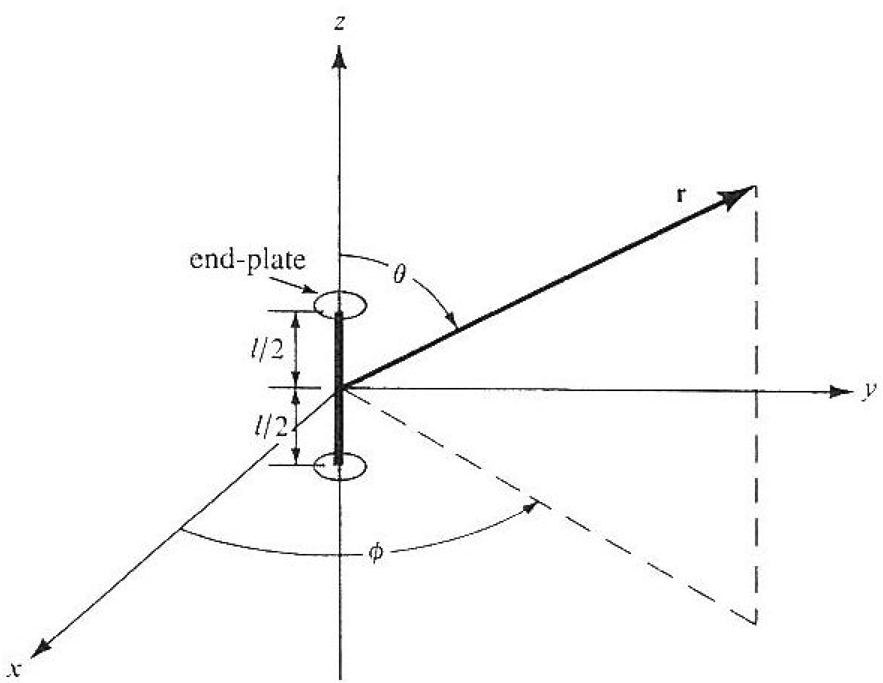
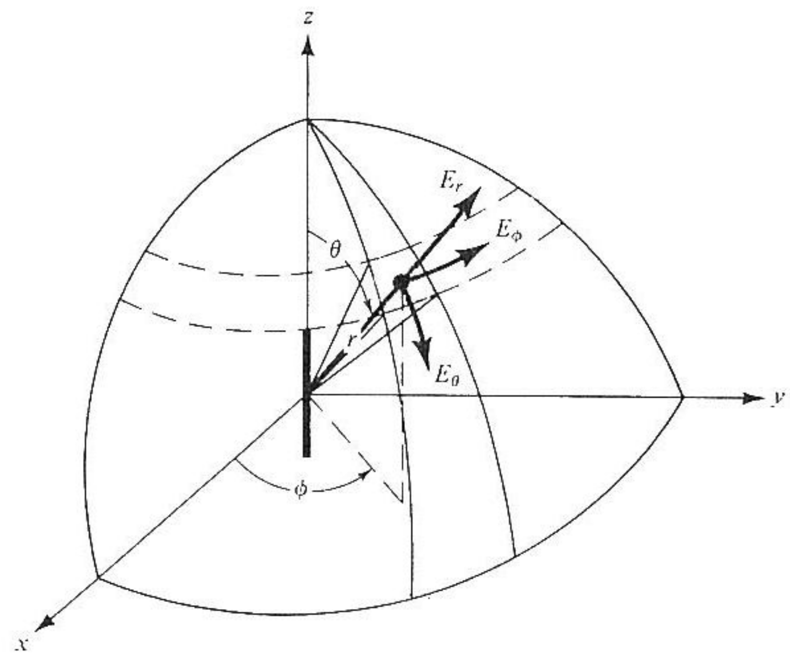
Chapter 4 : Linear Wire Antenna

- Infinitesimal Dipole
- Small Dipole
- Finite Length Dipole
- Half-Wavelength Dipole
- Linear Elements near or on Infinite Perfect Conductors

Infinitesimal Dipole

- Length $l << \lambda$
- Used to represent capacitor-plate (top-hat-loaded) antennas
- Capacitive loading to maintain the uniform current





Radiated Field

- **The current on the infinitesimal dipole is assumed to be constant, i.e.,**

$$\mathbf{I}(z') = \hat{z} I_0$$

- **Recall that**

$$\mathbf{A}(x, y, z) = \boxed{\frac{\mu}{4\pi} \int_C \mathbf{J}(x', y', z') \frac{e^{-jkR}}{R} dl'}$$

- **Since the dipole is infinitesimal, the following approximations hold:**

$$x' = y' = z' = 0 \qquad dl' = dz'$$

$$R = |\vec{r} - \vec{r}\,'| = r = \text{constant}$$

Radiated Field (2)

Thus

$$\mathbf{A}(x,y,z) = \hat{z}\frac{\mu I_0}{4\pi r}e^{-jkr}\int_{-l/2}^{l/2} dz' = \boxed{\hat{z}\frac{\mu I_0 l}{4\pi r}e^{-jkr}}$$

Since **A** has only z component,

$$A_r = A_z\cos\theta;\ A_\theta = -A_z\sin\theta; A_\phi = 0$$

and

$$\nabla\times\mathbf{A} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]$$

$$+\frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right],$$

The magnetic field becomes:

$$\boxed{\mathbf{H} = \hat{\phi}\frac{1}{\mu r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]}$$

Radiated Field (3)

Therefore,

$$H_r = H_\theta = 0$$

$$H_\phi = j\frac{kI_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

Likewise, the electric field can be found to be

$$E_r = \eta\frac{I_0 l \cos\theta}{2\pi r^2}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$E_\theta = j\eta\frac{kI_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}$$

$$E_\phi = 0$$

Power Density and Radiation Resistance

- **Poynting vector**

$$\mathbf{W} = \frac{1}{2}(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2}(\hat{r}E_r + \hat{\theta}E_\theta)\times(\hat{\phi}H_\phi^*)$$

$$= \frac{1}{2}(\hat{r}E_\theta H_\phi^* - \hat{\theta}E_r H_\phi^*)$$

Components of Poynting Vector

$$W_r = \frac{\eta}{8}\left|\frac{I_0 l}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}\left[1 - j\frac{1}{(kr)^3}\right]$$

$$W_\theta = j\eta\frac{k\,|I_0 l|^2\cos\theta\sin\theta}{16\pi^2 r^3}\left[1 + \frac{1}{(kr)^2}\right]$$

Outgoing Power

$$P = \oiint_S \mathbf{W}\cdot d\mathbf{s} = \int_0^{2\pi}\int_0^{\pi}(\hat{r}W_r + \hat{\theta}W_\theta)\cdot\hat{r}r^2\sin\theta d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} W_r r^2 \sin\theta d\theta d\phi = \eta\frac{\pi}{3}\left|\frac{I_0 l}{\lambda}\right|^2\left[1 - j\frac{1}{(kr)^3}\right]$$

Power Density and Radiation Resistance (2)

Complex Power

$$P = P_{rad} + j2\omega(\tilde{W}_m - \tilde{W}_e)$$

$P =$ power

$P_{rad} =$ time-average radiated power

$\tilde{W}_m =$ time-average magnetic energy

$\tilde{W}_e =$ time-average electric energy

$2\omega(\tilde{W}_m - \tilde{W}_e) =$ time-average imaginary (reactive) power; All in radial direction

$$P_{rad} = \eta \frac{\pi}{3}\left|\frac{I_0 l}{\lambda}\right|^2 = \frac{1}{2}|I_0|^2 R_r$$

$$R_r = \eta \frac{2\pi}{3}\left(\frac{l}{\lambda}\right)^2 = 80\pi^2\left(\frac{l}{\lambda}\right)^2$$

Radiation Resistance

$$2\omega(\tilde{W}_m - \tilde{W}_e) = -\eta \frac{\pi}{3}\left|\frac{I_0 l}{\lambda}\right|^2 \frac{1}{(kr)^3}$$

Reactive Power

Far Field (*kr* >> 1)

- **For *kr* >> 1, the fields can be approximated as**

$$\left.\begin{aligned} E_\theta &\cong j\eta \frac{kI_0 l \sin\theta}{4\pi r} e^{-jkr} \\ H_\phi &\cong j \frac{kI_0 l \sin\theta}{4\pi r} e^{-jkr} \\ E_r &\cong E_\phi = H_r = H_\theta = 0 \end{aligned}\right\} kr >> 1$$

→ **TEM Wave**

Ratio of E and H: $$Z_w = \frac{E_\theta}{H_\phi} = \eta$$

$$Z_w = \text{wave impedance}$$

$$\eta = \text{intrinsic impedance } (120\pi\ \Omega \text{ for free-space})$$

Directivity

Time-average power density:

$$\mathbf{W}_{av} = \frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = \hat{r}\frac{1}{2\eta}|E_\theta|^2 = \hat{r}\frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2 \frac{\sin^2\theta}{r^2}$$

Radiation intensity:

$$U = r^2 W_{av} = \frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2 \sin^2\theta = \frac{r^2}{2\eta}|E_\theta(r,\theta,\phi)|^2$$

Maximum radiation intensity:

$$U_{\max} = \frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2$$

Maximum directivity:

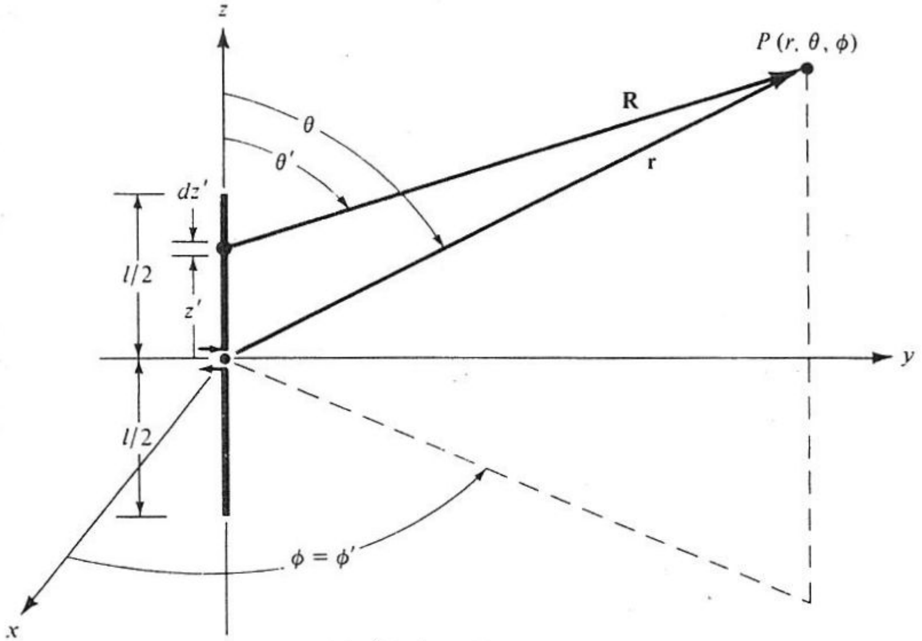
$$D_0 = 4\pi\frac{U_{\max}}{P_{rad}} = \frac{3}{2}$$

Maximum effective area:

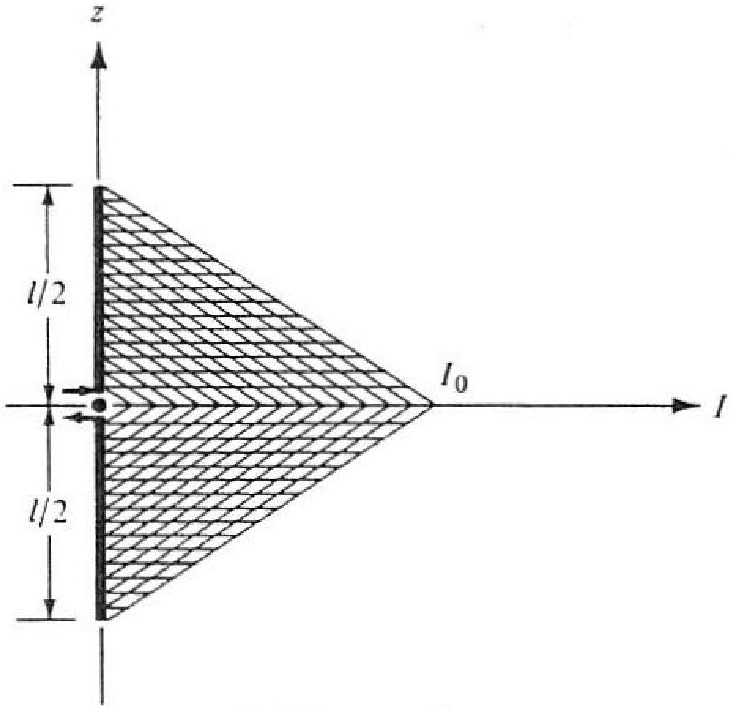
$$A_{em} = \frac{\lambda^2}{4\pi}D_0 = \frac{3\lambda^2}{8\pi}$$

Small Dipole

- **Length $\lambda/50 < l < \lambda/10$**



(a) Dipole and geometry



(b) Current distribution

Radiated Field

- **The current on the small dipole is assumed to be a triangular function, i.e.,**

$$\mathbf{I}(z') = \begin{cases} \hat{z} I_0 (1 - \frac{2}{l} z'), & 0 \le z' \le l/2 \\ \hat{z} I_0 (1 + \frac{2}{l} z'), & -l/2 \le z' \le 0 \end{cases}$$

where I_0 is a constant. Vector potential becomes:

$$\mathbf{A}(x, y, z) = \hat{z} \frac{\mu}{4\pi} \left[\int_{-l/2}^{0} I_0 (1 + \frac{2}{l} z') \frac{e^{-jkR}}{R} dz' + \int_{0}^{l/2} I_0 (1 - \frac{2}{l} z') \frac{e^{-jkR}}{R} dz' \right]$$

Approximating $R \sim r$ yields the maximum phase error $kl/2 = \pi/10$ for $l = \lambda/10$.

Radiated Field (2)

Using $R \sim r$:

$$\mathbf{A}(x, y, z) = \hat{z}A_z = \boxed{\hat{z}\frac{1}{2}\frac{\mu I_0 l}{4\pi r}e^{-jkr}}$$

which is one-half of that for the infinitesimal dipole.

The far-field can be given by

$$\left.\begin{aligned} E_\theta &\cong j\eta \frac{kI_0 l \sin\theta}{8\pi r}e^{-jkr} \\ H_\phi &\cong j\frac{kI_0 l \sin\theta}{8\pi r}e^{-jkr} \\ E_r &\cong E_\phi = H_r = H_\theta = 0 \end{aligned}\right\} kr >> 1$$

Radiation resistance:

$$R_r = \frac{2P_{rad}}{|I_0|^2} = 20\pi^2\left(\frac{l}{\lambda}\right)^2$$

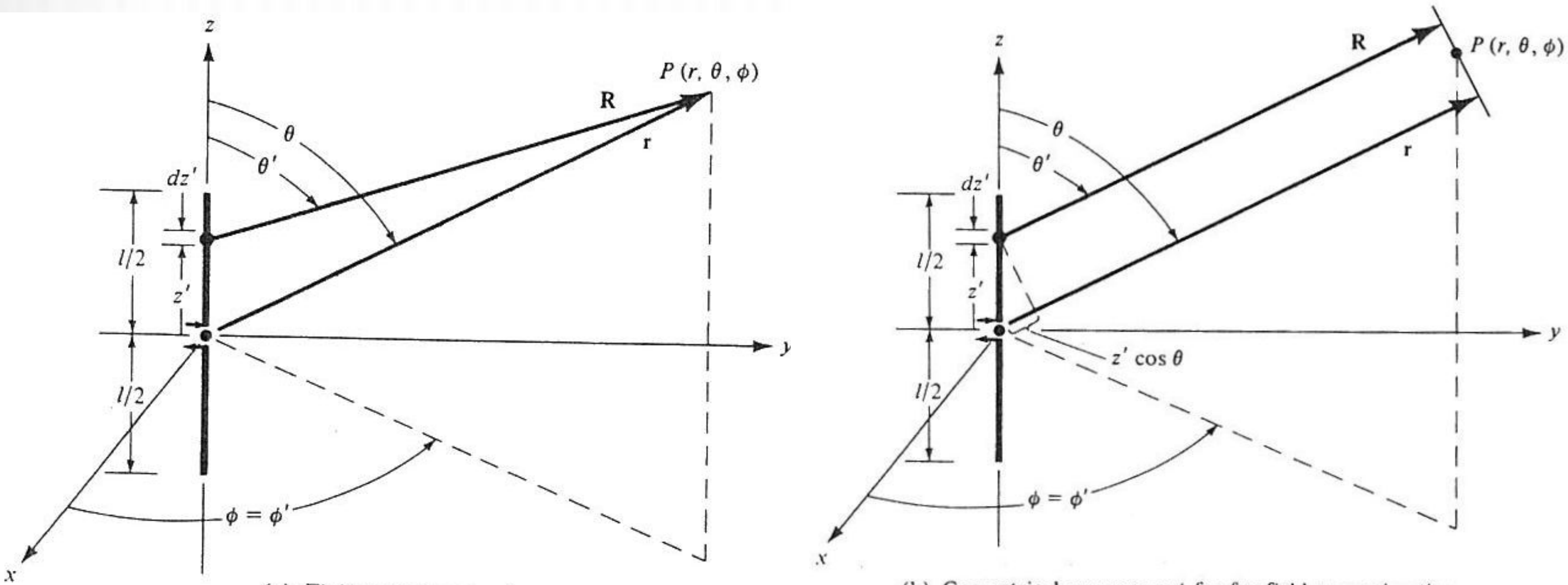
Field Separation

- **For a very thin dipole, *x'=y'*=0, thus**

$$R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}=\sqrt{x^2+y^2+(z-z')^2}$$

$$=\sqrt{(x^2+y^2+z^2)+(-2zz'+z'^2)}=\sqrt{r^2+(-2rz'\cos\theta+z'^2)}$$

where $r^2=x^2+y^2+z^2; z=r\cos\theta$



(a) Finite dipole geometry

(b) Geometrical arrangement for far-field approximations

Binomial Expansion

$$(a+b)^n = \frac{a^n b^0}{0!} + n\frac{a^{n-1}b^1}{1!} + n(n-1)\frac{a^{n-2}b^2}{2!} + n(n-1)(n-2)\frac{a^{n-3}b^3}{3!} + \cdots$$

$$(1+x)^n = 1 + n\frac{x}{1!} + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots$$

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^2}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{6} + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

Field Separation (2)

Recall also the Taylor expansion:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

which yields the same result:

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\cdots$$

Recall that

$$R = r\sqrt{1 + \frac{1}{r^2}(-2rz'\cos\theta + z'^2)}$$

Let $x = \dfrac{-2rz'\cos\theta + z'^2}{r^2}$, then

$$R = r - z'\cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right) + \frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right) + \cdots$$

Far Field

By retaining only the first two terms, i.e.,

$$R = r - z'\cos\theta$$

The most significant neglected term has the maximum value

$$\frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)_{\max} = \frac{z'^2}{2r} \quad \text{when } \theta = \frac{\pi}{2}$$

A maximum total phase error of π/8 is acceptable, thus

$$k\frac{z'^2}{2r} \le \frac{\pi}{8} \quad \xrightarrow{-l/2 \le z' \le l/2} \quad \boxed{r \ge \frac{2l^2}{\lambda}}$$

Far-field approximation

$$\text{when } r \ge \frac{2D^2}{\lambda} \quad \Longrightarrow \quad \begin{aligned} &R \approx r \ \text{ for amplitude term} \\ &R \approx r - r'\cos\theta = r - \hat{r}\cdot\vec{r}' \ \text{ for phase term} \end{aligned}$$

Radiating Near Field

By retaining only the first three terms, i.e.,

$$R = r - z'\cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)$$

The most significant neglected term is the fourth term. In order to find its maximum value, one can differentiate the fourth term with respect to θ, and the result is set to 0, i.e.,

$$\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right)\right] = \frac{z'^3}{2r^2}\sin\theta\left[-\sin^2\theta + 2\cos^2\theta\right] = 0$$

yields

$$\left[-\sin^2\theta + 2\cos^2\theta\right]_{\theta=\theta_1} = 0 \quad \Rightarrow \quad \theta_1 = \tan^{-1}(\pm\sqrt{2})$$

Radiating Near Field (2)

If the maximum total phase error is allowed to be π/8,

$$\left.\frac{kz'^3}{2r^2}\cos\theta\sin^2\theta\right|_{\substack{z'=l/2\\ \theta=\tan^{-1}\theta}} = \frac{\pi}{\lambda}\frac{l^3}{8r^2}\left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{3}\right) = \frac{\pi}{12\sqrt{3}}\left(\frac{l^3}{\lambda r^2}\right) \le \frac{\pi}{8}$$

which reduces to

$$r^2 \ge \frac{2}{3\sqrt{3}}\left(\frac{l^3}{\lambda}\right)$$

or

$$r \ge 0.62\sqrt{\frac{l^3}{\lambda}}$$

Near-field region

$$\frac{2D^2}{\lambda} \ge r \ge 0.62\sqrt{\frac{D^3}{\lambda}}$$

Finite Length Dipole

- Length > $\lambda/10$
- Current on the finite length dipole assuming the wire is very thin

$$\mathbf{I}_e(z') = \begin{cases} \hat{z} I_0 \sin\left[k\left(\frac{l}{2} - z'\right) \right], & 0 \le z' \le l/2 \\ \hat{z} I_0 \sin\left[k\left(\frac{l}{2} + z'\right) \right], & -l/2 \le z' \le 0 \end{cases}$$

where I_0 is a constant. This distribution assumes that the antenna is *center-fed* and the current vanishes at *the end points*.

Radiated Field

The electric and magnetic field components in the far field for the infinitesimal dipole *dz'* are given by

$$dE_\theta \cong j\eta \frac{kI_e(z')\sin\theta}{4\pi R} e^{-jkR} dz'$$

$$dH_\phi \cong j \frac{kI_e(z')\sin\theta}{4\pi R} e^{-jkR} dz'$$

$$dE_r \cong dE_\phi = dH_r = dH_\theta = 0$$

Using the far-field approximation yields

$$dE_\theta \cong j\eta \frac{kI_e(z')\sin\theta}{4\pi r} e^{-jkr} e^{jkz'\cos\theta} dz'$$

Radiated Field (2)

The total electric field can be obtained by summing up contributions from all infinitesimal dipoles, i.e.,

$$E_\theta = \int_{-l/2}^{l/2} dE_\theta = \underbrace{j\eta \frac{ke^{-jkr}\sin\theta}{4\pi r}}_{\text{element factor}} \underbrace{\left[\int_{-l/2}^{l/2} I_e(z')e^{jkz'\cos\theta} dz'\right]}_{\text{space factor}}$$

$$\boxed{\text{total field} = (\text{element factor}) \times (\text{space factor})}$$

Thus, the electric field of the finite length dipole can be given by

$$E_\theta = j\eta \frac{kI_0 e^{-jkr}\sin\theta}{4\pi r}\left\{\int_{-l/2}^{0} \sin\left[k\left(\frac{l}{2}+z'\right)\right] e^{jkz'\cos\theta} dz' + \int_{0}^{l/2} \sin\left[k\left(\frac{l}{2}-z'\right)\right] e^{jkz'\cos\theta} dz'\right\}$$

Radiated Field (3)

Using

$$\int e^{\alpha x}\sin(\beta x+\gamma)dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}[\alpha\sin(\beta x+\gamma)-\beta\cos(\beta x+\gamma)]$$

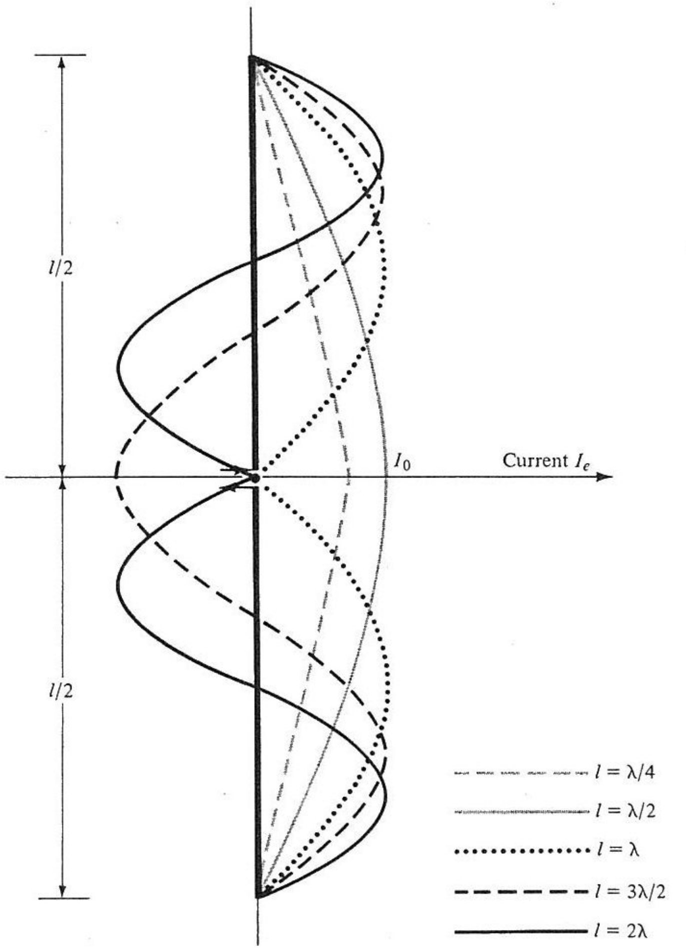
yields

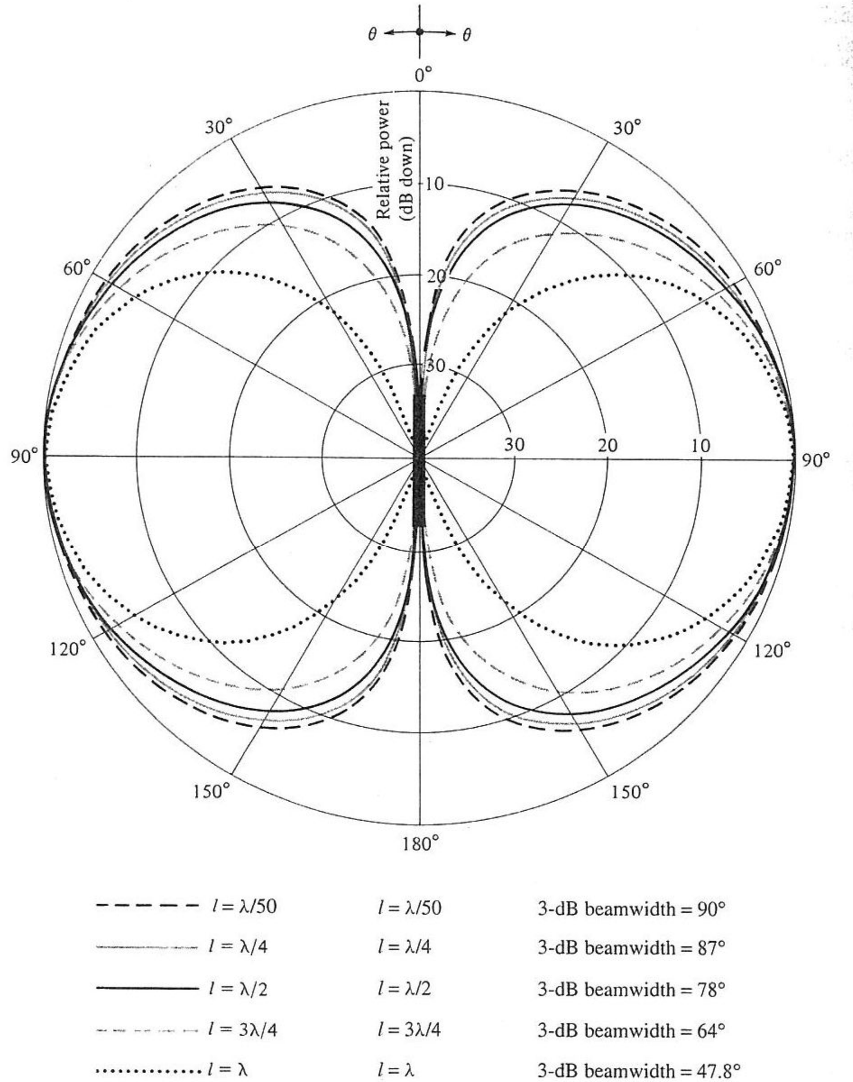
$$E_\theta \cong j\eta\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]$$

Likewise, the magnetic field can be given by

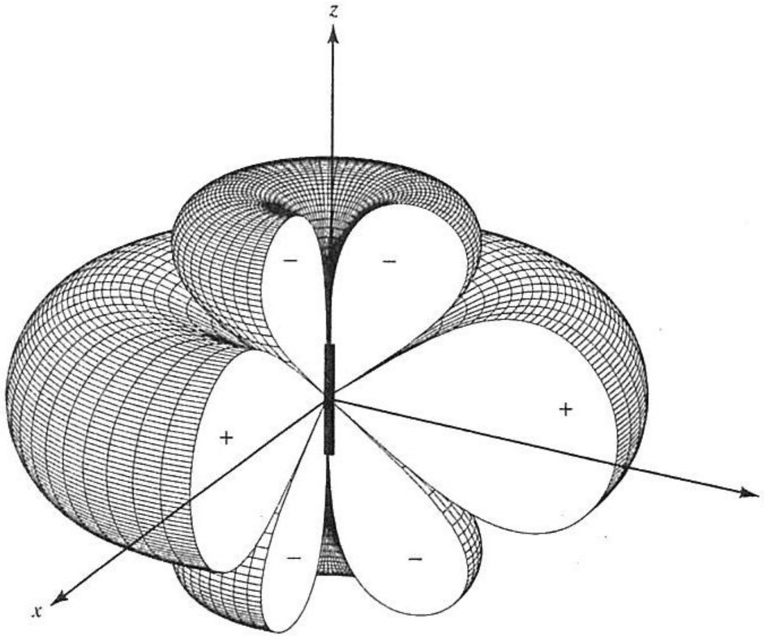
$$H_\phi \cong \frac{E_\theta}{\eta} = j\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]$$

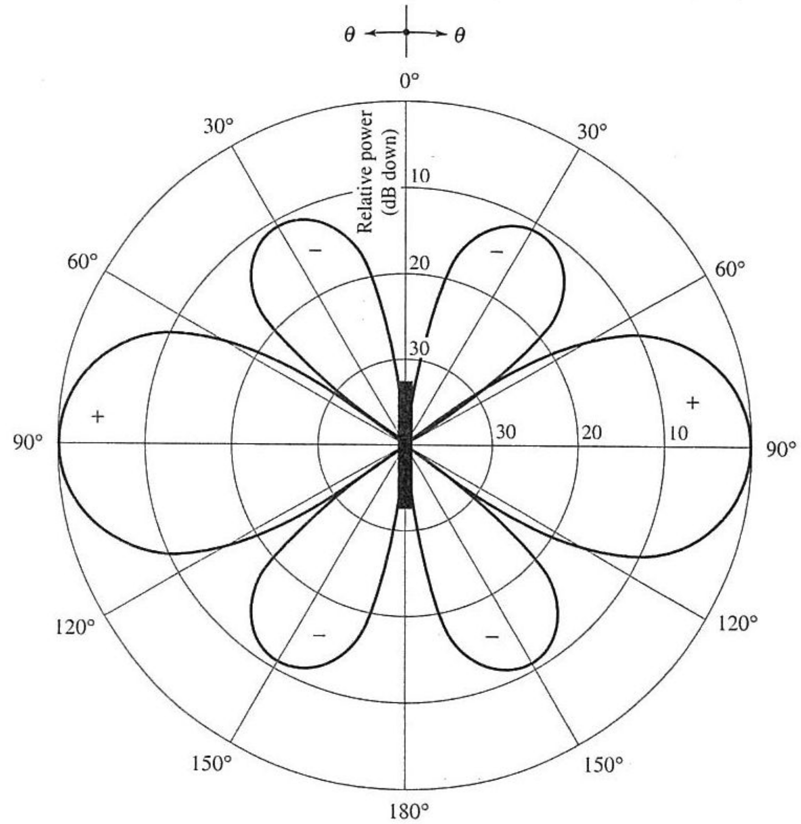
Current Distributions and Radiation Pattern





Radiation Pattern for l=1.25λ





Power Density and Radiation Intensity

Time-average power density:

$$\mathbf{W}_{av} = \frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2}\mathrm{Re}(\hat{\theta}E_\theta \times \hat{\phi}H_\phi^*)$$

$$= \hat{r}\frac{1}{2\eta}|E_\theta|^2 = \hat{r}\eta\frac{|I_0|^2}{8\pi^2 r^2}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]^2$$

Radiation intensity:

$$U = r^2 W_{av} = \eta\frac{|I_0|^2}{8\pi^2}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]^2$$

Radiated Power

Radiated power can be obtained by

$$P_{rad} = \oiint_{\Omega} U d\Omega = \int_0^{2\pi}\int_0^{\pi} U \sin\theta d\theta d\phi$$

$$= \eta \frac{|I_0|^2}{4\pi}\int_0^{\pi} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]^2}{\sin\theta} d\theta$$

which can be given by

$$P_{rad} = \eta \frac{|I_0|^2}{4\pi}\left\{C + \ln(kl) - C_i(kl) + \frac{1}{2}\sin(kl)[S_i(2kl) - 2S_i(kl)] + \frac{1}{2}\cos(kl)[C + \ln(kl/2) + C_i(2kl) - 2C_i(kl)]\right\} \quad (1)$$

where

$$C = 0.5772156649 \text{ (Euler's constant)}$$

Radiated Power (2)

$$C_i(x) = -\int_x^{\infty} \frac{\cos y}{y} dy = \int_{\infty}^{x} \frac{\cos y}{y} dy$$

Cosine integral

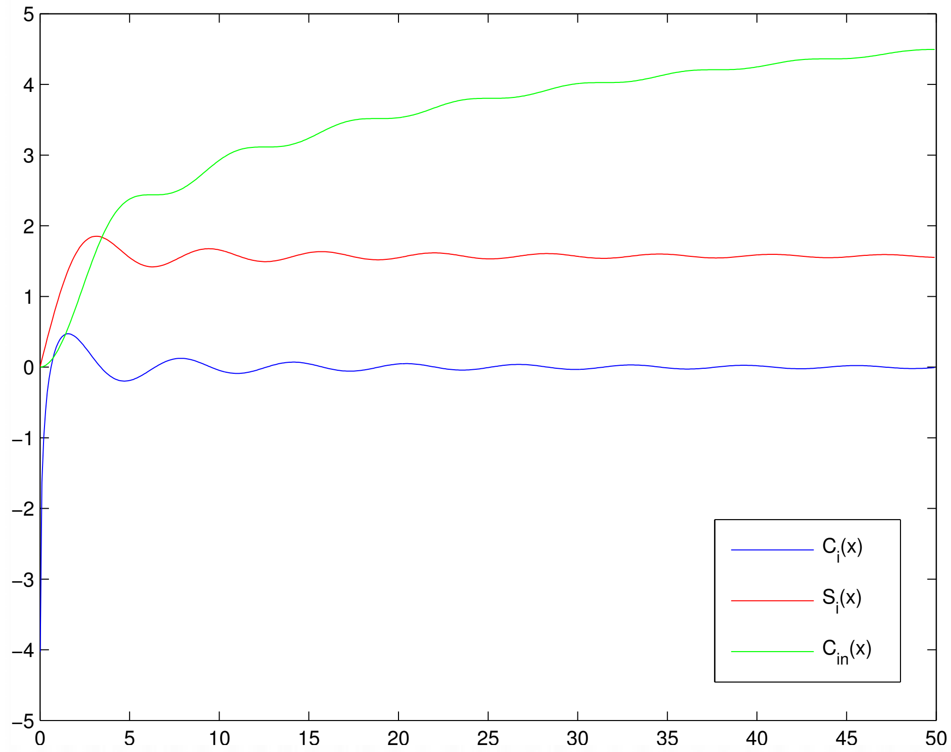
$$S_i(x) = \int_0^x \frac{\sin y}{y} dy$$

Sine integral

$C_i(x)$ is related to $C_{in}(x)$ by

$$C_{in}(x) = C + \ln(x) - C_i(x)$$

where $C_{in}(x) = \int_0^x \left(\frac{1-\cos y}{y} \right) dy$



Radiation Resistance and Input Resistance

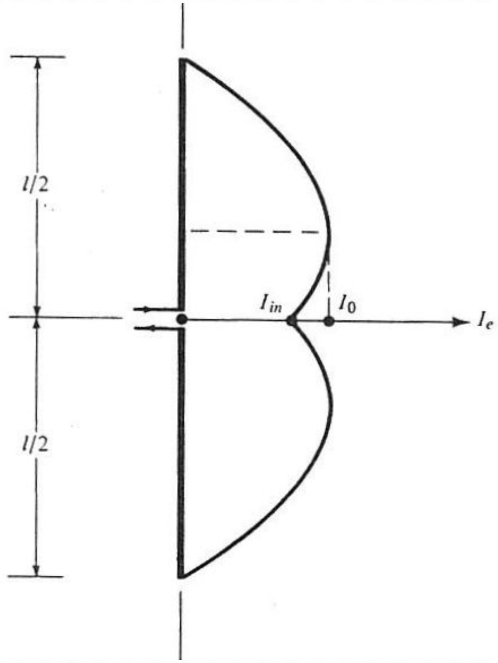
Radiation resistance becomes

$$R_r = \frac{2P_{rad}}{|I_0|^2} = \frac{\eta}{2\pi}\left\{C + \ln(kl) - C_i(kl) + \frac{1}{2}\sin(kl)[S_i(2kl) - 2S_i(kl)] + \frac{1}{2}\cos(kl)[C + \ln(kl/2) + C_i(2kl) - 2C_i(kl)]\right\}$$

Since $\frac{|I_{in}|^2}{2}R_{in} = \frac{|I_0|^2}{2}R_r$ assuming loss-less

Input resistance can be given by $R_{in} = \left[\frac{I_0}{I_{in}}\right]^2 R_r$

For a dipole of length l, $I_{in} = I_0 \sin\left(\frac{kl}{2}\right)$



Input Resistance

$$R_{in} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)}$$

Directivity

Directivity is given by

$$D_0 = 4\pi \frac{U_{\max}}{P_{rad}} = \frac{2F(\theta)\big|_{\max}}{Q}$$

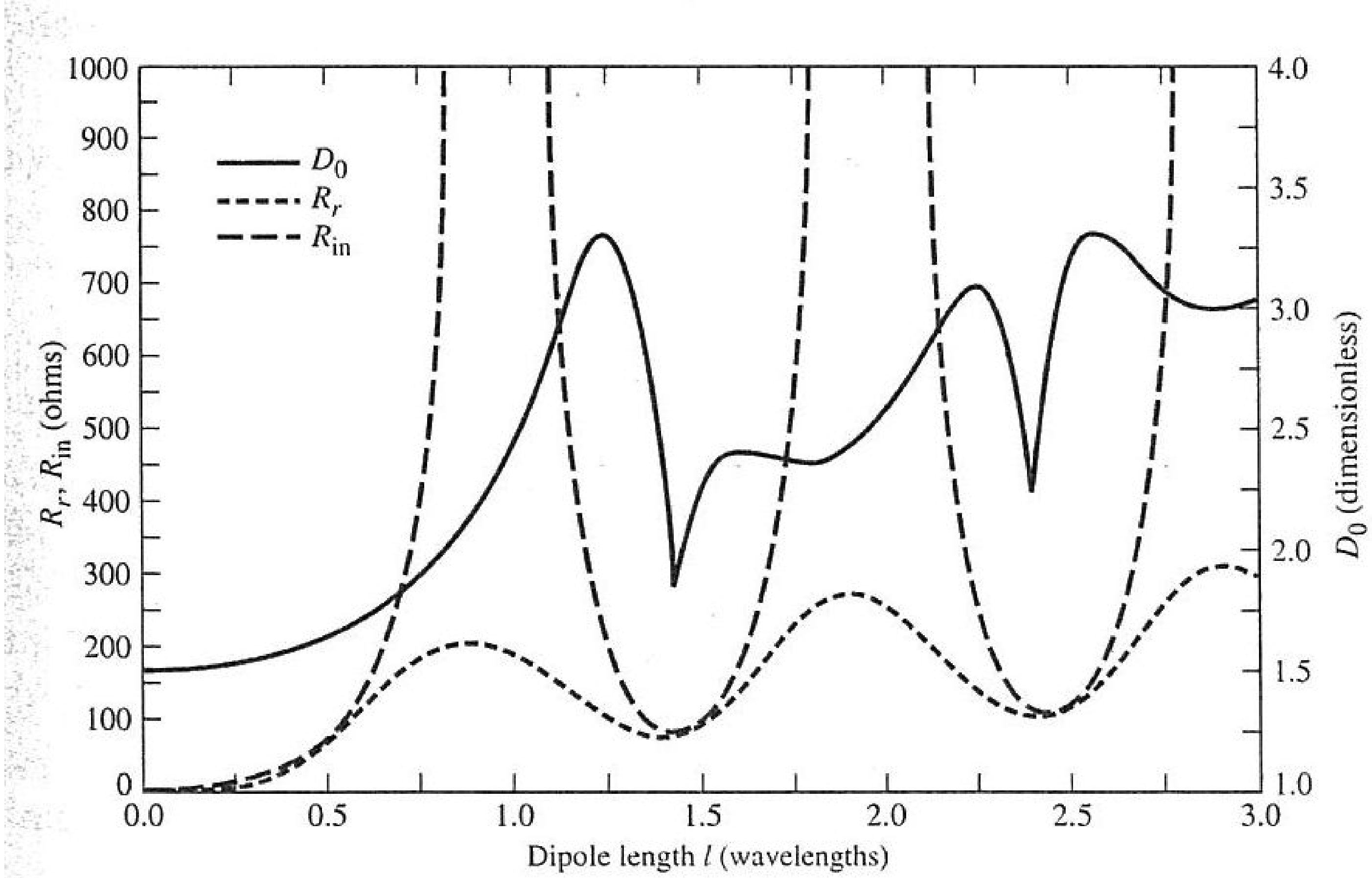
where

$$F(\theta) = \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right]^2$$

and

$$Q = \left\{ C + \ln(kl) - C_i(kl) + \frac{1}{2}\sin(kl)[S_i(2kl) - 2S_i(kl)] \right.$$
$$\left. + \frac{1}{2}\cos(kl)[C + \ln(kl/2) + C_i(2kl) - 2C_i(kl)] \right\}$$

Radiation resistance, input resistance and directivity



Half-wavelength Dipole

Let $l = \lambda/2$, then

$$E_\theta \cong j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

$$H_\phi \cong \frac{E_\theta}{\eta} = j\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

Power density

$$W_{av} = \eta \frac{|I_0|^2}{8\pi^2 r^2}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2 \cong \eta \frac{|I_0|^2}{8\pi^2 r^2}\sin^3\theta$$

Radiation Intensity

$$U = r^2 W_{av} = \eta \frac{|I_0|^2}{8\pi^2}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2 \cong \eta \frac{|I_0|^2}{8\pi^2}\sin^3\theta$$

Half-wavelength Dipole (2)

Radiated Power

$$P_{rad} = \eta \frac{|I_0|^2}{4\pi} \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta$$

$$= \eta \frac{|I_0|^2}{4\pi} \int_0^{2\pi} \left(\frac{1-\cos y}{y}\right) dy = \eta \frac{|I_0|^2}{8\pi} C_{in}(2\pi) \qquad (2)$$

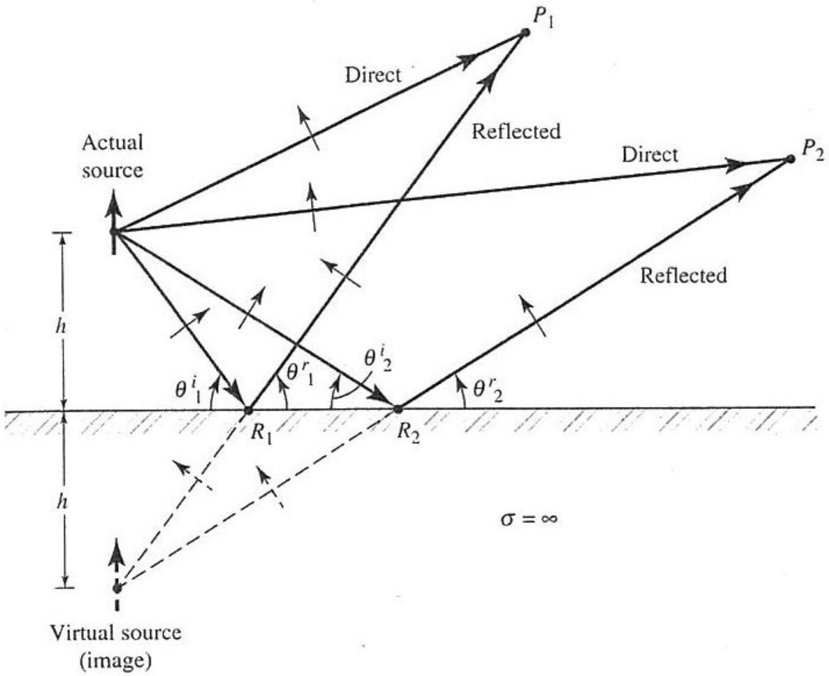
where $C_{in}(2\pi) = C + \ln(2\pi) - C_i(2\pi) \cong 2.435$

Directivity $D_0 = 4\pi \frac{U_{max}}{P_{rad}} = \frac{4}{2.435} \cong 1.643$

Radiation Resistance $R_r = \frac{2P_{rad}}{|I_0|^2} = \frac{\eta}{4\pi} C_{in}(2\pi) \cong 73$

Input Impedance $Z_{in} = 73 + j42.5$

Wire antennas near or on infinite perfect conductor



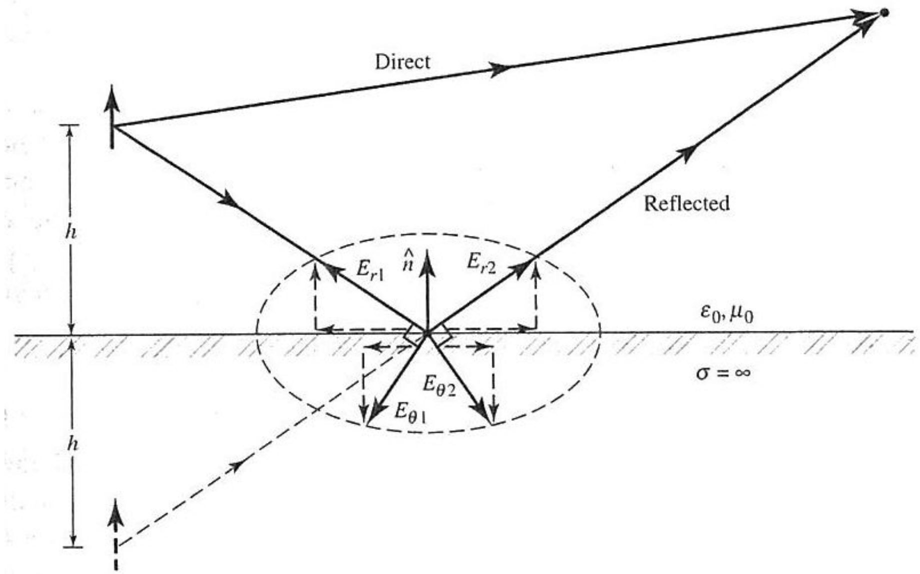
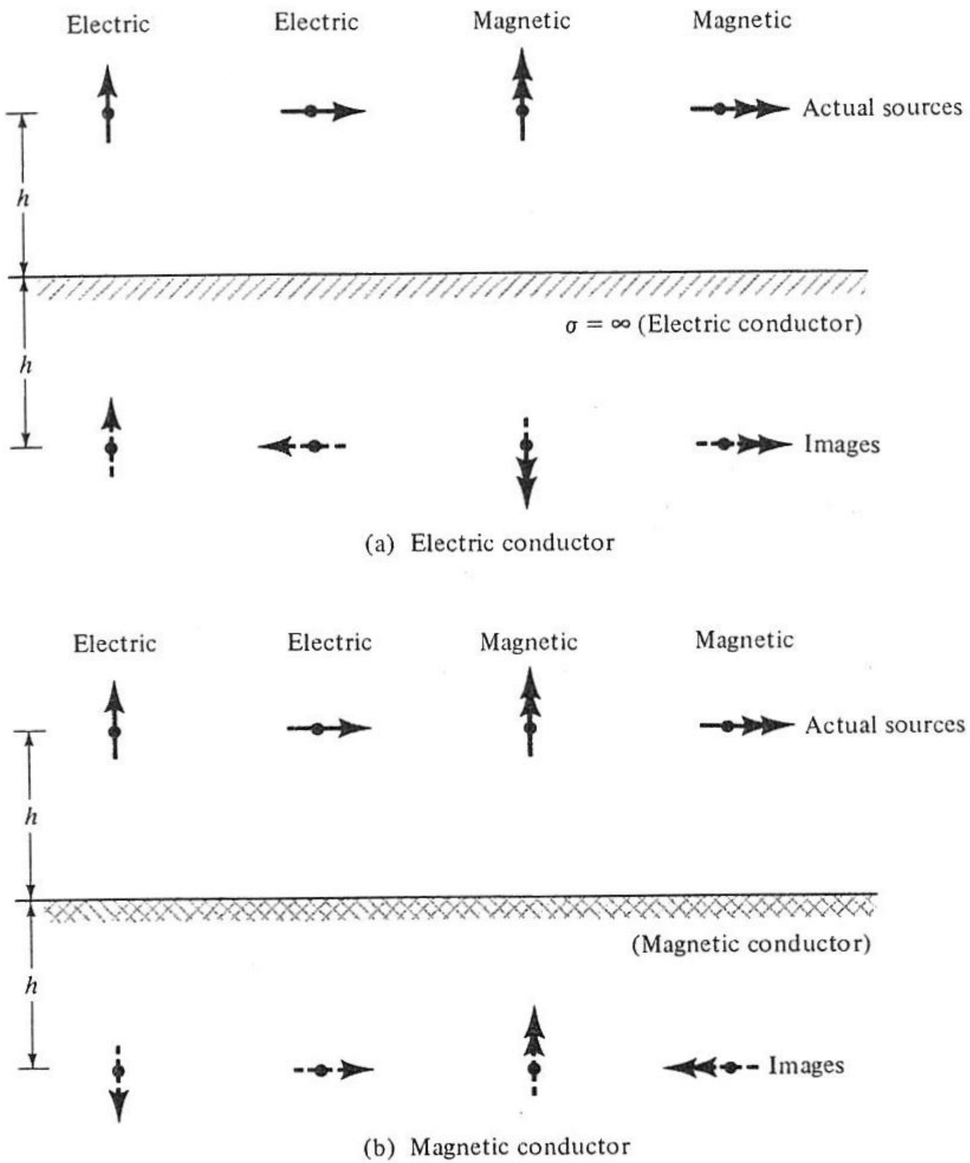


Image Theory

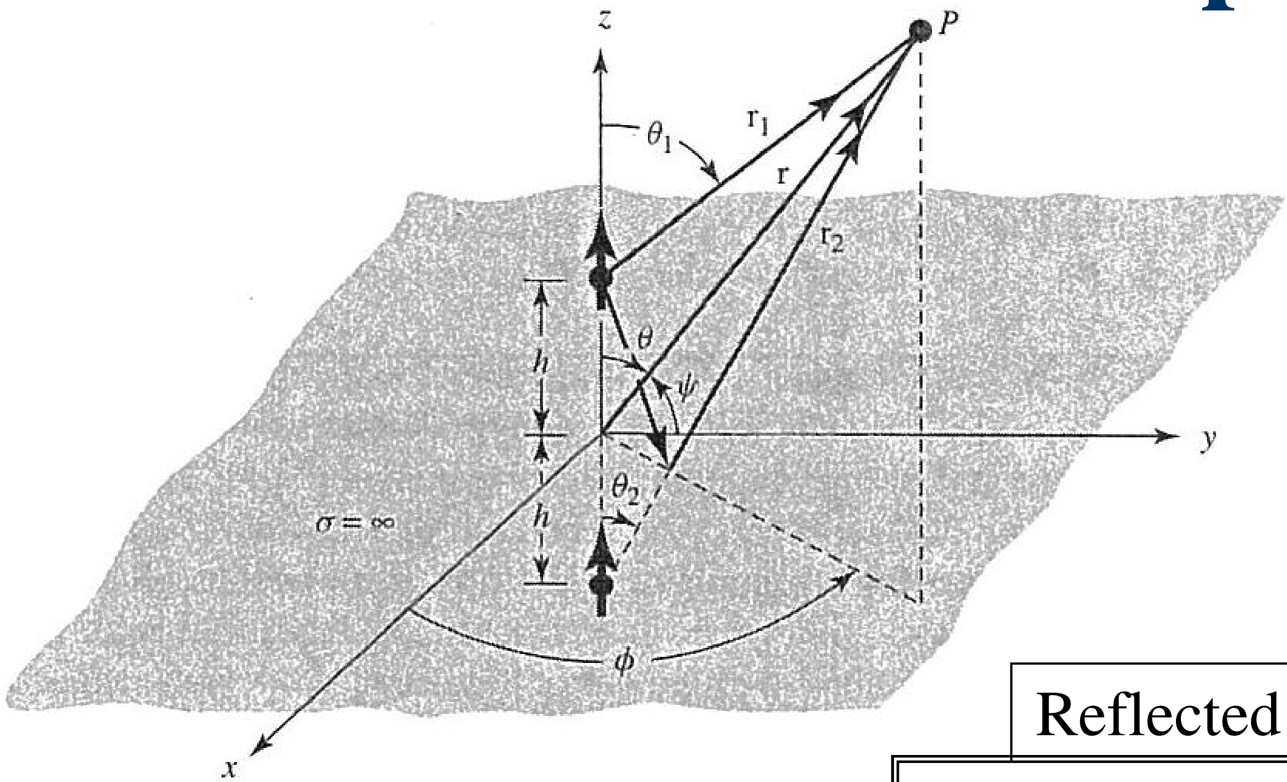


$$\overline{E}_{tan} = \overline{0} \rightarrow \hat{n} \times \mathbf{E} = \overline{\mathbf{0}}\Big|_{\text{on PEC}}$$

$$\overline{H}_{tan} = \overline{0} \rightarrow \hat{n} \times \mathbf{H} = \overline{\mathbf{0}}\Big|_{\text{on PMC}}$$

NOTE: The fields obtained are valid only in the top half-plane.

Vertical Electric Dipole



Reflected Component

$$E_\theta^r = jR_v\eta \frac{kI_0 l e^{-jkr_2}}{4\pi r_2}\sin\theta_2$$

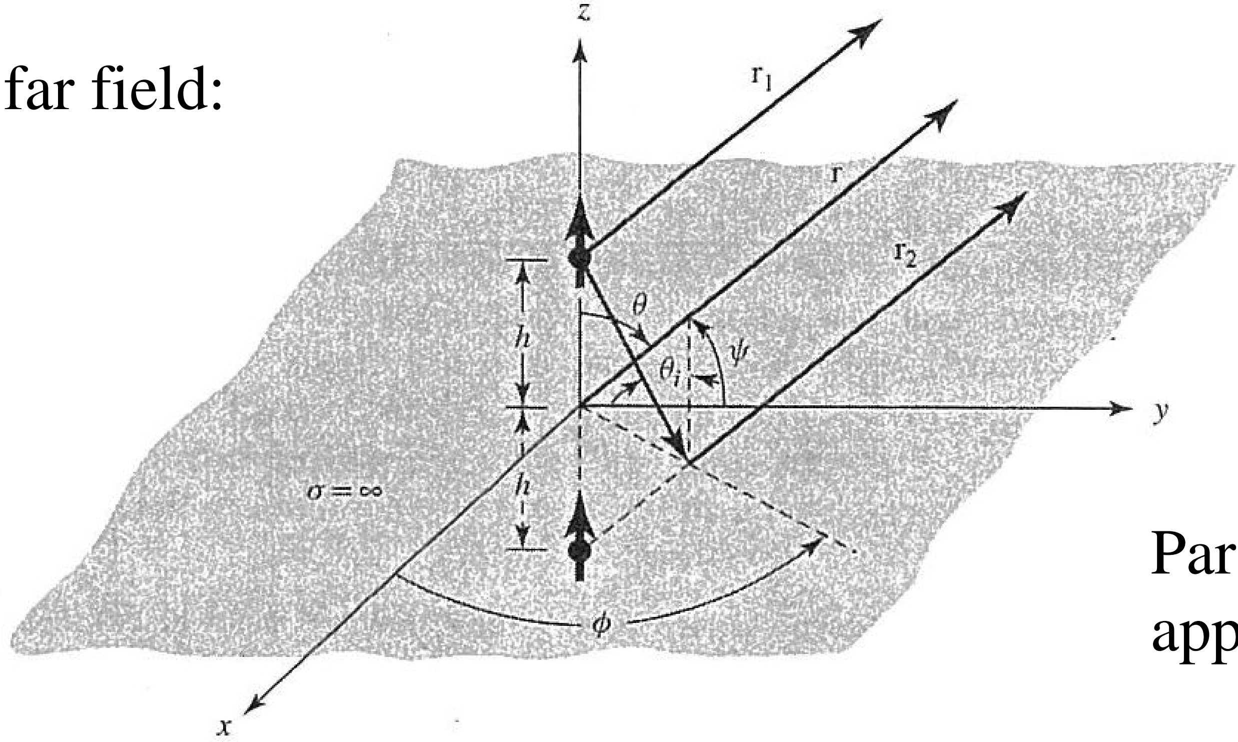
$$= j\eta \frac{kI_0 l e^{-jkr_2}}{4\pi r_2}\sin\theta_2$$

Direct Component

$$E_\theta^d = j\eta \frac{kI_0 l e^{-jkr_1}}{4\pi r_1}\sin\theta_1$$

Vertical Electric Dipole (2)

In far field:



Parallel ray approximation

$$r_1 = [r^2 + h^2 - 2rh\cos\theta]^2 \cong r - h\cos\theta$$
$$r_2 = [r^2 + h^2 + 2rh\cos\theta]^2 \cong r + h\cos\theta$$

Phase term

$$r_1 \cong r_2 \cong r$$

Amplitude term

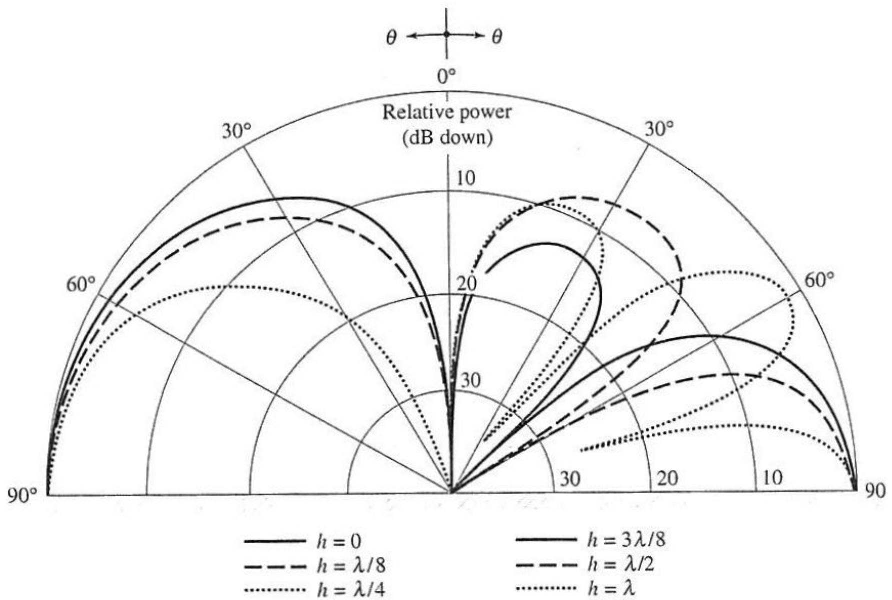
Vertical Electric Dipole (3)

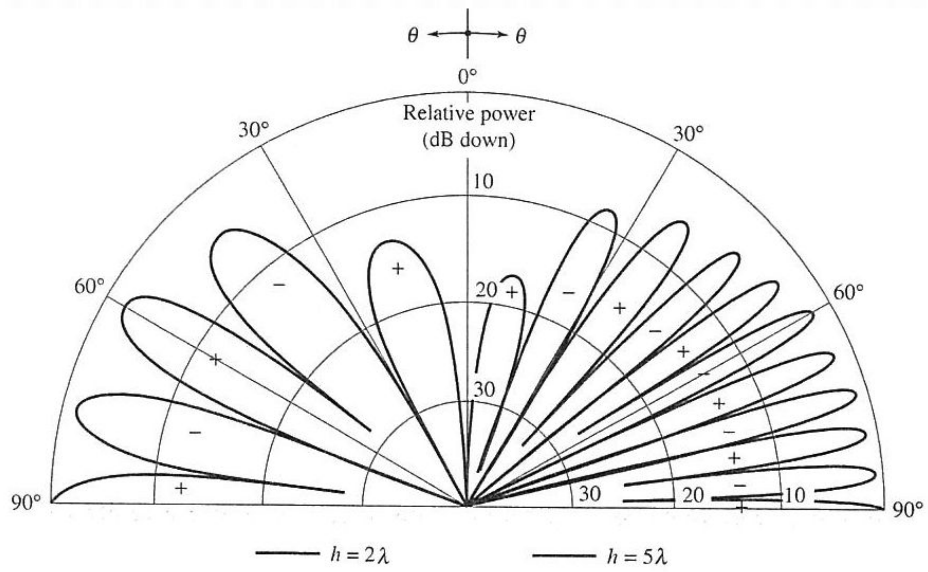
Total Electric Field

$$E_\theta \cong j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \sin\theta [2\cos(kh\cos\theta)] \quad z \geq 0$$

$$= 0 \quad z < 0$$

$$\text{number of lobes} \cong \frac{2h}{\lambda} + 1$$





Vertical Electric Dipole (4)

Power Density

$$\mathbf{W}_{av} = \frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = \hat{r}\frac{1}{2\eta}|E_\theta|^2 = \hat{r}\frac{\eta}{2r^2}\left|\frac{I_0 l}{\lambda}\right|^2 \sin^2\theta\cos^2(kh\cos\theta)$$

Radiation Intensity

$$U = \frac{r^2}{2\eta}|E_\theta|^2 = \frac{\eta}{2}\left|\frac{I_0 l}{\lambda}\right|^2 \sin^2\theta\cos^2(kh\cos\theta)$$

Maximum Radiation Intensity

$$U_{\max} = \frac{\eta}{2}\left|\frac{I_0 l}{\lambda}\right|^2$$

Radiated Power

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U\sin\theta d\theta d\phi = 2\pi\int_0^{\pi/2} U\sin\theta d\theta$$

$$= \pi\eta\left|\frac{I_0 l}{\lambda}\right|^2\left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right] \quad (3)$$

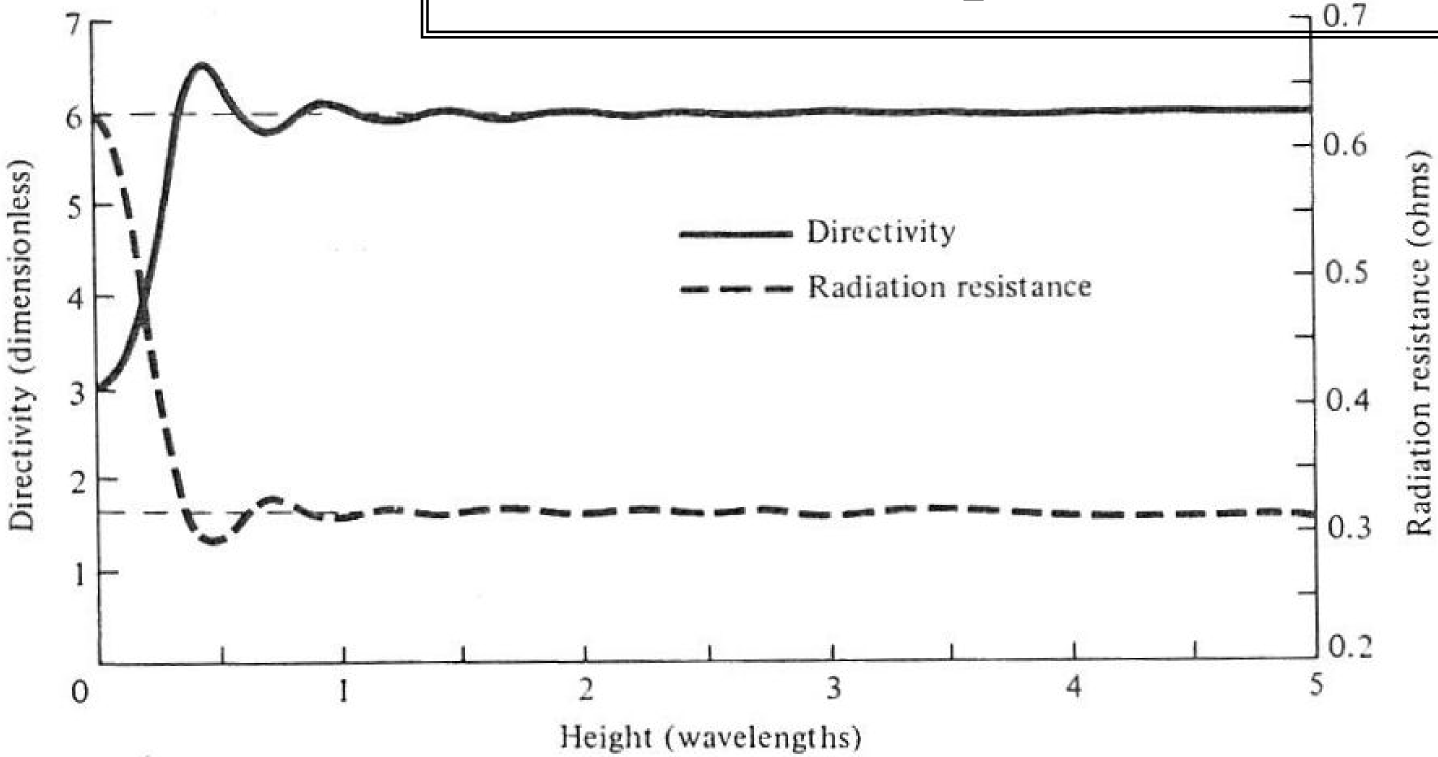
Vertical Electric Dipole (5)

Directivity

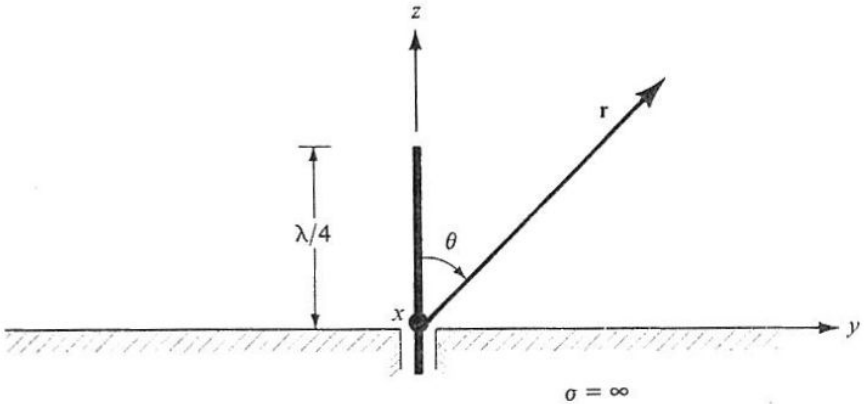
$$D_0 = 4\pi \frac{U_{max}}{P_{rad}} = 2\left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]^{-1}$$

Radiation Resistance

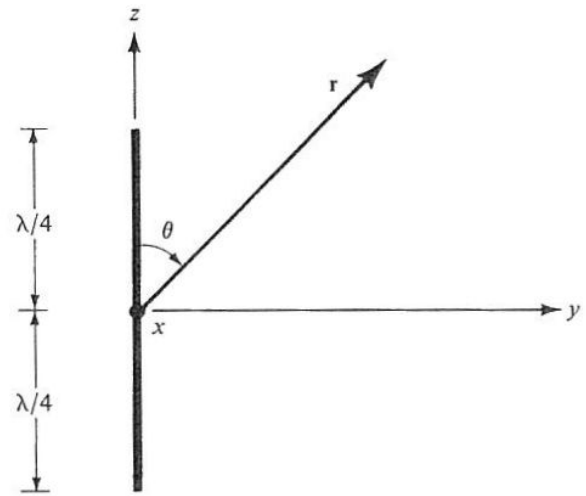
$$R_r = \frac{2P_{rad}}{|I_0|^2} = 2\pi\eta\left(\frac{l}{\lambda}\right)^2\left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$



$\lambda/4$ Monopole



(a) λ/4 monopole on infinite electric conductor



(b) Equivalent of λ/4 monopole on infinite electric conductor

$$
\begin{aligned}
Z_{in} &= \frac{1}{2} Z_{in}(\text{dipole}) \\
&= \frac{1}{2}(73 + j42.5) \\
&= 36.5 + j21.25
\end{aligned}
$$

Fields due to y-directed dipole

Vector Potential **A**

$$\mathbf{A} = \hat{y}\frac{\mu I_0 l e^{-jkr}}{4\pi r}$$

$$A_\theta = A_y \cos\theta \sin\phi = \frac{\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\sin\phi$$

$$A_\phi = A_y \cos\phi = \frac{\mu I_0 l e^{-jkr}}{4\pi r}\cos\phi$$

Recall that $\mathbf{E}(\vec{r}) = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{A})$

Far-field Electric Field

$$E_\theta \cong -j\omega A_\theta = -j\omega\frac{\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\sin\phi$$

$$E_\phi \cong -j\omega A_\phi = -j\omega\frac{\mu I_0 l e^{-jkr}}{4\pi r}\cos\phi$$

Fields due to y-directed dipole (2)

Introduce a “new” spherical coordinate system (r,ψ,ζ) such that

$$\hat{r} = \hat{z}\sin\psi\cos\zeta + \hat{x}\sin\psi\sin\zeta + \hat{y}\cos\psi$$

$$\hat{\psi} = \hat{z}\cos\psi\cos\zeta + \hat{x}\cos\psi\sin\zeta - \hat{y}\sin\psi$$

$$\hat{\zeta} = -\hat{z}\sin\psi + \hat{x}\cos\psi$$

(This can be obtained by letting (*x*,*y*,*z*)->(*z*,*x*,*y*) and (ψ,ζ)->(θ,ϕ))

then

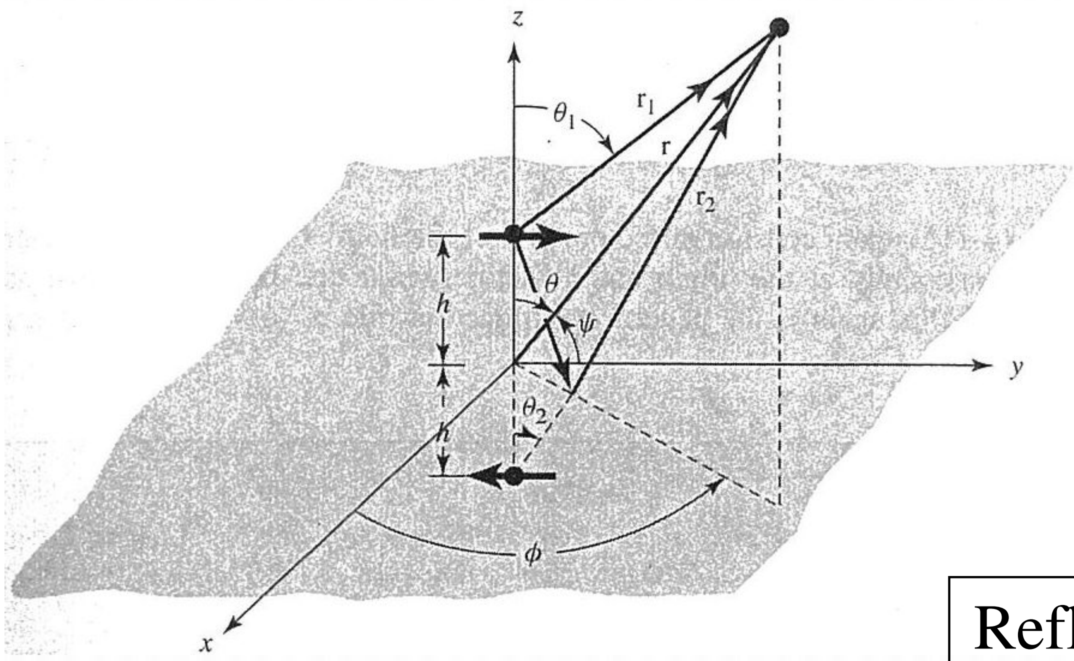
Far-field Electric Field

$$E_{\psi} \cong j\omega\frac{\mu I_0 l e^{-jkr}}{4\pi r}\sin\psi$$

Far-field Magnetic Field

$$H_{\zeta} \cong j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi\eta r}\sin\psi$$

Horizontal Electric Dipole



Direct Component

$$E_\psi^d = j\eta \frac{kI_0 l e^{-jkr_1}}{4\pi r_1} \sin\psi$$

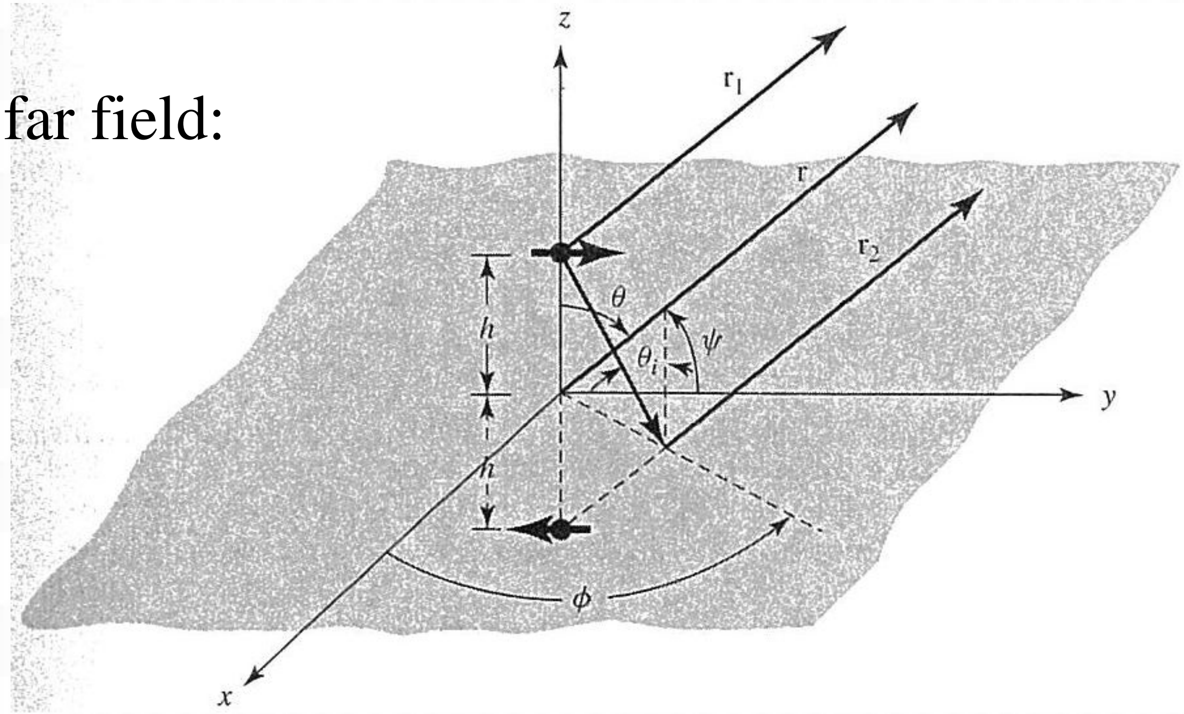
Reflected Component

$$E_\psi^r = jR_h \eta \frac{kI_0 l e^{-jkr_2}}{4\pi r_2} \sin\psi$$

$$= -j\eta \frac{kI_0 l e^{-jkr_2}}{4\pi r_2} \sin\psi$$

Horizontal Electric Dipole (2)

In far field:



$$r_1 = [r^2 + h^2 - 2rh\cos\theta]^2 \cong r - h\cos\theta$$

$$r_2 = [r^2 + h^2 + 2rh\cos\theta]^2 \cong r + h\cos\theta$$

Phase term

$$r_1 \cong r_2 \cong r$$

Amplitude term

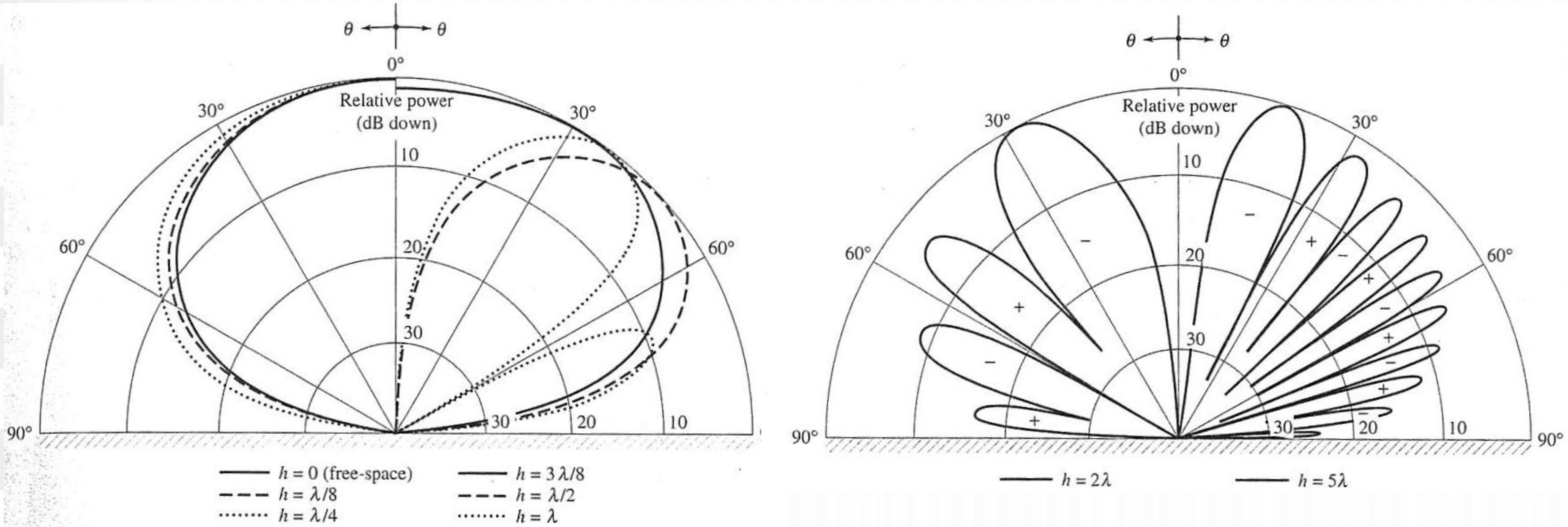
$$\cos\psi = \hat{y}\cdot\hat{r} = \sin\theta\sin\phi$$

Horizontal Electric Dipole (3)

Total Electric Field

$$E_{\psi} \cong j\eta \frac{kI_0 l e^{-jkr}}{4\pi r}\sqrt{1-\sin^2\theta\sin^2\phi}\,[2j\sin(kh\cos\theta)]$$

$$\text{number of lobes} \cong \frac{2h}{\lambda}$$



Horizontal Electric Dipole (4)

Power Density

$$\mathbf{W}_{av} = \hat{r}\frac{\eta}{2r^2}\left|\frac{I_0 l}{\lambda}\right|^2 (1-\sin^2\theta\sin^2\phi)\sin^2(kh\cos\theta)$$

Radiation Intensity

$$U = \frac{r^2}{2\eta}|E_\theta|^2 = \frac{\eta}{2}\left|\frac{I_0 l}{\lambda}\right|^2 (1-\sin^2\theta\sin^2\phi)\cos^2(kh\cos\theta)$$

Radiated Power

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U\sin\theta d\theta d\phi = 2\pi\int_0^{\pi/2} U\sin\theta d\theta$$

$$= \eta\frac{\pi}{2}\left|\frac{I_0 l}{\lambda}\right|^2 \left[\frac{2}{3} - \frac{\sin(2kh)}{2kh} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$

Radiation Resistance

$$R_r = \frac{2P_{rad}}{|I_0|^2} = \pi\eta\left(\frac{l}{\lambda}\right)^2 \left[\frac{2}{3} - \frac{\sin(2kh)}{2kh} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$

Horizontal Electric Dipole (5)

For small kh

$$R_r \overset{kh\to 0}{=} \pi\eta\left(\frac{l}{\lambda}\right)^2\left[\frac{2}{3}-\frac{2}{3}+\frac{8}{15}\left(\frac{2\pi h}{\lambda}\right)^2\right] = \eta\frac{32\pi^3}{15}\left(\frac{l}{\lambda}\right)^2\left(\frac{h}{\lambda}\right)^2$$

Maximum Radiation Intensity

$$U_{max} = \begin{cases} \dfrac{\eta}{2}\left|\dfrac{I_0 l}{\lambda}\right|^2 \sin^2(kh) & kh \le \pi/2, \theta = 0 \\ \dfrac{\eta}{2}\left|\dfrac{I_0 l}{\lambda}\right|^2 & kh > \pi/2, \phi = 0 \text{ and } \sin(kh\cos\theta_{max}) = 1 \end{cases}$$

Directivity

$$D_0 = \begin{cases} \dfrac{4\sin^2(kh)}{R(kh)} & kh \le \pi/2 \\ \dfrac{4}{R(kh)} & kh > \pi/2 \end{cases}$$

Horizontal Electric Dipole (6)

where $$R(kh) = \left[\frac{2}{3} - \frac{\sin(2kh)}{2kh} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$

For small *kh*

$$D_0 \overset{kh\to 0}{=} 4\sin^2(kh)\left[\frac{2}{3} - \frac{2}{3} + \frac{8}{15}(kh)^2\right]^{-1} = 7.5\left(\frac{\sin(kh)}{kh}\right)^2$$