical plot of reactances as a function of frequency for low-pass, high-pass, band-pass, and band-stop constant-k filters. Likewise, Fig. 19 shows reactance plot for *m*-derived filters when $m = 0.7$.

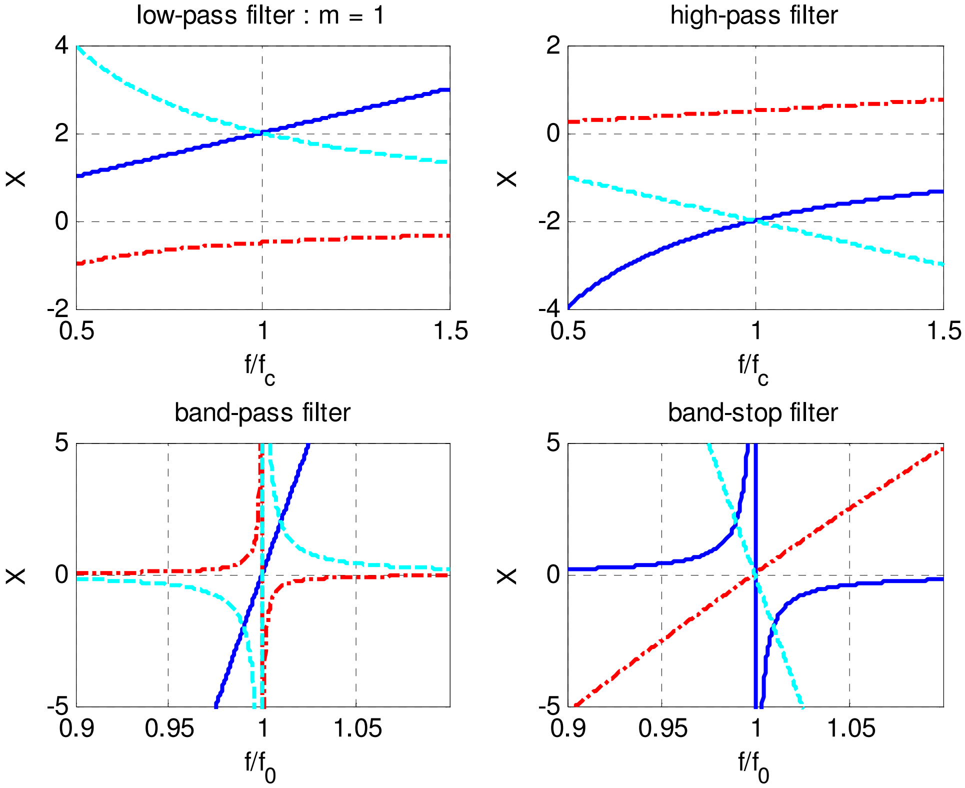


Fig. 18: Reactance plot for constant-k filters. The solid lines denote X_1, the dash-dot lines denote X_2 and the dash lines denote $-4Z_2$.

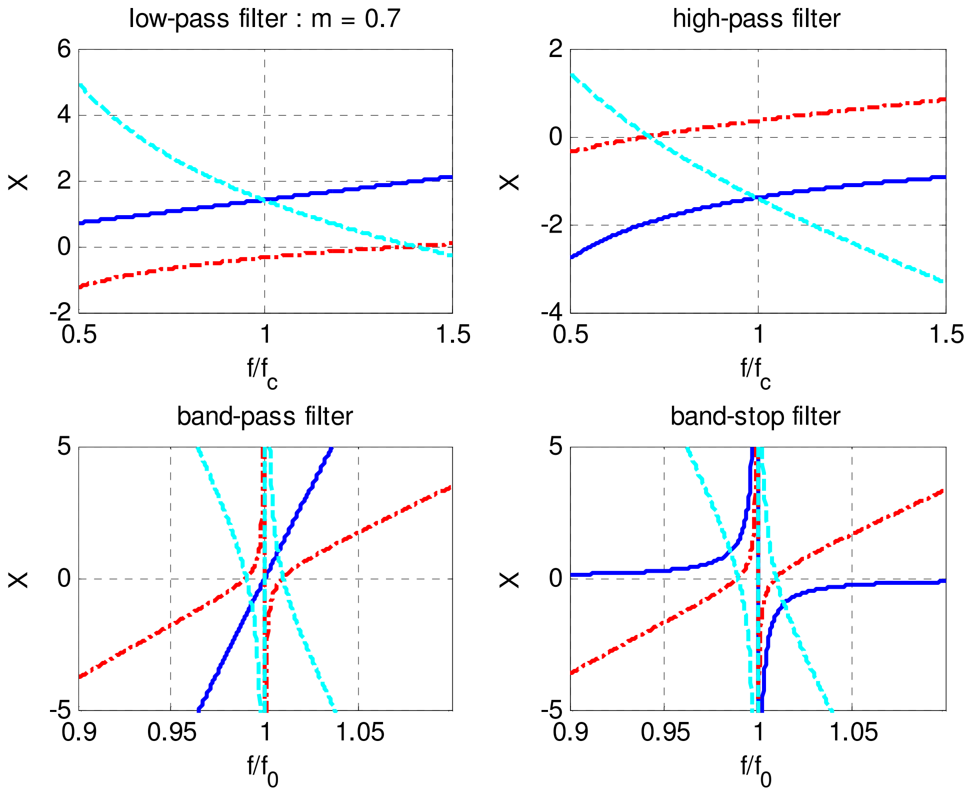


Fig. 19: Reactance plot for *m*-derived filters when $m = 0.7$. The solid lines denote X_1, the dash-dot lines denote X_2 and the dash lines denote $-4X_2$.

12 Insertion Loss Method

The insertion loss method is based on the attenuation response or insertion loss of a filter. The insertion loss or power loss ratio of a two-port network is given by:

$$P_{LR} = \frac{\text{Power available from the source}}{\text{Power delivered to load}} = \frac{P_{inc}}{P_{load}} = \frac{1}{1-|\Gamma(\omega)|^2}$$

where Γ is the reflection coefficient looking into the filter (assume no loss in the filter).

Design of a filter using the insertion-loss approach usually begins by designing a normalized low-pass prototype (LPP). The LPP is a low-pass filter with source resistance of 1Ω and cutoff frequency of 1 Radian/s. Impedance transformation and frequency scaling are then applied to denormalize the LPP and synthesize different type of filters with different cutoff frequencies. Fig. 20 summarizes the process of filter design by the insertion loss method.

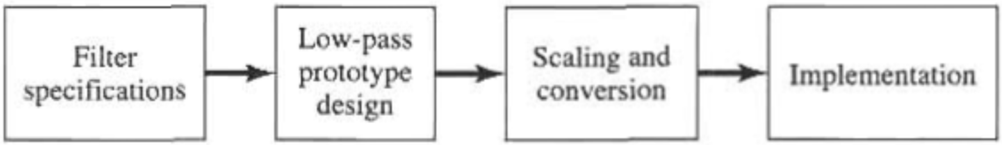


Fig. 20 Summary of filter design by insertion loss method

Now, consider the reflection coefficient at the input port, which is given by

$\Gamma(\omega) = \dfrac{Z(\omega)-1}{Z(\omega)+1}$, where the 1Ω source resistance is assumed.

Since $V(\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt$ and $v(t)$ is a real function, $V(-\omega) = V^*(\omega)$. Similar result holds for $I(\omega)$ as well. Thus, $Z(-\omega) = \dfrac{V(-\omega)}{I(-\omega)} = \dfrac{V^*(\omega)}{I^*(\omega)} = Z^*(\omega)$. Therefore,

$\Gamma(-\omega) = \dfrac{Z(-\omega)-1}{Z(-\omega)+1} = \dfrac{Z^*(\omega)-1}{Z^*(\omega)+1} = \Gamma^*(\omega)$. It follows that

$$|\Gamma(\omega)|^2 = \Gamma(\omega)\Gamma^*(\omega) = \Gamma(\omega)\Gamma(-\omega) = \Gamma^*(-\omega)\Gamma(-\omega) = |\Gamma(-\omega)|^2,$$

hence $|\Gamma(\omega)|^2$ is an even function of ω. Therefore, it can be written as a polynomial in ω^2:

$$|\Gamma(\omega)|^2 = \frac{M(\omega^2)}{M(\omega^2)+N(\omega^2)}$$

where M and N are real polynomials in ω^2. Thus, the insertion loss can be rewritten as

$$P_{LR} = 1 + \frac{M(\omega^2)}{N(\omega^2)},$$

which is the form of *physically realizable* power loss ratio. This equation is used to specify desirable filter responses.

<u>Maximally Flat (Butterworth or binomial filter)</u> This type of filter is optimum in the sense that it provides the *flattest* possible passband response for a given filter complexity, or order (i.e., number of passive elements). For a low-pass filter, it is specified by

$$P_{LR} = 1 + k^2\omega^{2N},$$

where N denotes the order of the filter. The pass band extends from $\omega = 0$ to $\omega = 1$; at the band edge the power loss ratio is $1 + k^2$. Typically, k is chosen to be 1 in order to make the band edge the -3 dB point. Note that the first ($2N$-1) derivatives of the power loss ratio are zero at $\omega = 0$, resulting in the maximally flat response.

<u>Equal Ripple (Chebyshev filter)</u> The insertion loss of this low-pass Chebyshev filter is specified by Chebyshev polynomial as follows

$P_{LR} = 1 + k^2 T_N^2(\omega)$.

This leads to a sharp cutoff with the expense of amplitude ripples in the pass band. The maximum pass band ripples are given by $1 + k^2$, thus the pass band ripple level is specified by k^2.

Maximally Flat Delay (Bessel-Thomson filter) Flat delay simply implies constant phase velocity, which in turn implies linear phase, since $\beta = \omega/u_p$. The greatest advantage of this filter is that the output signal is not *distorted*, which is desirable in most applications. The insertion loss of this low-pass filter is specified by

$$P_{LR} = k^2 B_N^2(\omega),$$

where k is chosen such that the insertion loss at $\omega = 0$ is unity and $B_n(x)$ denotes the Bessel polynomial of order n. The first 4 Bessel polynomials are

$$B_1(x) = 1 + x; B_2(x) = x^2 + 3x + 3; B_3(x) = x^3 + 6x^2 + 15x + 15; B_4(x) = x^4 + 10x^3 + 45x^2 + 105x + 105$$

Higher-order polynomials can be found using the following recurrence formula:

$$B_n(x) = (2n-1)B_{n-1}(x) + x^2 B_{n-2}(x).$$

In fact, the coefficients of $B_n(x)$ can be found directly by formula

$$c_k = \frac{(2n-k)!}{2^{n-k}k!(n-k)!} \text{ and } B_n(x) = \sum_{k=0}^{n} c_k x^k.$$

The above insertion loss specification is obtained by setting $x = j\omega$.

Now, consider the transfer function of the third-order low-pass filter given by

$$H(s) = 15/(s^3 + 6s^2 + 15s + 15);\ s=j\omega.$$

The phase is given by

$$\phi(\omega) = \arg H(j\omega) = -\tan^{-1}\frac{15\omega - \omega^3}{15 - 6\omega^2}$$ and the group delay becomes

$$D(\omega) = -\frac{d\phi(\omega)}{d\omega} = \frac{6\omega^4 + 45\omega^2 + 225}{\omega^6 + 6\omega^4 + 45\omega^2 + 225},$$

which is approximately 1 for small ω, i.e., low frequency range. The higher the order, the broader the frequency range where group delay is flat.

Low-pass filter prototype

Fig. 21 shows the ladder circuit for low-pass filter prototype with the source impedance of 1 Ω, where their elements are defined as follows:

$$g_0 = \begin{cases} \text{generator resistance (Fig. 21a)} \\ \text{generator conductance (Fig. 21b)} \end{cases};\quad \underset{k=1 \text{ to } N}{g_k} = \begin{cases} \text{inductance for series inductors} \\ \text{capacitance for shunt capacitors} \end{cases};$$

$$g_{N+1} = \begin{cases} \text{load resistance if } g_N \text{ is a shunt capacitor} \\ \text{load conductance if } g_N \text{ is a series inductor} \end{cases}$$

Element values in the figure are different depending on the filter type. They can be determined by using tables 1, 2, 3, for maximally-flat filter, equal-ripple filter, and maximally-flat time delay, respectively.

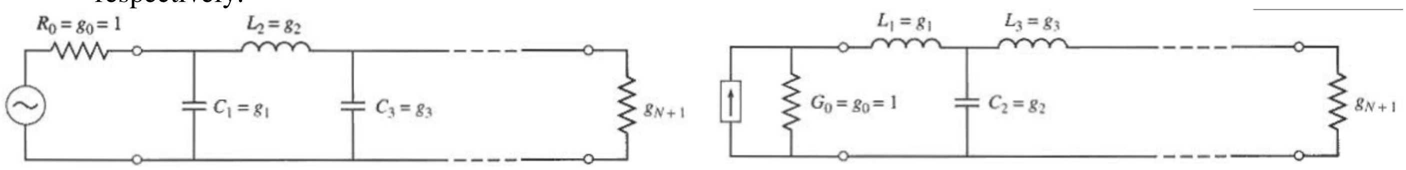


Fig. 21 Ladder circuits for low-pass filter prototype

Also, Fig. 22 shows attenuation versus normalized frequency for maximally flat filter prototypes.

Table 1[1]:

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	2.0000	1.0000									
2	1.4142	1.4142	1.0000								
3	1.0000	2.0000	1.0000	1.0000							
4	0.7654	1.8478	1.8478	0.7654	1.0000						
5	0.6180	1.6180	2.0000	1.6180	0.6180	1.0000					
6	0.5176	1.4142	1.9318	1.9318	1.4142	0.5176	1.0000				
7	0.4450	1.2470	1.8019	2.0000	1.8019	1.2470	0.4450	1.0000			
8	0.3902	1.1111	1.6629	1.9615	1.9615	1.6629	1.1111	0.3902	1.0000		
9	0.3473	1.0000	1.5321	1.8794	2.0000	1.8794	1.5321	1.0000	0.3473	1.0000	
10	0.3129	0.9080	1.4142	1.7820	1.9754	1.9754	1.7820	1.4142	0.9080	0.3129	1.0000

Table 2a[1]:

0.5 dB Ripple

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	0.6986	1.0000									
2	1.4029	0.7071	1.9841								
3	1.5963	1.0967	1.5963	1.0000							
4	1.6703	1.1926	2.3661	0.8419	1.9841						
5	1.7058	1.2296	2.5408	1.2296	1.7058	1.0000					
6	1.7254	1.2479	2.6064	1.3137	2.4758	0.8696	1.9841				
7	1.7372	1.2583	2.6381	1.3444	2.6381	1.2583	1.7372	1.0000			
8	1.7451	1.2647	2.6564	1.3590	2.6964	1.3389	2.5093	0.8796	1.9841		
9	1.7504	1.2690	2.6678	1.3673	2.7239	1.3673	2.6678	1.2690	1.7504	1.0000	
10	1.7543	1.2721	2.6754	1.3725	2.7392	1.3806	2.7231	1.3485	2.5239	0.8842	1.9841

Table 2b:

3.0 dB Ripple

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	1.9953	1.0000									
2	3.1013	0.5339	5.8095								
3	3.3487	0.7117	3.3487	1.0000							
4	3.4389	0.7483	4.3471	0.5920	5.8095						
5	3.4817	0.7618	4.5381	0.7618	3.4817	1.0000					
6	3.5045	0.7685	4.6061	0.7929	4.4641	0.6033	5.8095				
7	3.5182	0.7723	4.6386	0.8039	4.6386	0.7723	3.5182	1.0000			
8	3.5277	0.7745	4.6575	0.8089	4.6990	0.8018	4.4990	0.6073	5.8095		
9	3.5340	0.7760	4.6692	0.8118	4.7272	0.8118	4.6692	0.7760	3.5340	1.0000	
10	3.5384	0.7771	4.6768	0.8136	4.7425	0.8164	4.7260	0.8051	4.5142	0.6091	5.8095

Table 3[1]:

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	2.0000	1.0000									
2	1.5774	0.4226	1.0000								
3	1.2550	0.5528	0.1922	1.0000							
4	1.0598	0.5116	0.3181	0.1104	1.0000						
5	0.9303	0.4577	0.3312	0.2090	0.0718	1.0000					
6	0.8377	0.4116	0.3158	0.2364	0.1480	0.0505	1.0000				
7	0.7677	0.3744	0.2944	0.2378	0.1778	0.1104	0.0375	1.0000			
8	0.7125	0.3446	0.2735	0.2297	0.1867	0.1387	0.0855	0.0289	1.0000		
9	0.6678	0.3203	0.2547	0.2184	0.1859	0.1506	0.1111	0.0682	0.0230	1.0000	
10	0.6305	0.3002	0.2384	0.2066	0.1808	0.1539	0.1240	0.0911	0.0557	0.0187	1.0000

Likewise, Fig. 23 (a), 23 (b) show attenuation versus normalized frequency for equal-ripple filter prototypes with (a) 0.5 dB ripple level and (b) 3.0 dB ripple level. Fig. 24 shows delay versus normalized frequency for maximally flat group delay filter prototypes.

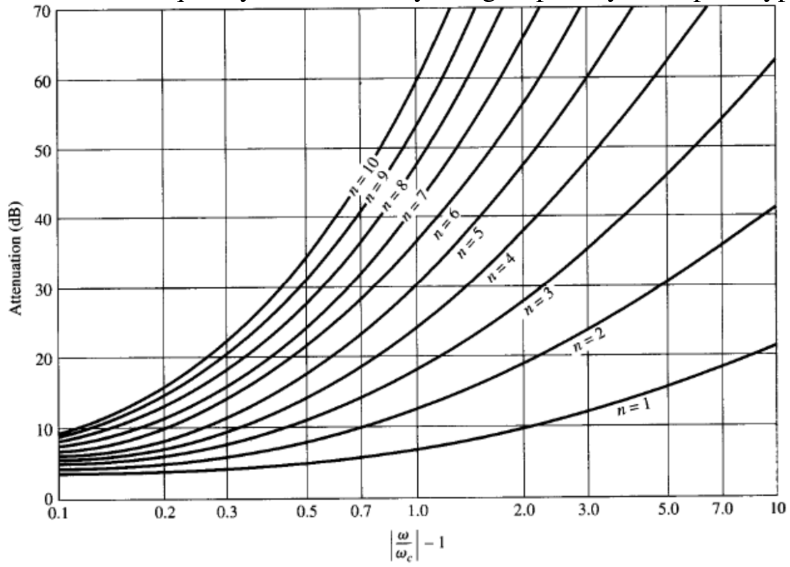


Fig. 22[2]

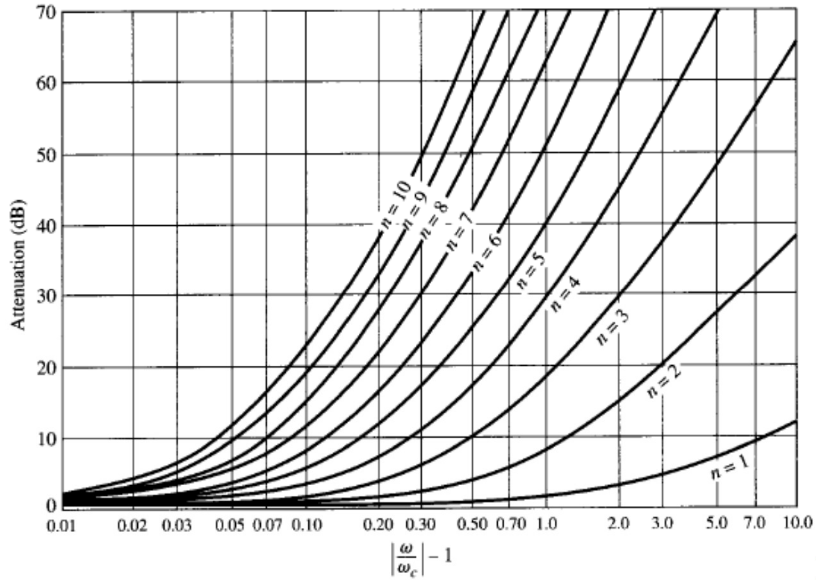


Fig. 23 (a) [2]

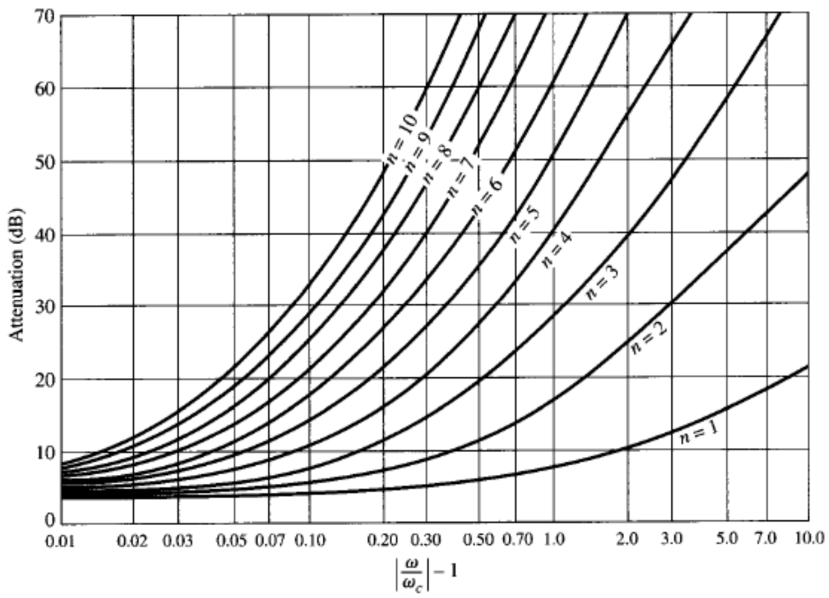


Fig. 23 (b) [2]

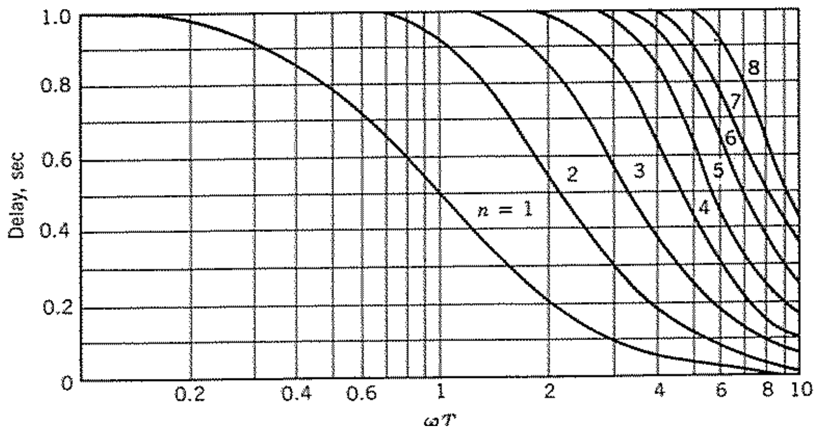


Fig. 24 [3]

13 Filter Transformations

The low-pass filter prototypes are designed assuming that R_s = 1 Ω and ω_c = 1 rad/s. For practical filters, these prototypes have to be scaled in terms of impedance and frequency. Furthermore, they can also be converted to give high-pass, bandpass or bandstop characteristics.

Impedance and Frequency Scaling for Low-Pass Filters

Assuming the source impedance of R_0, the new filter component values are obtained by

$$L' = R_0 L\,;\; C' = C / R_0\,;\; R'_s = R_0\,;\; R'_L = R_0 R_L\,,$$

where L, C and R_L are the component values for the original prototype.

To change the cutoff frequency of a low-pass filter prototype from unity to ω_c, the following frequency scaling is required:

$$\omega \leftarrow \omega / \omega_c\,;\; P'_{LR}(\omega) = P_{LR}(\omega / \omega_c)\,,$$

where ω_c denotes the new cutoff frequency. This transformation can be viewed as a stretching of the original passband. The new element values are determined by applying the frequency scaling to the series reactances and shunt susceptances as follows:

$jX_k = j\dfrac{\omega}{\omega_c}L_k = j\omega L'_k\,;\; jB_k = j\dfrac{\omega}{\omega_c}C_k = j\omega C'_k$, which shows that the new element values are

$$L'_k = \frac{L_k}{\omega_c}\,;\; C'_k = \frac{C_k}{\omega_c}\,.$$

Combining the results due to both impedance and frequency scaling yields

$$L'_k = \frac{R_0 L_k}{\omega_c}\,;\; C'_k = \frac{C_k}{R_0 \omega_c}\,.$$

Low-pass to high-pass transformation The frequency substitution

$$\omega \leftarrow -\omega_c / \omega\,;\; P'_{LR}(\omega) = P_{LR}(-\omega_c / \omega)\,,$$

can be used to convert a low-pass response to a high-pass response. This substitution maps ω= 0 to ω = ±∞, and vice versa; cutoff occurs when ω= ±ω_c. The negative sign is needed to convert inductors (and capacitors) to realizable capacitors (and inductors). Applying frequency substitution yields

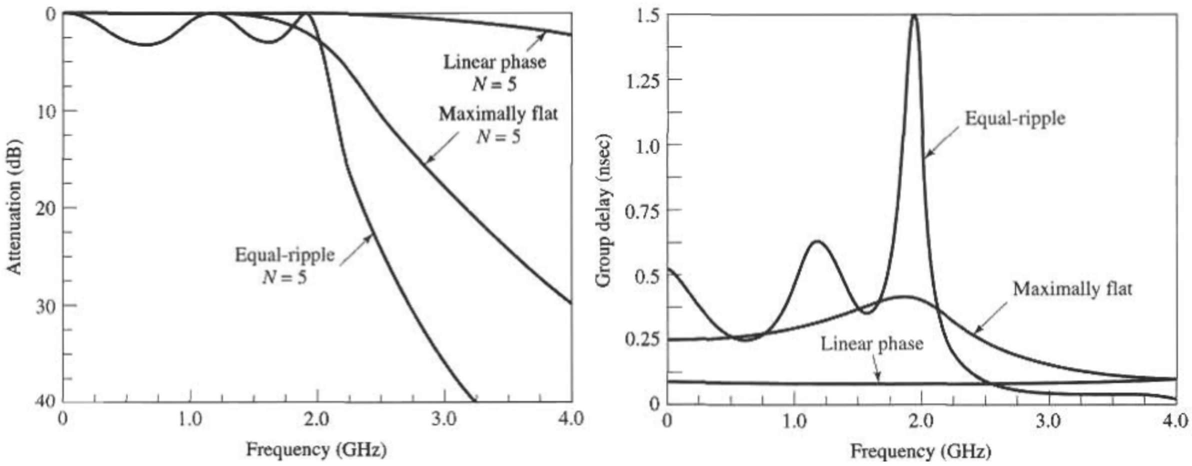
$jX_k = -j\dfrac{\omega_c}{\omega}L_k = \dfrac{1}{j\omega C'_k}\,;\; jB_k = -j\dfrac{\omega_c}{\omega}C_k = \dfrac{1}{j\omega L'_k}$, which shows that the new element values are

$$C'_k = \frac{1}{\omega_c L_k}\,;\; L'_k = \frac{1}{\omega_c C_k}\,.$$

Combining the results due to both impedance and frequency substitution yields

$$C'_k = \frac{1}{R_0 \omega_c L_k}; \; L'_k = \frac{R_0}{\omega_c C_k}.$$

Example (Low-pass filter design) Design a maximally flat low-pass filter with a cutoff frequency of 2 GHz, impedance of 50 Ω, and at least 15 dB insertion loss at 3 GHz. Compute and plot the amplitude response and group delay for $f = 0$ to 4 GHz, and compare with an equal-ripple (3.0 dB ripple) and linear phase filter having the same order. [2]



Bandpass filter transformation Let ω_1, ω_2 denote the edges of the passband, then the low-pass to bandpass transformation can be accomplished by the following frequency substitution:

$$\omega \leftarrow \frac{\omega_0}{\omega_2 - \omega_1}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right); \Delta = \frac{\omega_2 - \omega_1}{\omega_0},$$

where Δ denotes the fractional bandwidth of the passband and ω_0 denotes the center frequency, which is chosen to be the geometric mean of ω_1 and ω_2, i.e.,

$$\omega_0 = \sqrt{\omega_1 \omega_2},$$

for simplification purposes. It follows that

when $\omega = \omega_0$, $\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = 0$; when $\omega = \omega_1$, $\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{\Delta}\left(\frac{\omega_1^2 - \omega_0^2}{\omega_0 \omega_1}\right) = -1$;

when $\omega = \omega_2$, $\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{\Delta}\left(\frac{\omega_2^2 - \omega_0^2}{\omega_0 \omega_2}\right) = 1$.

Therefore, the pass band exists in the range where the normalized frequency is between -1 and 1, as in the case of low-pass filter prototype. Applying the frequency substitution in the expressions for series reactances and shunt susceptances yields

$$jX_k = j\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)L_k = j\frac{\omega L_k}{\Delta \omega_0} - j\frac{\omega_0 L_k}{\Delta \omega} = j\omega L'_k - j\frac{1}{\omega C'_k}, L'_k = \frac{L_k}{\Delta \omega_0}, \; C'_k = \frac{\Delta}{\omega_0 L_k};$$

$$jB_k = j\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)C_k = j\frac{\omega C_k}{\Delta \omega_0} - j\frac{\omega_0 C_k}{\Delta \omega} = j\omega C'_k - j\frac{1}{\omega L'_k}, C'_k = \frac{C_k}{\Delta \omega_0}, \; L'_k = \frac{\Delta}{\omega_0 C_k}.$$

Thus, the low-pass filter elements are converted to series resonant circuits (low impedance at resonance) in the series arms, and to parallel resonant circuits (high impedance at resonance) in the shunt arms. Notice that both series and parallel resonator elements have a resonant frequency of ω_0.

Bandstop filter transformation The inverse transformation can be used to obtain a bandstop response. Thus, the frequency substitution is given by

$\omega \leftarrow \Delta\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^{-1}$, where Δ and ω_0 have the same definitions as before. Then following the procedure used previously for the bandpass filter, the series inductors of the low-pass filter prototype are converted to parallel *LC* circuits having element values given by

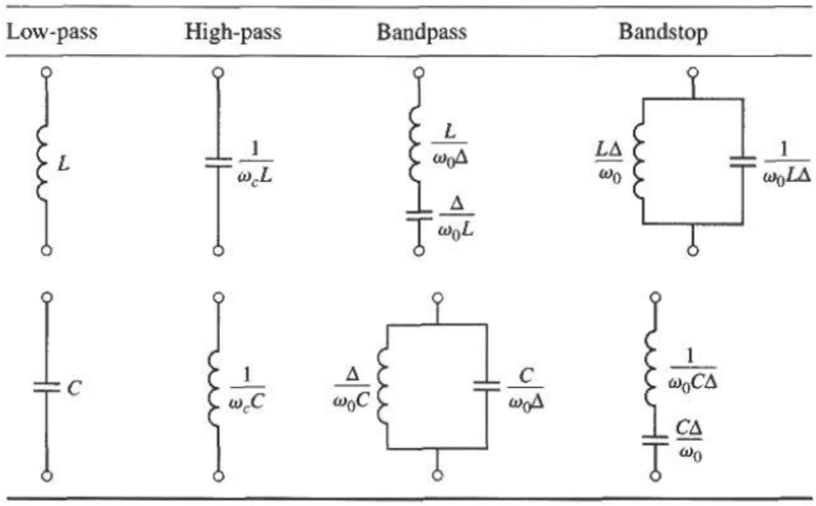
$$L'_k=\frac{\Delta L_k}{\omega_0},\ C'_k=\frac{1}{\omega_0\Delta L_k}.$$

Likewise, the shunt capacitors are converted to series *LC* circuits having element values given by

$$C'_k=\frac{\Delta C_k}{\omega_0},\ L'_k=\frac{1}{\omega_0\Delta C_k}.$$

Table 4 summarizes the element transformations from a low-pass filter prototype to a high-pass, bandpass, or bandstop filter. Notice that the impedance scaling is not included.

Table 4[2]



Example (Bandpass filter design) Design a bandpass filter having a 0.5 dB equal-ripple response, with $N = 3$. The center frequency is 1 GHz, the bandwidth is 10%, and the impedance is 50 Ω.[2]

Reference

1. G. L. Mathaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures* (Dedham, Mass.: Artech House, 1980)
2. D. M. Pozar, *Microwave Engineering Third Edition* (Haboken, NJ: John Wiley & Sons, Inc., 2005)
3. W.-K. Chen, *Passive and Active Filters* (Haboken, NJ: John Wiley & Sons, Inc., 1986)