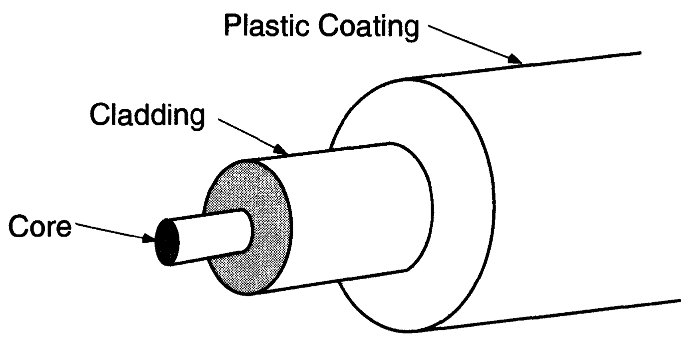


Figure 5.1 The cylindrical step waveguide consists of a high-index core surrounded by a lower-index cladding.

To find the modes of the circular step-index fiber, we must solve the wave equation in cylindrical coordinates. The modes of the cylindrical structure are more abstract than those of the rectangular or planar structure. Once the mode concept is established, we will develop useful formulas for mode-cutoff conditions, numerical aperture, and normalized frequency. As before, the eigenvalue equations will require graphical or numerical solution.

5.2 THE WAVE EQUATION IN CYLINDRICAL COORDINATES

We have already derived the homogeneous wave equation,

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{5.1}$$

for the planar waveguide structure (Equation 1.24). If we assume a time-harmonic field, and use $|k| = \sqrt{\mu_0 \epsilon}\,\omega$ (assuming $\mu = \mu_0$), the wave equation takes the familiar form

$$\nabla^2 \mathbf{E} + k_0^2 n^2 \mathbf{E} = 0 \tag{5.2}$$

To solve this equation in a cylindrical waveguide, we must write this equation in cylindrical coordinates. The electric field is a vector, and there are three components, each of which is a function of r, ϕ, and z:

$$\mathbf{E}(r, \phi, z) = \hat{r}E_r(r, \phi, z) + \hat{\phi}E_\phi(r, \phi, z) + \hat{z}E_z(r, \phi, z) \tag{5.3}$$

In cylindrical coordinates, the *vector Laplacian* (∇^2) is a rather unwieldy expression (see for example reference [1]). The cylindrical wave equation must be evaluated in the following form:

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla \times \nabla \times \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{5.4}$$

Unlike the vector Laplacian in rectilinear coordinates, Equation 5.4 cannot be easily decomposed into three individual components. The transverse components of the field are tightly coupled. Imagine for example a linearly polarized field traveling at a slight angle to the axis of a cylindrical waveguide, as shown in Figure 5.2.

At $z = 0$, the field is purely radial, but as it travels down the axis, it becomes an azimuthal (ϕ) field. It is impossible to decouple the E_r or E_ϕ components in this example.

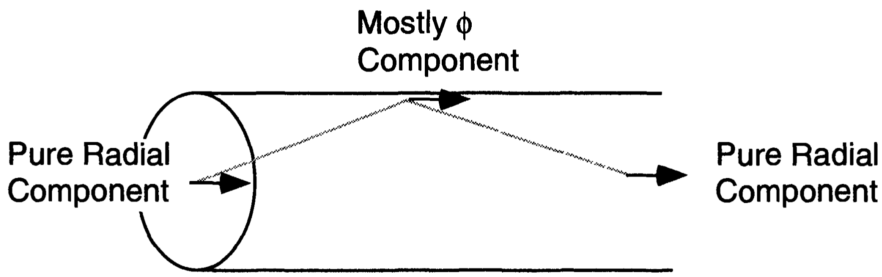


Figure 5.2 A radial field at one point in the waveguide will become an azimuthal field at another location. Notice that the field is not converted between the components by reflection, but by propagation through the coordinate system.

Here is an important point: the $\hat{z}$ component of a field, E_z, does not couple to the two other components as it propagates. Even after reflection at a cylindrical surface, the E_z component remains oriented along the z axis. Figure 5.3 shows an example of how the field component remains pure.

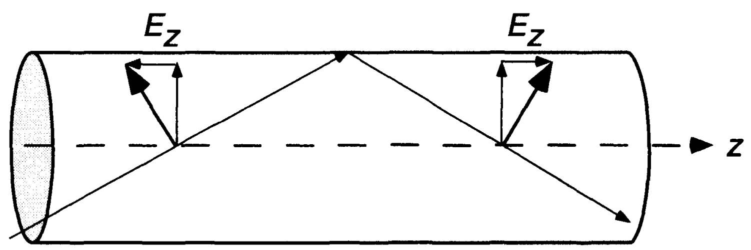


Figure 5.3 The longitudinal component of the electric field does not change through either propagation or reflection at the cylindrical surface.

Since E_z couples only to itself, it is possible to write the scalar wave equation for E_z directly in cylindrical coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k_0^2 n^2 E_z = 0 \tag{5.5}$$

and to solve this equation for E_z. Once we have a solution for $E_z(r, \theta, \phi)$, we can use Maxwell's equations to relate E_z to E_r and E_ϕ. In this indirect fashion, all field components within a circular waveguide are derived.

5.3 SOLUTION OF THE WAVE EQUATION FOR E_z

Since E_z is a function of r, ϕ, and z, we can employ separation of variables to solve the scalar equation, Equation 5.5. Setting $E_z(r, \phi, z) = R(r)\Phi(\phi)Z(z)$, and substituting this into Equation 5.5 results in

$$R''\Phi Z + \frac{1}{r}R'\Phi Z + \frac{1}{r^2}R\Phi''Z + R\Phi Z'' + k_0^2 n^2 R\Phi Z = 0 \tag{5.6}$$

Multiply Equation 5.6 by $r^2/R\Phi Z$ to get

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Phi''}{\Phi} + r^2\frac{Z''}{Z} + k_0^2 n^2 r^2 = 0 \tag{5.7}$$

Due to the translational invariance along the z axis, we can assume a phase term describes the z dependence,

$$Z(z) = e^{-j\beta z} \tag{5.8}$$

where β is (again) the z component of the wavevector k in the waveguide. Using Equation 5.8, we find that $Z''/Z = -\beta^2$, which can be substituted into the wave equation

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Phi''}{\Phi} - r^2\beta^2 + k_0^2 n^2 r^2 = 0 \tag{5.9}$$

Now we can use standard separation techniques to find

$$r^2\frac{R''}{R} + r\frac{R'}{R} - r^2\beta^2 + k_0^2 n^2 r^2 = -\frac{\Phi''}{\Phi} = \nu^2 \tag{5.10}$$

The term ν is called the separation constant. Equation 5.10 can be solved directly for $\Phi(\phi)$:

$$\Phi''(\phi) = -\nu^2\Phi \tag{5.11}$$

which has the solution

$$\Phi(\phi) = Ae^{j\nu\phi} + c.c. \tag{5.12}$$

where A is a normalization constant. Since circular symmetry requires $\Phi(\phi) = \Phi(\phi + 2\pi)$, we can infer that ν must be an integer.

Substituting Equation 5.11 into Equation 5.9 yields an equation that only contains $R(r)$:

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + r^2\left(k_0^2 n^2 - \beta^2 - \frac{\nu^2}{r^2}\right)R = 0 \tag{5.13}$$

The solution to this differential equation is given by Bessel functions [2]. There are many different types of Bessel functions, and to the uninitiated the choice can look formidable. Bessel functions share these properties with *sine* and *cosine* functions: 1) the value of the function must be calculated or looked up in a table; 2) the functions are orthogonal to one another; and 3) they are defined everywhere. It is primarily the lack of familiarity with Bessel functions that causes trepidation. Appendix B, "Bessel Functions," reviews useful relations and properties of relevant Bessel functions.

Two types of Bessel functions solve Equation 5.13. When the argument $\left(k_0^2 n^2 - \beta^2 - \frac{\nu^2}{r^2}\right)$ is positive, *Bessel functions of the first kind of order* ν, symbolized by $J_\nu(\kappa r)$ are the proper solution. For all cases that we will examine, ν is an integer. κ is defined through the expression

$$\kappa^2 = k_0^2 n^2 - \beta^2 \tag{5.14}$$

Note that this is the same definition used in Chapter 3 for the transverse wavevector. The symbol κ has the same meaning in these cylindrical waveguide equations.

When the argument $\left(k_0^2 n^2 - \beta^2 - \frac{\nu^2}{r^2}\right)$ in Equation 5.13 is negative, *modified Bessel functions of the second kind of order* ν, symbolized by $K_\nu(\gamma r)$, are the proper solution. γ is defined as

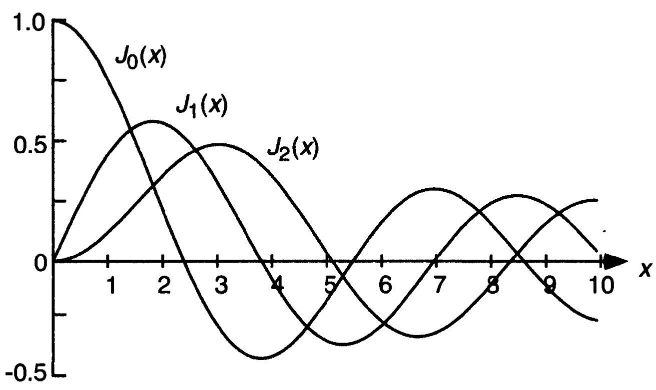
$$\gamma^2 = \beta^2 - k_0^2 n^2 \tag{5.15}$$

Again, the notation is intentionally chosen to correspond to the decay parameter γ used in Chapter 2. As with κ, the function γ plays the same role in cylindrical waveguides that it did in planar waveguides.

Plots of both types of Bessel functions are shown in Figure 5.4 below. The $J_\nu(\kappa r)$ functions are periodic along the radial axis. Only $J_0(\kappa r)$ has finite value at $r = 0$; all other $J_{\nu \neq 0}(\kappa r)$ functions are zero at the origin. For large arguments, the Bessel function of the first kind can be approximated as

$$J_\nu(\kappa r) \approx \sqrt{\frac{2}{\pi \kappa r}} \cos\left(\kappa r - \frac{\nu \pi}{2} - \frac{\pi}{4}\right) \quad \text{for } \kappa r \text{ large} \tag{5.16}$$

These Bessel functions can be viewed as damped sine waves. The amplitude decreases slowly with radial distance, much like the amplitude of a spreading wave in a pond. As we shall see, the J_ν Bessel functions describe the radial standing wave in a cylindrical structure.



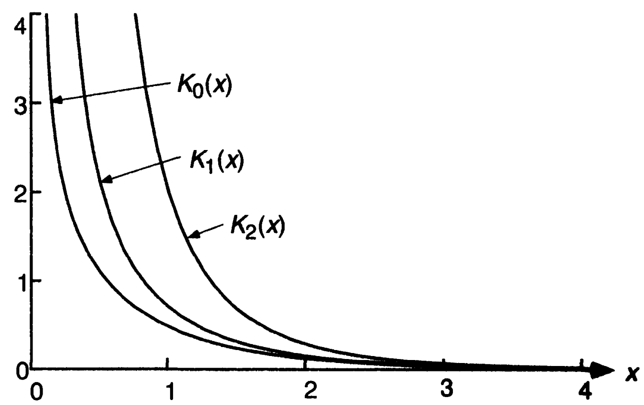


Figure 5.4 Graphs show the first three Bessel functions of the first kind, $J_\nu(\kappa r)$, and of the second kind, $K_\nu(\gamma r)$.

The modified Bessel functions $K_\nu(\gamma r)$ display a monotonic decreasing characteristic. The higher orders of the function decrease at a slower rate, but all orders have the same functional form. In the limit of large γr, the function can be approximated as

$$K_\nu(\gamma r) \approx \frac{e^{-\gamma r}}{\sqrt{2\pi\gamma r}} \tag{5.17}$$

Again, this looks like a radially damped, exponentially decreasing function. Note that at large distance, all orders of $K_\nu(\gamma r)$ look approximately the same. The $\sqrt{1/2\pi\gamma r}$ dependence is the natural decrease of a wave as it expands with radius, while the exponent represents decay due to evanescent interference. $K_\nu(\gamma r)$ functions are used to describe evanescent fields in the optical waveguide.

5.4 FIELD DISTRIBUTIONS IN THE STEP-INDEX FIBER

In this and the next section, we derive expressions for the fields and the characteristic equation for the cylindrical dielectric waveguide. Consider the fiber waveguide shown in Figure 5.5. The fiber waveguide has a core of radius a surrounded by a cladding with lower index. Since we expect oscillatory solutions to the transverse wave equation in the core, $J_\nu(\kappa r)$ solutions will be sought in this region. From Equation 5.14, β must satisfy

$$k_0 n_{core} > \beta > k_0 n_{clad} \tag{5.18}$$

which is the standard criteria for guided wave modes.

In the cladding, the field exponentially decays, so we choose the $K_\nu(\gamma r)$ solutions for $r > a$. The only criteria on the size of the cladding is that the evanescent field should decay to negligible values long before the outer radius of the cladding is reached.

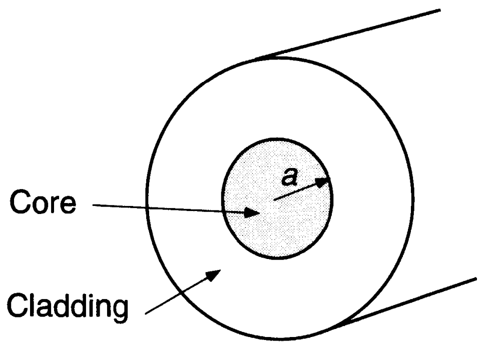


Figure 5.5 The cylindrical waveguide has a core radius of dimension a.

Let's construct a solution to the wave equation. The complete longitudinal fields (E_z and H_z) in both regions can be written as

$$
\begin{aligned}
\text{for } r < a \qquad & E_z(r, \phi, z) = AJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c. \\
& H_z(r, \phi, z) = BJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c. \\
\text{for } r > a \qquad & E_z(r, \phi, z) = CK_\nu(\gamma r)e^{j\nu\phi}e^{-j\beta z} + c.c. \\
& H_z(r, \phi, z) = DK_\nu(\gamma r)e^{j\nu\phi}e^{-j\beta z} + c.c.
\end{aligned}
\tag{5.19}
$$

Note that the electric and magnetic fields have the same spatial dependence. Also note that ν is a mode number, or eigenvalue. Determining the coefficients A, B, C, and D requires application of the boundary conditions, specifically, continuity of the tangential E and H fields. The next few steps involve a lot of mathematics, but are necessary to derive the eigenvalue equation for the step-index fiber. Matching boundary conditions requires that we know the azimuthal field components E_ϕ and H_ϕ, in addition to the longitudinal components E_z and H_z. We can get the azimuthal components in terms of E_z from Maxwell's equations:

$$
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu\frac{\partial \mathbf{H}}{\partial t} = -\mu j\omega\mathbf{H} \tag{5.20}
$$

Expanding the $\nabla \times \mathbf{E}$ terms in cylindrical components, and then collecting terms, the field components H_r, H_ϕ, E_r, and E_ϕ can be described [3] in terms of the longitudinal components:

$$
\begin{aligned}
E_\phi &= \frac{-j}{\alpha^2}\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial r}\right) \\
E_r &= \frac{-j}{\alpha^2}\left(\frac{\mu\omega}{r}\frac{\partial H_z}{\partial \phi} + \beta\frac{\partial E_z}{\partial r}\right) \\
H_\phi &= \frac{-j}{\alpha^2}\left(\omega\epsilon\frac{\partial E_z}{\partial r} + \frac{\beta}{r}\frac{\partial H_z}{\partial \phi}\right) \\
H_r &= \frac{-j}{\alpha^2}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\omega\epsilon}{r}\frac{\partial E_z}{\partial \phi}\right)
\end{aligned}
\tag{5.21}
$$

where α^2 stands for $k_0^2 n^2 - \beta^2$. Note that α^2 is a positive quantity in the core, and a negative quantity in the cladding for allowed values of β.

Using the longitudinal fields described in Equation 5.19, the field components in Equations 5.21 can be exactly calculated. In the core region ($r < a$) we get

$$E_r = \frac{-j\beta}{\kappa^2}\left[A\kappa J'_\nu(\kappa r) + \frac{j\omega\mu\nu}{\beta r}BJ_\nu(\kappa r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$E_\phi = \frac{-j\beta}{\kappa^2}\left[\frac{j\nu}{r}AJ_\nu(\kappa r) - \frac{\omega\mu}{\beta}B\kappa J'_\nu(\kappa r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$H_r = \frac{-j\beta}{\kappa^2}\left[B\kappa J'_\nu(\kappa r) - \frac{j\omega\epsilon_{core}\nu}{\beta r}AJ_\nu(\kappa r)\right]e^{j\nu\phi}e^{-j\beta z} \tag{5.22}$$

$$H_\phi = \frac{-j\beta}{\kappa^2}\left[\frac{j\nu}{r}BJ_\nu(\kappa r) + \frac{\omega\epsilon_{core}}{\beta}A\kappa J'_\nu(\kappa r)\right]e^{j\nu\phi}e^{-j\beta z}$$

where $J'_\nu(\kappa r) = dJ_\nu(\kappa r)/d(\kappa r)$.

In the cladding region ($r > a$) we get

$$E_r = \frac{j\beta}{\gamma^2}\left[C\gamma K'_\nu(\gamma r) + \frac{j\omega\mu\nu}{\beta r}DK_\nu(\gamma r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$E_\phi = \frac{j\beta}{\gamma^2}\left[\frac{j\nu}{r}CK_\nu(\gamma r) - \frac{\omega\mu}{\beta}D\gamma K'_\nu(\gamma r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$H_r = \frac{j\beta}{\gamma^2}\left[D\gamma K'_\nu(\gamma r) - \frac{j\omega\epsilon_{clad}\nu}{\beta r}CK_\nu(\gamma r)\right]e^{j\nu\phi}e^{-j\beta z} \tag{5.23}$$

$$H_\phi = \frac{j\beta}{\gamma^2}\left[\frac{j\nu}{r}DK_\nu(\gamma r) + \frac{\omega\epsilon_{clad}}{\beta}C\gamma K'_\nu(\gamma r)\right]e^{j\nu\phi}e^{-j\beta z}$$

where $K'_\nu(\gamma r) = dK_\nu(\gamma r)/d(\gamma r)$.

5.5 BOUNDARY CONDITIONS FOR THE STEP-INDEX WAVEGUIDE

To determine the propagation constant β and the amplitude coefficients A, B, C, and D of Equation 5.19, we need to apply the boundary conditions. The boundary conditions at $r = a$ require that the four tangential components E_z, E_ϕ, H_z, and H_ϕ be continuous at the core-cladding boundary. For example, the longitudinal electric field must satisfy $AJ_\nu(\kappa a)e^{j\nu\phi}e^{-j\beta z} = CK_\nu(\gamma a)e^{j\nu\phi}e^{-j\beta z}$. The simplest way to simultaneously satisfy all four boundary-value equations is to write the four linear equations in matrix form, and then set the determinant of the matrix equal to zero.

$$\begin{bmatrix} J_\nu(\kappa a) & 0 & -K_\nu(\gamma a) & 0 \\ 0 & J_\nu(\kappa a) & 0 & -K_\nu(\gamma a) \\ \frac{\beta\nu}{a\kappa^2}J_\nu(\kappa a) & j\frac{\omega\mu}{\kappa}J'_\nu(\kappa a) & \frac{\beta\nu}{a\gamma^2}K_\nu(\gamma a) & j\frac{\omega\mu}{\gamma}K'_\nu(\gamma a) \\ -j\frac{\omega\epsilon_{core}}{\kappa}J'_\nu(\kappa a) & \frac{\beta\nu}{a\kappa^2}J_\nu(\kappa a) & -j\frac{\omega\epsilon_{clad}}{\gamma}K'_\nu(\gamma a) & \frac{\beta\nu}{a\gamma^2}K_\nu(\gamma a) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0 \tag{5.24}$$

For nontrivial solutions (i.e., nonzero amplitudes), the four equations will simultaneously equal zero if and only if the determinant of the matrix equals zero. Expansion of the determinant yields the "characteristic equation" for the step-index fiber.

$$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right] \tag{5.25}$$

This formidable equation requires numerical or graphical solution. There is only one unknown: β. As with the slab waveguide, the terms κ and γ are functions of β and the local index. Due to the oscillatory nature of $J_\nu(\kappa a)$, there can be several values of β for a given structure. Since there are two dimensional degrees of freedom in the cylindrical waveguide, solutions to the wave equation are labeled with two indices, ν and m. Both numbers are integers. The m value is called the *radial mode number*, and represents the number of radial nodes that exist in the field distribution. The integer ν is called the *angular mode number*, and represents the number of angular nodes that exist in the field distribution.

Once β is determined from Equation 5.25, three of the coefficients (A, B, C, and D) can be determined in terms of the fourth by solving the individual equations of the matrix. For example, from the boundary condition for continuity of E_z at $r = a$,

$$AJ_\nu(\kappa a) = CK_\nu(\gamma a) \tag{5.26}$$

One can solve for coefficient C in terms of A

$$C=\frac{J_\nu(\kappa a)}{K_\nu(\gamma a)}A \tag{5.27}$$

Similarly D can be solved in terms of B

$$D=\frac{J_\nu(\kappa a)}{K_\nu(\gamma a)}B \tag{5.28}$$

The coefficients A and B can be related to one another using the continuity of E_ϕ or H_ϕ, and Equations 5.27 and 5.28. Using the electric field continuity, one gets

$$B=\frac{j\nu\beta}{\omega\mu a}\left[\frac{1}{\kappa^2}+\frac{1}{\gamma^2}\right]\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]^{-1}A \tag{5.29}$$

If the magnetic field continuity is used, one gets

$$B=\frac{j\omega a}{\beta\nu}\left[\frac{n_{core}^2}{\kappa}\frac{J_\nu'(\kappa a)}{J_\nu(\kappa a)}+\frac{n_{clad}^2}{\gamma}\frac{K_\nu'(\gamma a)}{K_\nu(\gamma a)}\right]\left[\frac{1}{\kappa^2}+\frac{1}{\gamma^2}\right]^{-1}A \tag{5.30}$$

The choice of which equation to use depends on the type of mode carried in the waveguide. This is explained in the next section. Note that B/A is purely imaginary in both cases, indicating that the two longitudinal fields are $\pi/2$ out of phase. On an instantaneous basis, there is radial power flow, but due to the $\pi/2$ phase shift the power is reactive, so it averages to zero.

5.6 THE SPATIAL MODES OF THE STEP-INDEX WAVEGUIDE

Unlike the slab waveguide with only two possible types of mode (TE or TM), the circular waveguide has four types of mode. The quantity $|B/A|$ is of particular interest in determining the relative size of the longitudinal components of the E and H fields. These, in turn, characterize the type of mode. We will start with the simplest mode.

5.6.1 Transverse Electric and Transverse Magnetic Modes

Consider the characteristic equation (Equation 5.25) for the case where $\nu = 0$. Since ν represents angular dependence of the solution, the field solutions to E_z when $\nu = 0$ will be rotationally invariant. The equation simplifies to

$$\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right] = 0 \qquad (5.31)$$

Either term on the left-hand side can be set to zero to satisfy the equation. The two terms in Equation 5.31 appeared individually in Equations 5.29 and 5.30, where the amplitude A was related to amplitude B. If the first term of Equation 5.31 is set to zero, then A must also be zero to keep the magnitude of B in Equation 5.29 finite. If $A = 0$, then $E_z = 0$, and the electric field will be transverse. Such modes are called TE modes.

Conversely, if the second term in Equation 5.31 is zero, then the amplitude B will be zero (see Equation 5.30), and the longitudinal component of the H field will be zero. The solution will therefore be a TM mode. Thus, if $\nu = 0$, the allowed modes will be either TE or TM.

The problem of finding the allowed values of the propagation vector β reduces to finding the roots of Equation 5.31. These equations for the TE and TM modes can be further simplified using the Bessel function relations (see Appendix B):

$$\frac{J'_\nu}{\kappa J_\nu} = \pm\frac{J_{\nu\mp 1}}{\kappa J_\nu} \mp \frac{\nu}{\kappa^2}$$
$$\frac{K'_\nu}{\gamma K_\nu} = \mp\frac{K_{\nu\pm 1}}{\gamma K_\nu} \mp \frac{\nu}{\gamma^2} \qquad (5.32)$$

Consider first the TE mode. The first term of Equation 5.31 should be set equal to zero. Using the relation in Equation 5.32, the eigenvalue equation for TE modes becomes

$$-\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)} - \frac{K_1(\gamma a)}{\gamma K_0(\gamma a)} = 0 \qquad (5.33)$$

The other half of Equation 5.31 is the eigenvalue equation for TM modes. These can be solved numerically or graphically. We will use *Mathematica* to do both in the following example of a TE mode.

Example 5.1 Eigenvalues for the TE Modes in a Step-Index Fiber

Let's analyze a step-index circular fiber with a core index $n_{core} = 1.5$, a cladding index $n_{clad} = 1.45$, and a core radius $a = 5\mu$m. The wavelength of the light is 1.3μm. We want to determine the allowed eigenvalues for β for the TE modes. A simple *Mathematica* command evaluates and plots the two terms in Equation 5.33.

```
k=2 Pi /(1.3 10^(-4));
a=5 10^(-4);
n1=1.5;
n2=1.45;
kappamax=Sqrt[k^2(n1^2-n2^2)];
gamma = Sqrt[ kappamax^2-kappa^2];
Plot[{BesselJ[1, kappa a]/(kappa BesselJ[0, kappa a]), -BesselK[1, gamma a]/
     (gamma BesselK[0, gamma a])}, {kappa, 0, kappamax}]
```

The graphical output is presented in Figure 5.6. As in previous chapters, we chose to plot the functions against the transverse wavevector κ, instead of against β. The plot extends from $\kappa = 0$ to κ_{max}, which is given by

$$\kappa_{max} a = \sqrt{k_0^2 n_{core}^2 - k_0^2 n_{clad}^2}\, a \tag{5.34}$$

The $J_1(\kappa a)/\kappa J_0(\kappa a)$ term explodes to infinity at every root of $J_0(\kappa a)$. Since the roots of $J_0(\kappa a)$ occur (almost) periodically, the ratio J_1/J_0 regularly sweeps from $-\infty$ to $+\infty$. The K_1/K_0 term monotonically decreases as κ increases.

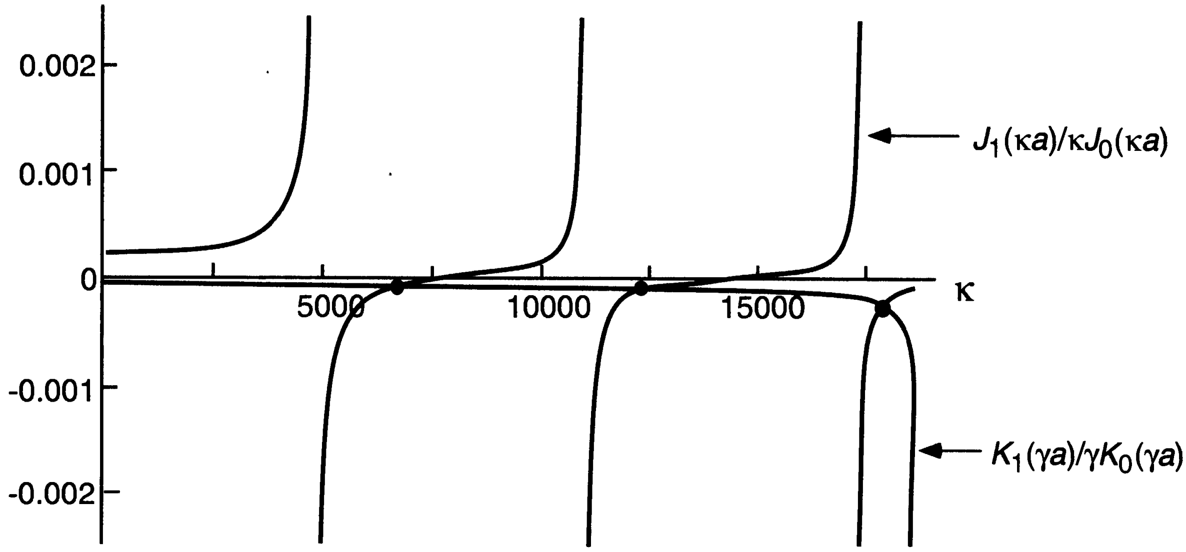


Figure 5.6 The eigenvalue equation is plotted against κ for a waveguide with core index 1.5, cladding index 1.45, and wavelength 1.3μm.

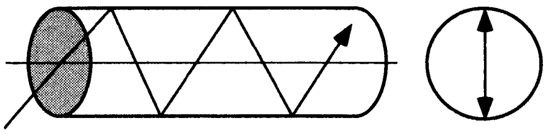
Every time the two lines cross in Figure 5.6, there is an allowed TE mode. In this case, three TE modes are allowed, with approximate κ values of 7,000, 12,500, and 17,500cm^{-1}. The exact values are easily found using a root-finding command. In *Mathematica*, the appropriate command is

```
FindRoot[-BesselK[1, gamma a]/(gamma BesselK[0, gamma a])==
```

```
BesselJ[1,kappa a]/(kappa BesselJ[0, kappa a]), {kappa, 5200}]
```

The exact values for this example are $\kappa = 6{,}902$, 12,549, and 17,795cm^{-1}. The corresponding values of β can be determined from Equation 5.14.

The transverse modes (TE and TM) have no azimuthal structure ($\nu = 0$). We will look at the field solutions in a later section, but in the ray picture these modes are geometrically represented by meridional rays. As seen in Figure 5.7, the ray associated with these modes travels through the origin, $r = 0$.



Meridional Ray (TE or TM Modes)

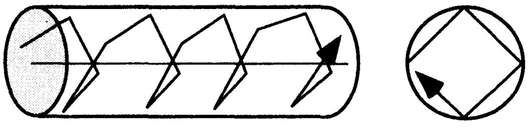
Figure 5.7 A meridional ray zigzags down the fiber, passing through the origin. There is no angular rotation of the ray path as it propagates.

5.6.2 The Hybrid Modes

When $\nu \neq 0$, the characteristic equation is a little more complicated to solve. The values of β will correspond to modes which have finite components of both E_z and H_z, and are therefore neither TE nor TM modes. These modes are called EH or HE modes, depending on the relative magnitude of the longitudinal E and H components [4, 5].

if $A = 0$	then the mode is called a	TE mode
if $B = 0$	then the mode is called a	TM mode
if $A > B$	then the mode is called an	HE mode (E_z dominates H_z)
if $A < B$	then the mode is called an	EH mode (H_z dominates E_z)

The EH and HE modes are called *hybrid* modes because they have both longitudinal H and E components in the waveguide. The EH and HE modes exist only for $\nu \geq 1$, so they have azimuthal structure. In the ray picture, these modes are called *skew* rays, because they travel down the waveguide in a screw-like pattern (Figure 5.8), glancing off the interface as they spiral down the axis. The azimuthal structure is apparent from the cyclical path of the ray.



Skew Ray (EH or HE Mode)

Figure 5.8 A skew ray travels in a spiral path down the fiber. The ray does not go through the origin.

The EH and HE modes have complicated field patterns. These patterns are not only difficult to determine, but they are hard to visualize. Because of this, and the limited utility derived in actually graphing such distributions, we will not pursue their description. Instead, the next subsection develops a useful approximation that simplifies both the calculation and visualization of the hybrid modes.

5.6.3 The Linearly Polarized Modes (LP Modes)

The characteristic equation for the hybrid modes is difficult to solve for β. Fortunately, a very simple and reasonable approximation makes solution straightforward [6]. Consider again the characteristic equation, Equation 5.25:

$$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right] \left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right] \tag{5.25}$$

For $\nu = 1, 2, \ldots$, HE and EH modes are possible. Unfortunately, even with powerful software, finding the roots of this equation is very difficult. Dramatic simplification occurs if we make the *weakly guiding* approximation. For many practical optical fibers, the core and cladding index are nearly identical. Typical commercial fibers have $\Delta n = n_{core} - n_{clad}$ on the order of 0.001–0.005. In view of this, it is not unreasonable (at least for the purpose of finding roots) to approximate that the core and cladding index are identical, $n_{core} \approx n_{clad} = n$. This approximation will introduce an error on the order of less than 1 part per thousand in the actual value of the propagation vector, but will enable easy solution of the problem. In the weakly guiding approximation, Equation 5.25 reduces to

$$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]^2 k_0^2 n^2 \tag{5.35}$$

This can be further simplified, noting that if $n_{core} = n_{clad}$, then $\beta^2 = k_0^2 n^2$, and these terms can be canceled from both sides. Taking advantage of some Bessel function identities,

$$\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} = \pm\frac{J_{\nu\mp 1}}{\kappa a J_\nu(\kappa a)} \mp \frac{\nu}{\kappa^2 a} \quad \text{and} \quad \frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)} = \frac{K_{\nu\pm 1}(\gamma a)}{\gamma a K_\nu(\gamma a)} \mp \frac{\nu}{\gamma^2 a} \tag{5.36}$$

simplifies Equation 5.35, leaving only

$$\frac{J_{\nu\pm1}(\kappa a)}{\kappa J_\nu(\kappa a)} = \mp\frac{K_{\nu\pm1}(\gamma a)}{\gamma K_\nu(\gamma a)} \tag{5.37}$$

These are the characteristic equations for the EH (top sign) and HE (bottom sign) modes. Solution will yield the eigenvalues for the allowed modes. A little more manipulation with Bessel function identities reduces these two equations into one single equation [7]:

$$\kappa\frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\gamma\frac{K_{j-1}(\gamma a)}{K_j(\gamma a)} \tag{5.38}$$

The indices define the mode as follows:

$j = 1$	TE, TM modes
$j = \nu + 1$	EH_ν modes
$j = \nu - 1$	HE_ν modes

More than one mode has the same eigenvalue, or, mathematically speaking, different modes are degenerate. In the weakly guiding approximation, the TE_{0m} is degenerate with the TM_{0m} mode. These modes will have the same eigenvalue, β, and will propagate at the same velocity (at least to the accuracy of the weakly guiding approximation). Also, the $HE_{\nu+1,m}$ modes and $EH_{\nu-1,m}$ modes are degenerate.

Since degenerate modes travel at the same velocity, this degeneracy in β makes it possible to define stable superpositions of different modes. Certain combinations of degenerate modes can be found which are *linearly polarized.* Furthermore, the superpositions are primarily transverse, meaning E_z is negligible. This is best illustrated by example. We will take a back door approach to creating a superposition that leads to a linearly polarized mode by initially assuming that a mode has a *transverse* field configuration, and then deriving what the *longitudinal* mode structure must be.

5.7 THE NORMALIZED FREQUENCY (*V* NUMBER) AND CUTOFF

Often we are concerned whether a given mode will propagate within a fiber. For example, we might need a single mode fiber for an experiment using a visible laser, such as the HeNe laser operating at $\lambda = 633$nm, but all we can find is single-mode fiber that is designed for operation at 1.3μm. How can we determine if this fiber will be satisfactory? To answer this, we need to develop what are known as *cutoff* conditions, which determine under what circumstances a mode will propagate in a fiber.

The characteristic equation (Equation 5.25) contains a term with the ratio of Bessel functions, $J_{\nu\pm1}/J_\nu$. This term explodes to infinity at each root of J_ν. This was demonstrated in Example 5.1. To ensure that there is at least one solution to the equation, the argument κa must extend beyond the first of these roots. Each time κa increases beyond another root of $J_{\nu\pm1}/J_\nu$, another mode will be allowed. The roots of the Bessel functions are good signposts for establishing mode cutoff conditions.

We can generalize the cutoff conditions for the modes in terms of the roots of the appropriate Bessel function. For example, referring back to Figure 5.6, it is clear that no TE mode will exist if $\kappa a < 2.405$. The TE_{01} can only exist if $\kappa a > 2.405$, so we say that the cutoff condition for the TE_{01} mode is $\kappa a = 2.405$. The cutoff condition for the TE_{02} mode occurs at the second root of the Bessel function, $J_0(\kappa a)$, which occurs 5.520. The cutoff conditions for every variety of mode can be found in a similar fashion. These cutoff conditions are:

TE_{0m} modes	$\kappa a > m^{th}$ root of $J_0(\kappa a)$
HE_{1m} mode	$\kappa a > m^{th}$ root of $J_1(\kappa a)$
$EH_{\nu m}$ mode	$\kappa a > m^{th}$ root of $J_\nu(\kappa a)$ with the added constraint that the first root is not 0
$HE_{\nu m}$ modes	$\left(\frac{\epsilon_{core}}{\epsilon_{clad}} + 1\right) J_{\nu-1}(\kappa a) = \frac{\kappa a}{\nu - 1} J_\nu(\kappa a)$

Figure 5.11(a) shows a plot of the first three Bessel functions, with notations on the cutoff points for a few modes. For example, if κa is greater than 2.405, then

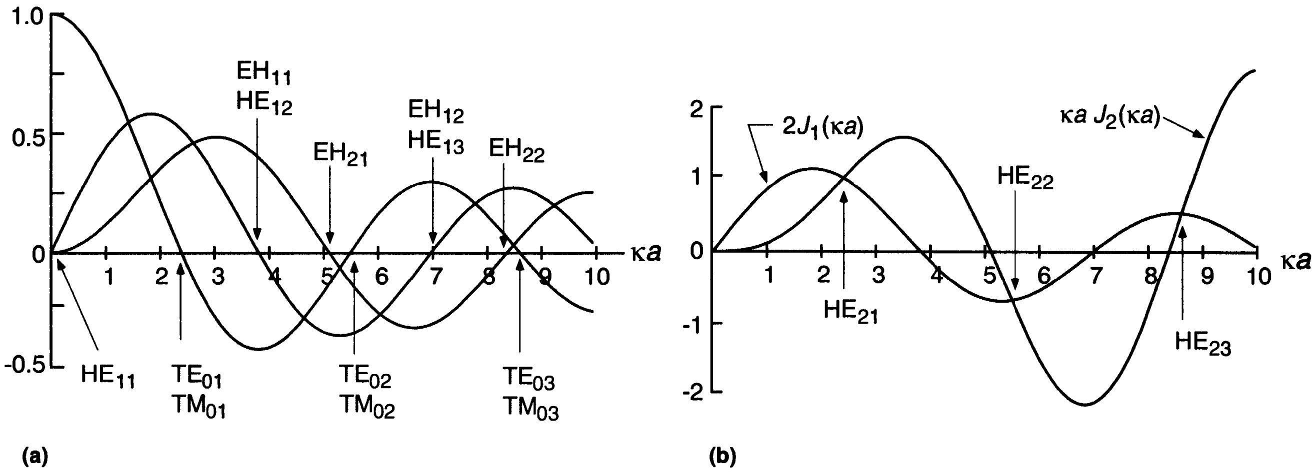


Figure 5.11 The first three J_ν Bessel functions are plotted, with the mode cutoff conditions of a few modes indicated at the various roots of the curves. The condition $2J_1(\kappa a) = \kappa a J_2(\kappa a)$ is plotted for the HE_{2m} mode cutoff conditions. Cutoff occurs where the curves cross.

the TE_{01}, TM_{01}, and HE_{21} modes will be allowed. This is in addition to the HE_{11} mode, which is always allowed. The HE_{11} mode is a special case which is described in the next section. The $HE_{\nu m}$ modes have a complicated cutoff formula which requires knowledge of the refractive indices of the core and cladding. In most cases, the ratio can be approximated as unity. Figure 5.11(b) shows the cutoff conditions for the HE_{2m} modes.

The parameter used to characterize a waveguide is the *normalized frequency* or the *V number.* For a cylindrical fiber, the V number is defined as $\kappa_{max}a$.

$$V\text{ number} = ak_0\sqrt{n_{core}^2 - n_{clad}^2} = \frac{2\pi a}{\lambda}\sqrt{n_{core}^2 - n_{clad}^2} \tag{5.50}$$

where a is the core radius. The normalized frequency provides a quick way to determine the number of modes in a waveguide, and is often used as a specification for optical fibers and devices. The cutoff conditions can all be evaluated once the V number of a fiber is given.

Example 5.3 Number of TE Modes in a Step-Index Fiber

Consider a step-index fiber that has a core index $n_{core} = 1.45$, a cladding index $n_{clad} = 1.44$, and a core radius of 25μm. If the excitation wavelength is 1.5μm, how many TE and TM modes will exist in the waveguide?

Solution:

First calculate the normalized frequency for the fiber:

$$\begin{aligned} V &= \frac{2\pi\, 25\mu\text{m}}{1.5\mu\text{m}}\sqrt{1.45^2 - 1.44^2} \\ &= 17.802 \end{aligned} \tag{5.51}$$

The zeros of the $J_0(\kappa a) = 0$ occur at 2.405, 5.520, 8.654, 11.791, 14.931, 18.071, etc. (See Appendix B, "Bessel Functions".) Clearly, V is larger than the first five roots, but is smaller than the sixth root at 18.071. So five TE modes (and five TM modes) will be allowed in this waveguide at that wavelength.

The V number is useful for determining cutoff conditions, as well as a number of other parameters such as the total number of allowed modes and power profiles. The V number is often specified in the purchase of optical single-mode fiber. For example, the cutoff condition for a single-mode fiber occurs when the V number reaches 2.405 (the first root of the J_0 Bessel function). The term *cutoff* refers to the point where the TE_{01}, TM_{01}, and HE_{21} modes cease to propagate if V becomes smaller. The wavelength at which a single-mode fiber suddenly becomes multimode is called the *cutoff* wavelength λ_c.

5.8 THE FUNDAMENTAL HE_{11} MODE

A mode which deserves special attention is the HE_{11} mode, sometimes called the fundamental mode or the LP_{01} mode. It has no cutoff condition; every step-index fiber will support at least this mode. The transverse field of the HE_{11} mode is described by the J_0 Bessel function (see Problem 5.11) in the core region, but because Bessel functions are not convenient to mathematically manipulate, the mode field distribution is often approximated by a Gaussian shape,

$$E(r) = E_0 \exp\left[-\left(\frac{r}{w}\right)^2\right] \tag{5.52}$$

The parameter w is adjusted to give the best fit between the actual Bessel function and the Gaussian approximation. For a fiber with a core radius of a, w is chosen to be [8]

$$\frac{w}{a} = 0.65 + 1.619V^{-\frac{3}{2}} + 2.87V^{-6} \tag{5.53}$$

This approximation provides a good overlap (better than 96 percent) between the Bessel solution and Gaussian function over the range from $0.8\lambda_c$ to $2\lambda_c$, where λ_c is the cutoff wavelength.

The amplitude profile for the HE_{11} mode is shown in Figure 5.12. The distance between the $1/e$ points of the amplitude profile define the *mode field diameter* (MFD), which is twice the mode field radius, w. When coupling between two single-mode waveguides, matching the MFD is a critical parameter to minimize loss. When the mode is not well described by a Gaussian parameter, definition of the MFD becomes less clear. Several techniques have been proposed, and are still being considered for standards [9].

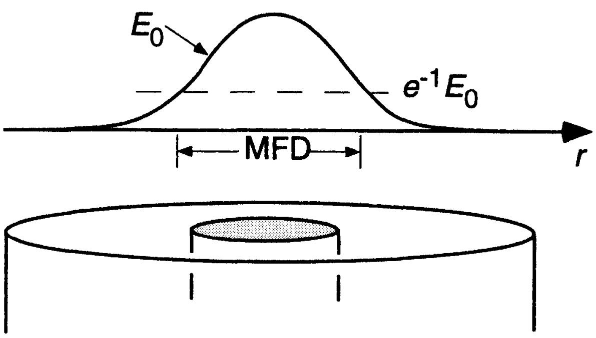


Figure 5.12 The electric field of the HE_{11} mode is transverse, and approximately Gaussian. The mode field diameter is determined by the points where the *power* is down by e^{-2}, or where the amplitude is down by e^{-1}. The MFD is not necessarily the same dimension as the core.

The cutoff wavelength defines the boundary between single-mode and multimode operation of a fiber. Wavelengths shorter than the cutoff wavelength can excite more than one spatial mode. The cutoff wavelength is defined in terms of the cutoff parameter for the onset of the TE and TM modes, namely $V = 2.405$,

$$\lambda_c = \frac{2\pi a}{2.405}\sqrt{n_{core}^2 - n_{clad}^2} \tag{5.54}$$

The HE_{11} mode can be polarized in any arbitrary direction in the x–y plane, so it has a degeneracy of two.

5.9 TOTAL NUMBER OF MODES IN A STEP-INDEX WAVEGUIDE

For large-core-diameter fibers with many modes, it is possible to provide an approximate formula describing the total number of modes that will propagate. Recall the characteristic equation for the LP modes (Equation 5.38):

$$\kappa\frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\gamma\frac{K_{j-1}(\gamma a)}{K_j(\gamma a)} \tag{5.38}$$

For values of κa far from cutoff, the term γa will be large, and the asymptotic value of the K_ν functions can be used. Since $K_j(\gamma a) \rightarrow \sqrt{\pi/2\gamma a}\, e^{-\gamma a}$ for large κa, the ratio K_{j-1}/K_j goes to unity for large arguments. The characteristic equation reduces to

$$\frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\frac{\gamma}{\kappa} \tag{5.55}$$

For a given value of ν, the number of allowed modes will be proportional to the number of roots of $J_j(\kappa a)$ between 0 and $\kappa a = V$. In the approximation that κa is large, the asymptotic expansion of $J_j(\kappa a)$ can be used:

$$J_j(\kappa a) \approx \sqrt{\frac{2}{\pi\kappa a}}\cos\left(\kappa a - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{5.56}$$

There will be one root every time the ratio goes to infinity, i.e., each time the argument increases by π. For a given value of ν, the number of roots will be approximately

$$m = \left(\kappa a - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\frac{1}{\pi} \tag{5.57}$$

Solving this in terms of the normalized frequency $V = \kappa_{max}a$, and ignoring the $\pi/4$ term,

$$V = \kappa_{max}a = (2m + \nu)\frac{\pi}{2} \tag{5.58}$$

This equation, while only an approximation, shows the general relationship between the azimuthal number ν and the number of radial nodes in the mode, m. As ν increases, indicating more angular lobes, the maximum value of m must decrease, implying that the radial structure becomes smoother.

Since there is an allowed mode for each value of m and ν, we can graphically plot the number of modes. The largest possible value for m, from Equation 5.58, is V/π when $\nu = 0$. Likewise, the maximum value for ν is $2V/\pi$. These allowed values are plotted in Figure 5.13.

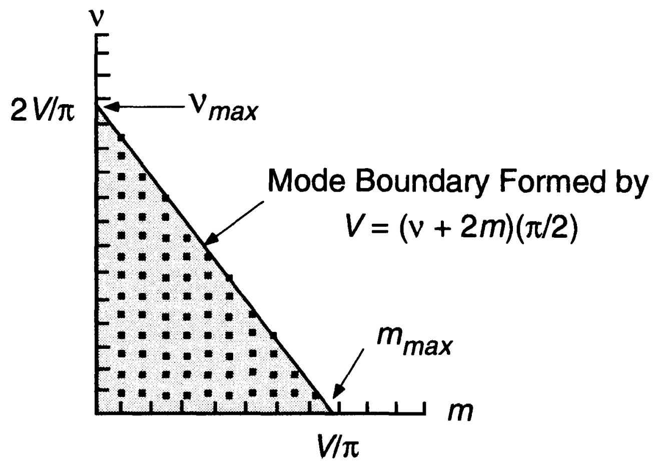


Figure 5.13 In this graphical plot of the allowed values of ν and m for the step-index fiber, the boundary is determined by the condition listed in Equation 5.58. The number of points beneath the curve is proportional to the area of the shaded region.

Each dot in the figure represents an allowed combination of m and ν. The total number of allowed modes is geometrically determined from the area of the triangle, which will be $(1/2)m_{max}\nu_{max} = V^2/\pi^2$. We must recall that, for each mode, there are two angular orientations (*cosine* or *sine* solution), and two possible polarizations (x or y in the LP mode approximation). The number of modes is increased by a factor of four. So the number of allowed modes in a fiber waveguide is given by the approximation

$$N = 4\frac{V^2}{\pi^2} \tag{5.59}$$

Again, we stress this formula is an approximation, and is only good when V is large.

5.10 POWER CONFINEMENT IN A STEP-INDEX FIBER

The electromagnetic energy of a mode is not totally confined within the core of the step-index fiber. A fraction of the total mode energy is carried in the evanescent field in the cladding. Calculation of the total power in a mode is determined by integration of the Poynting vector

$$\langle S_z \rangle = \frac{1}{2} Re(\mathbf{E} \times \mathbf{H}) \tag{5.60}$$

over the area of the waveguide. Substituting expressions for the electric and magnetic fields of the modes, the integrals can be evaluated. In the weakly guiding approximation, the relative core and cladding powers can be shown to be [6]

$$\frac{P_{core}}{P_{total}} = \left(1 - \frac{\kappa^2 a^2}{V^2}\right)\left[1 - \frac{K_\nu^2(\gamma a)}{K_{\nu+1}(\gamma a)K_{\nu-1}(\gamma a)}\right] \tag{5.61}$$

and

$$\frac{P_{clad}}{P_{total}} = 1 - \frac{P_{core}}{P_{total}} \tag{5.62}$$

where P_{total} is the total power of the mode carried by the waveguide. A plot of the fractional power carried in the core as a function of the V parameter for several modes is plotted in Figure 5.14. The two integers labeling each line correspond to the mode eigenvalue $\nu\ell$. As each mode approaches cutoff, the amount of power in the cladding dramatically increases; the mode becomes less confined near cutoff. Well above cutoff, most of the power is almost entirely contained in the core.

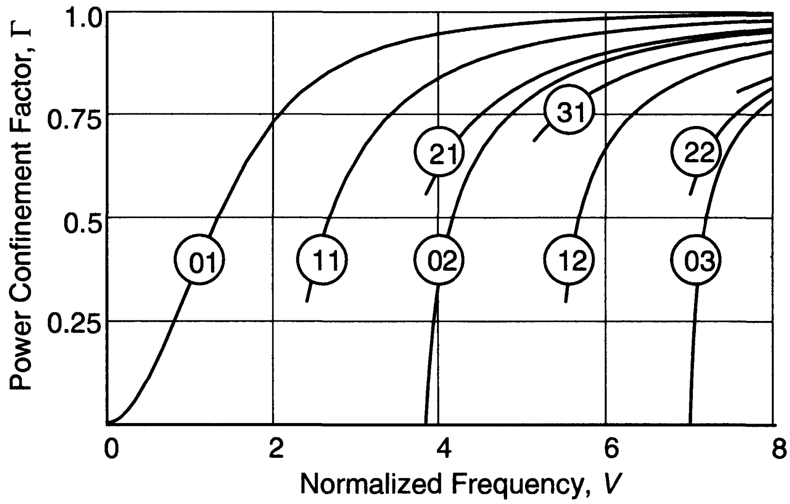


Figure 5.14 A graph of the fractional power of a mode that is carried in the core as a function of the fiber V number shows that, near cutoff, the fraction contained in the cladding increases dramatically. The modes are labeled in LP notation.

5.11 SUMMARY

In this chapter, we developed the fundamental concepts of the circular dielectric waveguide. Solution of the wave equation in cylindrical coordinates led to mode solutions in the form of cylinder functions such as $\sin\phi$ and the Bessel functions $J_\nu(\kappa r)$. As was found in planar waveguides, the propagation parameters β for the modes were found from solution of a transcendental equation, and the values of β were restricted to lie between $k_0 n_{core} > \beta > k_0 n_{clad}$. We developed explicit solutions to the longitudinal electric fields of the modes, and, using Maxwell's equations, we found expressions for all field components. The formal modes are complicated in terms of their field structure, so a picture based on the weakly guided mode approximation was developed which simplified both the characteristic equation for finding β, and the physical description of modes as linear superpositions which were linearly polarized.

We concluded the chapter with a number of short topics, such as the cutoff conditions, the V parameter, the number of modes, and the power confinement of the modes. One topic that was not discussed was dispersion, which is a very important topic for any long-distance optical waveguide system. We defer a complete discussion on dispersion in circular fibers until the next chapter, where graded-index fibers are described. The motivation for graded-index circular fibers, exactly as in the case of graded-index planar waveguides, is to reduce modal dispersion. Chapter 7 develops this important topic.