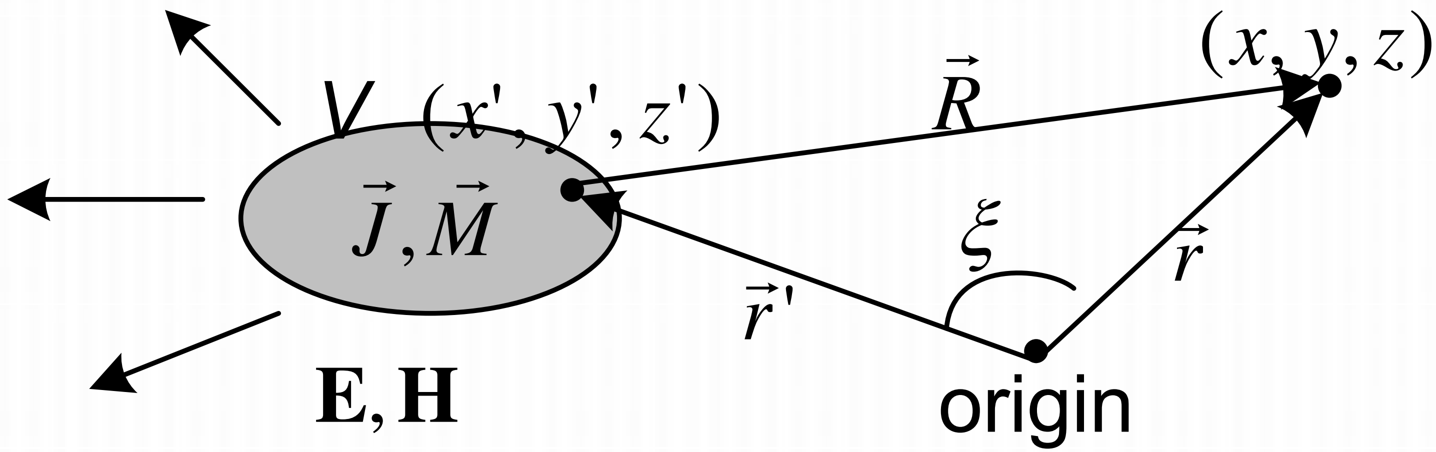
Chapter 3 : Vector potential and radiation integrals

- Vector potentials
- Far-field radiation
- Duality theorem

Vector Potentials

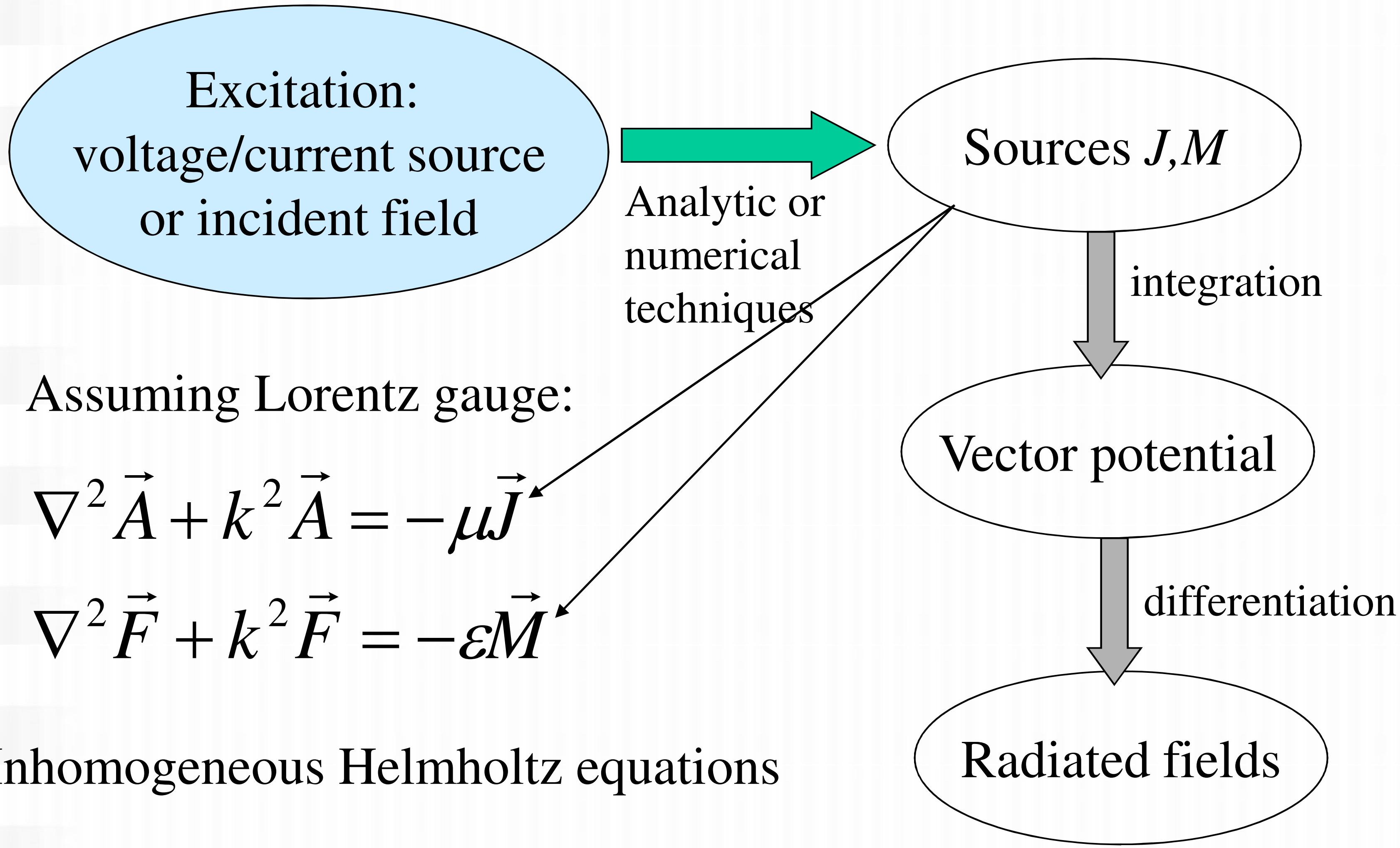


Problem : given electric and magnetic currents $(\vec{J}, \vec{M})$

find electric and magnetic fields $(\mathbf{E}, \mathbf{H})$

First, evaluate electric and magnetic vector potentials $(\vec{A}, \vec{F})$

Vector differential equations



Assuming Lorentz gauge:

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

Inhomogeneous Helmholtz equations

Solution for unbounded homogeneous regions

For volume currents:

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\iiint_V d\vec{A}(\vec{r}\,') = \frac{\mu}{4\pi}\iiint_V \vec{J}_V(\vec{r}\,')\frac{e^{-jkR}}{R}dv'$$

$$\vec{F}(\vec{r}) = \frac{\varepsilon}{4\pi}\iiint_V d\vec{F}(\vec{r}\,') = \frac{\varepsilon}{4\pi}\iiint_V \vec{M}_V(\vec{r}\,')\frac{e^{-jkR}}{R}dv'$$

$\vec{J}_V$ [A/m^2], $\vec{M}_V$ [V/m^2]

$$\vec{R} = \vec{r} - \vec{r}\,';\ \hat{R} = \frac{\vec{R}}{R};\ R = |\vec{R}| = (r^2 + r'^2 - 2rr'\cos\xi)^{1/2}$$

For surface currents:

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\iint_S d\vec{A}(\vec{r}\,') = \frac{\mu}{4\pi}\iint_S \vec{J}_S(\vec{r}\,')\frac{e^{-jkR}}{R}ds'$$

$$\vec{F}(\vec{r}) = \frac{\varepsilon}{4\pi}\iint_s d\vec{F}(\vec{r}\,') = \frac{\varepsilon}{4\pi}\iint_S \vec{M}_S(\vec{r}\,')\frac{e^{-jkR}}{R}ds'$$

$\vec{J}_S$ [A/m], $\vec{M}_S$ [V/m]

Solution for unbounded homogeneous regions (2)

For line currents:

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\int_L d\vec{A}(\vec{r}\,') = \frac{\mu}{4\pi}\int_L \vec{J}_l(\vec{r}\,')\frac{e^{-jkR}}{R}dl'$$

$$\vec{F}(\vec{r}) = \frac{\varepsilon}{4\pi}\int_L d\vec{F}(\vec{r}\,') = \frac{\varepsilon}{4\pi}\int_L \vec{M}_l(\vec{r}\,')\frac{e^{-jkR}}{R}dl'$$

$\vec{J}_l$ [A],
$\vec{M}$ [V]

Once the vector potentials are found, the fields can be obtained from

$$\mathbf{E}(\vec{r}) = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{A}) - \frac{1}{\varepsilon}\nabla\times\vec{F} \qquad (1)$$

$$\mathbf{H}(\vec{r}) = \frac{1}{\mu}\nabla\times\vec{A} - j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{F}) \qquad (2)$$

Solution for unbounded homogeneous regions (3)

From (1) and (2), it can be shown that

$$\mathbf{E}(\vec{r}) = \frac{j\omega\mu}{4\pi}\int[\hat{R}\times\hat{R}\times\vec{J}(\vec{r}\,')]\left(1-\frac{j}{kR}-\frac{1}{(kR)^2}\right)\frac{e^{-jkR}}{R} \qquad (3)$$

$$-\frac{j\omega\mu}{2\pi}\int[\hat{R}(\hat{R}\cdot\vec{J}(\vec{r}\,')]\left(\frac{j}{kR}+\frac{1}{(kR)^2}\right)\frac{e^{-jkR}}{R}$$

$$\mathbf{H}(\vec{r}) = -\frac{jk}{4\pi}\int[\hat{R}\times\vec{J}(\vec{r}\,')]\left(1-\frac{1}{kR}\right)\frac{e^{-jkR}}{R} \qquad (4)$$

when there is only electric current source.

Far-field radiation

In far-field: $R > \max[3\lambda, \frac{2D^2}{\lambda}]$

the following approximation can be applied:

$$R \approx r \quad \text{for amplitude term;} \qquad \hat{R} \approx \hat{r}$$

$$R = (r^2 + r'^2 - 2rr'\cos\xi)^{1/2}$$

$$= r\left(1 + \left(\frac{r'}{r}\right)^2 - \frac{2rr'}{r^2}\cos\xi\right)^{1/2}$$

$$\approx r\left(1 - \frac{r'}{r}\cos\xi + \ldots\right)$$

$$\approx r - r'\cos\xi = r - \hat{r}\cdot\vec{r}' \qquad \text{for phase term}$$

Far-field radiation (2)

Retaining only 1/R terms in (3) and (4) yields

$$\mathbf{E}(\vec{r}) \approx \frac{j\omega\mu}{4\pi}\frac{e^{-jkr}}{r}\int[\hat{r}\times\hat{r}\times\vec{J}(\vec{r}\,')]e^{-jk\hat{r}\cdot\vec{r}\,'} \quad (5)$$

$$\mathbf{H}(\vec{r}) \approx -\frac{jk}{4\pi}\frac{e^{-jkr}}{r}\int[\hat{r}\times\vec{J}(\vec{r}\,')]e^{-jk\hat{r}\cdot\vec{r}\,'} \quad (6)$$

Note that in far-field, $\mathbf{H} = \dfrac{\hat{r}\times\mathbf{E}}{\eta}; \mathbf{E} = \eta\mathbf{H}\times\hat{r}$

and $\hat{r}\cdot\mathbf{E} = \hat{r}\cdot\mathbf{H} = 0 = \mathbf{E}\cdot\mathbf{H}$

i.e., electric field and magnetic field are perpendicular to each other and to the direction of propagation. Thus, Poynting vector becomes

$$\mathbf{W} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \hat{r}\frac{|\mathbf{E}|^2}{2\eta}$$

Table 3.1 DUAL EQUATIONS FOR ELECTRIC (J) AND MAGNETIC (M) CURRENT SOURCES

Electric Sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$)	Magnetic Sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$)
$\nabla \times \mathbf{E}_A = -j\omega\mu\mathbf{H}_A$	$\nabla \times \mathbf{H}_F = j\omega\epsilon\mathbf{E}_F$
$\nabla \times \mathbf{H}_A = \mathbf{J} + j\omega\epsilon\mathbf{E}_A$	$-\nabla \times \mathbf{E}_F = \mathbf{M} + j\omega\mu\mathbf{H}_F$
$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}$	$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\epsilon\mathbf{M}$
$\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-jkR}}{R}\,dv'$	$\mathbf{F} = \frac{\epsilon}{4\pi}\iiint_V \mathbf{M}\frac{e^{-jkR}}{R}\,dv'$
$\mathbf{H}_A = \frac{1}{\mu}\nabla \times \mathbf{A}$	$\mathbf{E}_F = -\frac{1}{\epsilon}\nabla \times \mathbf{F}$
$\mathbf{E}_A = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{A})$	$\mathbf{H}_F = -j\omega\mathbf{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{F})$

Table 3.2 DUAL QUANTITIES FOR ELECTRIC (**J**) AND MAGNETIC (**M**) CURRENT SOURCES

Electric Sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$)	Magnetic Sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$)
$\mathbf{E}_A$	$\mathbf{H}_F$
$\mathbf{H}_A$	$-\mathbf{E}_F$
$\mathbf{J}$	$\mathbf{M}$
$\mathbf{A}$	$\mathbf{F}$
ϵ	μ
μ	ϵ
k	k
η	$1/\eta$
$1/\eta$	η