Chapter 5 : Loop Antennas

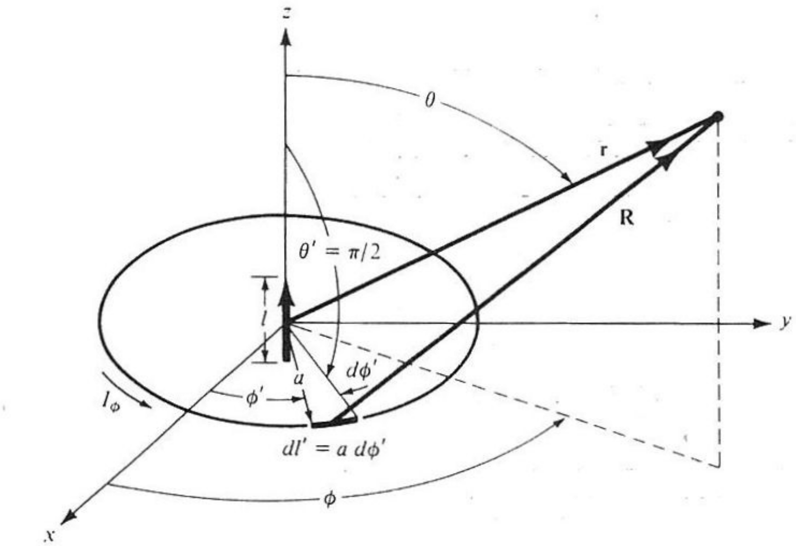
- Infinitesimal small circular loop
- Small circular loop

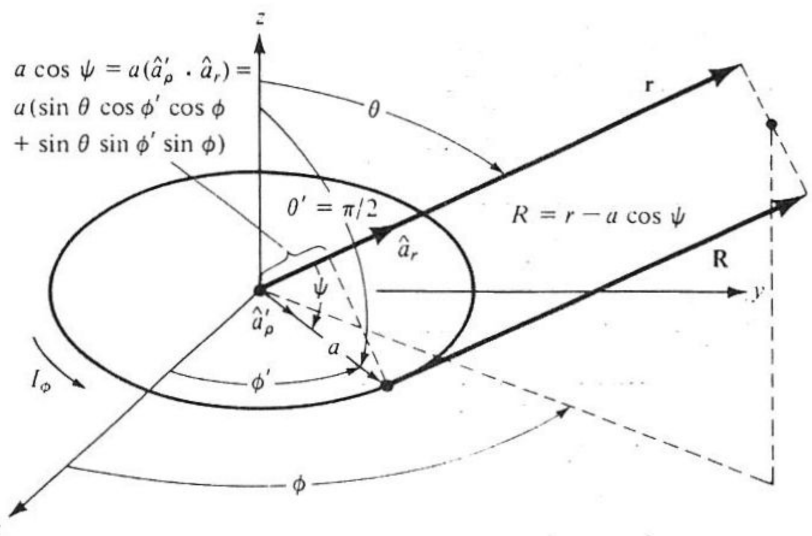
Infinitesimal Circular Loop

- Radius $a << \lambda$ ($a < \lambda/20$)
- Constant current along the azimuthal direction

$$\mathbf{I}_e = \hat{\phi}' I_\phi(\phi') = \hat{\phi}' I_0$$

Where I_0 is a constant.





Vector Potential

Vector potential **A** can be given by

$$\mathbf{A} = \frac{\mu}{4\pi}\int_C \mathbf{I}_e(\bar{r}')\frac{e^{-jkR}}{R}dl'$$

$$= \frac{\mu}{4\pi}\int_0^{2\pi} \mathbf{I}_e(\phi')\frac{e^{-jkR}}{R}ad\phi'$$

$$= \frac{\mu}{4\pi}\int_0^{2\pi} \hat{\phi}' I_0 \frac{e^{-jkR}}{R}ad\phi'$$

Since $\hat{\phi}' = -\hat{x}\sin\phi' + \hat{y}\cos\phi'$

$$\mathbf{A} = \frac{\mu I_0 a}{4\pi}\left[-\hat{x}\int_0^{2\pi}\sin\phi'\frac{e^{-jkR}}{R}d\phi' + \hat{y}\int_0^{2\pi}\cos\phi'\frac{e^{-jkR}}{R}d\phi'\right]$$

Vector Potential (2)

Notice that $$R = |\bar{r} - \bar{r}'| = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{1/2}$$
$$= \left[r^2 + a^2 - 2ar\sin\theta\cos(\phi - \phi')\right]^{1/2}$$

For small *a*, let $$f(a) = \frac{e^{-jkR}}{R}$$

and using $$f(a) = f(0) + f'(0)a + f''(0)\frac{a^2}{2!} + \cdots$$
$$\approx f(0) + f'(0)a$$

together with $$f(0) = \frac{e^{-jkr}}{r}$$
$$f'(0) = \left(\frac{1}{r^2} + \frac{jk}{r}\right)e^{-jkr}\sin\theta\cos(\phi - \phi')$$

Vector Potential (3)

one obtains
$$f(a) \approx \frac{e^{-jkr}}{r} + a\sin\theta\cos(\phi-\phi')\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}$$

and
$$\int_0^{2\pi}\sin\phi'\frac{e^{-jkR}}{R}d\phi' \approx a\pi\sin\theta\sin\phi\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}$$

$$\int_0^{2\pi}\cos\phi'\frac{e^{-jkR}}{R}d\phi' \approx a\pi\sin\theta\cos\phi\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}$$

thus
$$\mathbf{A} = \frac{\mu I_0 a^2}{4}\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta(-\hat{x}\sin\phi+\hat{y}\cos\phi)$$

$$= \hat{\phi}\frac{\mu I_0 a^2}{4}\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta$$

Radiated Field

Therefore, the magnetic field can be given by

$$H_r = \frac{jkI_0 a^2 \cos\theta}{2r^2}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$H_\theta = -\frac{(ka)^2 I_0 \sin\theta}{4r}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}$$

$$H_\phi = 0$$

Likewise, the electric field can be found to be

$$E_r = E_\theta = 0$$

$$E_\phi = \eta(ka)^2 \frac{I_0 \sin\theta}{4r}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

Infinitesimal Magnetic Dipole

- **Length $l << \lambda$**
- **Constant current, directed along the *z*-axis**

$$\mathbf{M}(z') = \hat{z} M_0$$

Using duality theorem

$$\mathbf{F}(x, y, z) = \frac{\varepsilon}{4\pi} \int_C \mathbf{M}(x', y', z') \frac{e^{-jkR}}{R} dl'$$

along with the same infinitesimal dipole approximations yields

$$\mathbf{F}(x, y, z) = \hat{z} \frac{\varepsilon M_0}{4\pi r} e^{-jkr} \int_{-l/2}^{l/2} dz' = \hat{z} \frac{\varepsilon M_0 l}{4\pi r} e^{-jkr}$$

Then, the electric field can be found from

$$\mathbf{E} = -\frac{1}{\varepsilon} \nabla \times \mathbf{F}$$

Radiated Field

Therefore,

$$E_r = E_\theta = 0$$

$$E_\phi = -j\frac{kM_0 l \sin\theta}{4\pi r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

Likewise, the magnetic field can be found to be

$$H_r = \frac{M_0 l \cos\theta}{2\pi r^2 \eta}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$H_\theta = j\frac{kM_0 l \sin\theta}{4\pi r \eta}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$

$$H_\phi = 0$$

Radiated Field of infinitesimal dipole

Therefore,

$$H_r = H_\theta = 0$$

$$H_\phi = j\frac{kI_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

Likewise, the electric field can be found to be

$$E_r = \eta\frac{I_0 l \cos\theta}{2\pi r^2}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$E_\theta = j\eta\frac{kI_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}$$

$$E_\phi = 0$$

Loop antenna and magnetic dipole

- **Comparing the fields from loop antenna and magnetic dipole, one find that by letting**

$$M_0 l = jk\eta(\pi a^2)I_0 = jk\eta S I_0$$

where $S = \pi a^2$

the fields due to the loop antenna is the same as those due to the equivalent magnetic dipole.

Power Density and Radiation Resistance

- **Poynting vector**

$$\mathbf{W} = \frac{1}{2}(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2}\left(\hat{\phi}E_\phi\right)\times\left(\hat{r}H_r^* + \hat{\theta}H_\theta^*\right)$$

$$= \frac{1}{2}(-\hat{r}E_\phi H_\theta^* + \hat{\theta}E_\phi H_r^*)$$

Components of Poynting Vector

$$W_r = \eta\frac{(ka)^4}{32}|I_0|^2\frac{\sin^2\theta}{r^2}\left[1 - j\frac{1}{(kr)^3}\right]$$

$$W_\theta = j\eta\frac{k^3a^4|I_0|^2\cos\theta\sin\theta}{16r^3}\left[1 + \frac{1}{(kr)^2}\right]$$

Outgoing Power

$$P = \oiint_S \mathbf{W}\cdot d\mathbf{s} = \int_0^{2\pi}\int_0^{\pi}(\hat{r}W_r + \hat{\theta}W_\theta)\cdot\hat{r}r^2\sin\theta d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}W_r r^2\sin\theta d\theta d\phi = \eta\frac{\pi}{12}(ka)^4|I_0|^2\left[1 + j\frac{1}{(kr)^3}\right]$$

Power Density and Radiation Resistance (2)

Radiated Power

$$P_{rad} = \mathrm{Re}[P] = \eta \frac{\pi}{12}(ka)^4 |I_0|^2$$

Since $$P_{rad} = \frac{1}{2} | I_0 |^2 R_r$$

$$R_r = \eta \frac{\pi}{6}(ka)^4 = \eta \frac{\pi}{6}\left(\frac{2\pi a}{\lambda}\right)^4 = \eta \frac{8}{3}\frac{\pi^3}{\lambda^4}\left(\pi a^2\right)^2$$

$$= 320\left(\frac{\pi}{\lambda}\right)^4 S^2$$

Radiation Resistance

$$R_r = 31{,}171\left(\frac{S^2}{\lambda^4}\right)$$

Radiation Resistance example

Let $a = \lambda/65$, $$R_r = 31{,}171\frac{\pi^2\lambda^4}{65^4\lambda^4} = 0.0172\ \Omega$$

Since $$R_{ohmic} = \frac{a}{b}R_s$$

$a =$ loop radius

$b =$ wire radius

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \text{surface impedance of conductor}$$

Let f=1 GHz, b=3.85×10$^{-5}\lambda$ and σ=5.7×10^{7}, then $R_{ohmic} = 3.33\ \Omega$

$$e_{cd} = \frac{R_r}{R_r + R_{ohmic}} = 0.00514$$

→ Impractical

N-turn circular loop

- **To increase the radiation resistance, many turns are required.**

$$\boxed{R_r = 31{,}171\left(\frac{S^2}{\lambda^4}\right)N^2} \quad N = \text{Number of turns}$$

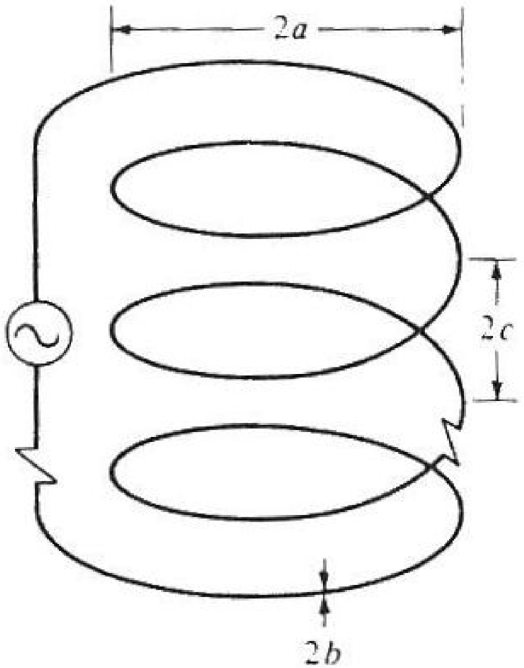
Let $a = \lambda/65$, N=54 $\quad R_r = 54^2 \times 0.0172 = 50.62\ \Omega$

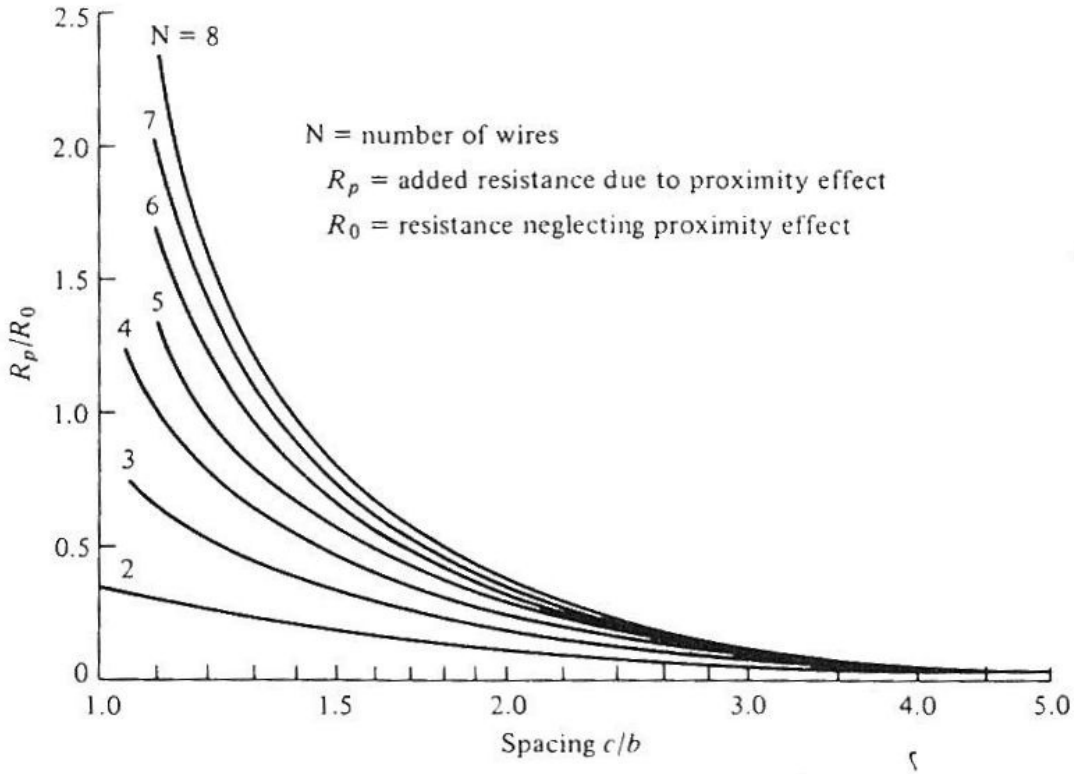
$$R_{ohmic} = \frac{Na}{b}R_s\left(\frac{R_p}{R_0}+1\right)$$

R_p = ohmic resistance per unit length due to proximity effect

$R_0 = \dfrac{NR_s}{2\pi b}$ ohmic skin effect resistance per unit length

N-turn circular loop (2)





Far Field (*kr* >> 1)

- **For *kr* >> 1, the fields can be approximated as**

$$\left.\begin{aligned} H_\theta &\cong -\frac{(ka)^2 I_0 \sin\theta}{4r} e^{-jkr} \\ E_\phi &\cong \eta \frac{(ka)^2 I_0 \sin\theta}{4r} e^{-jkr} \\ E_r &\cong E_\theta = H_r = H_\phi = 0 \end{aligned}\right\} kr >> 1$$

→ **TEM Wave**

Ratio of E and H: $Z_w = -\dfrac{E_\phi}{H_\theta} = \eta$

$Z_w = \text{wave impedance}$

$\eta = \text{intrinsic impedance} \; (120\pi \; \Omega \text{ for free-space})$

Directivity

Time-average power density:

$$\mathbf{W}_{av} = \frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = \hat{r}\frac{1}{2\eta}|E_\phi|^2 = \hat{r}\frac{\eta}{32}(ka)^4|I_0|^2\frac{\sin^2\theta}{r^2}$$

Radiation intensity:

$$U = r^2 W_{av} = \frac{\eta}{32}(ka)^4|I_0|^2\sin^2\theta$$

Maximum radiation intensity:

$$U_{\max} = \frac{\eta}{32}(ka)^4|I_0|^2$$

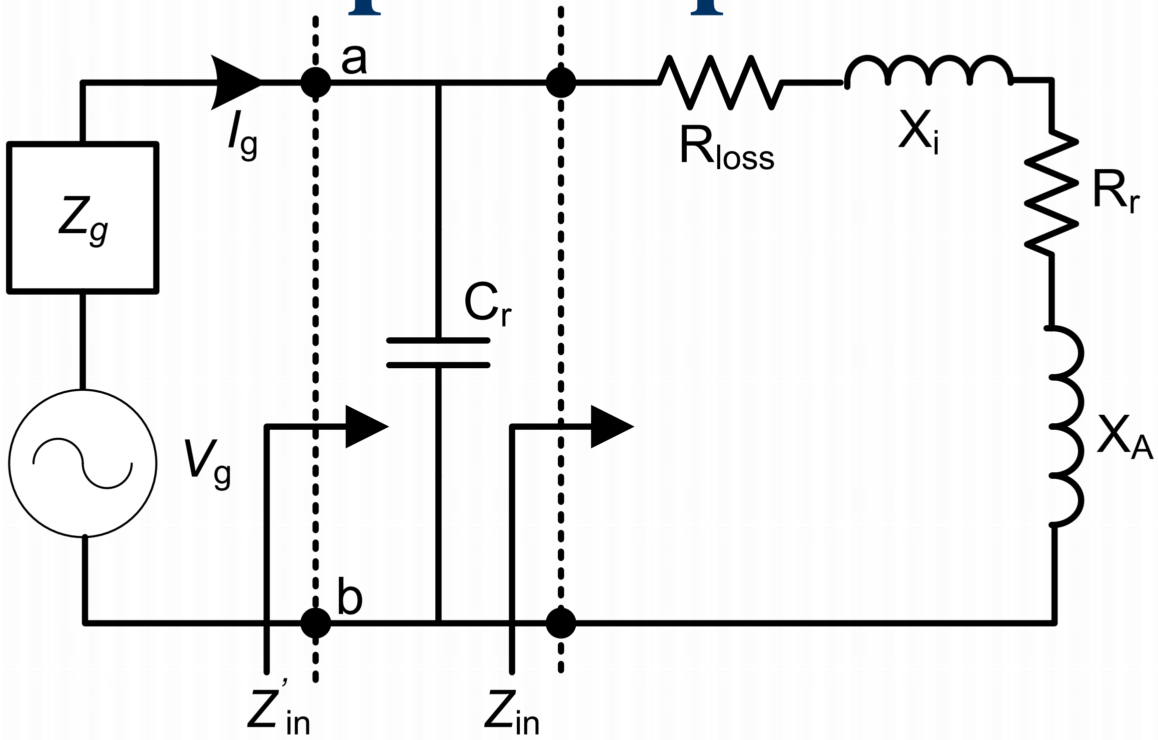
Maximum directivity:

$$D_0 = 4\pi\frac{U_{\max}}{P_{rad}} = \frac{3}{2}$$

Maximum effective area:

$$A_{em} = \frac{\lambda^2}{4\pi}D_0 = \frac{3\lambda^2}{8\pi}$$

Input Impedance



R_r = radiation resistance

R_{loss} = loss resistance of loop conductor

X_A = external inductive reactance of loop antenna $= \omega L_A$

X_i = internal high-frequency reactance of loop conductor $= \omega L_i$

C_r = matching capacitor

Input Impedance (2)

Input impedance is given by

$$Z_{in} = R_{in} + X_{in} = (R_r + R_{loss}) + j(X_A + X_i)$$

thus

$$Y_{in} = G_{in} + B_{in} = \frac{R_{in}}{R_{in}^2 + X_{in}^2} - j\frac{X_{in}}{R_{in}^2 + X_{in}^2}$$

Matching capacitor is chosen to be

$$C_r = -\frac{B_{in}}{2\pi f} = \frac{1}{2\pi f}\frac{X_{in}}{R_{in}^2 + X_{in}^2}$$

Under resonance, input impedance is equal to

$$Z_{in}^{'} = R_{in}^{'} = \frac{R_{in}^2 + X_{in}^2}{R_{in}}$$

Input Impedance (3)

For circular loop of radius *a* and wire radius *b*

$$L_A = \mu_0 a\left[\ln\left(\frac{8a}{b}\right) - 2\right]$$

For square loop with sides *a* and wire radius *b*

$$L_A = 2\mu_0 \frac{a}{\pi}\left[\ln\left(\frac{a}{b}\right) - 0.774\right]$$

Internal reactance of the loop conductor:

$$L_i = \frac{l}{\omega P}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{a}{\omega b}\sqrt{\frac{\omega\mu_0}{2\sigma}}$$

where *l* is the length and *P* is the perimeter of the wire of the loop.

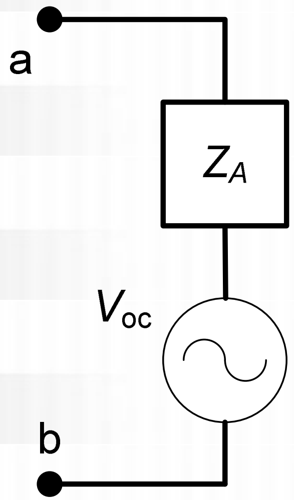
Input Impedance (4)

From $$\vec{\mathbf{E}}_a = -j\eta \frac{kI_{in}}{4\pi r} e^{-jkr} \vec{l}_e$$

and $$E_\phi = \eta \frac{(ka)^2 I_0}{4r} e^{-jkr} \sin\theta$$

Effective length $$\bar{l}_e = \hat{\phi} jkA \sin\theta$$

Let incident plane wave be at origin: $\mathbf{H}^i = \hat{\theta} H_0, \mathbf{E}^i = \hat{\phi} H_0 \eta$



$$V_{oc} = \mathbf{E}^i \cdot \bar{l}_e = \hat{\phi} H_0 \eta \cdot \hat{\phi} jkS \sin\theta = jH_0 \eta kS \sin\theta$$

since $$H_\theta^i = H_0, H_z^i = -H_\theta^i \sin\theta$$

$$V_{oc} = -jH_z^i \eta kS = -jH_z^i S\omega\mu$$

Input Impedance (5)

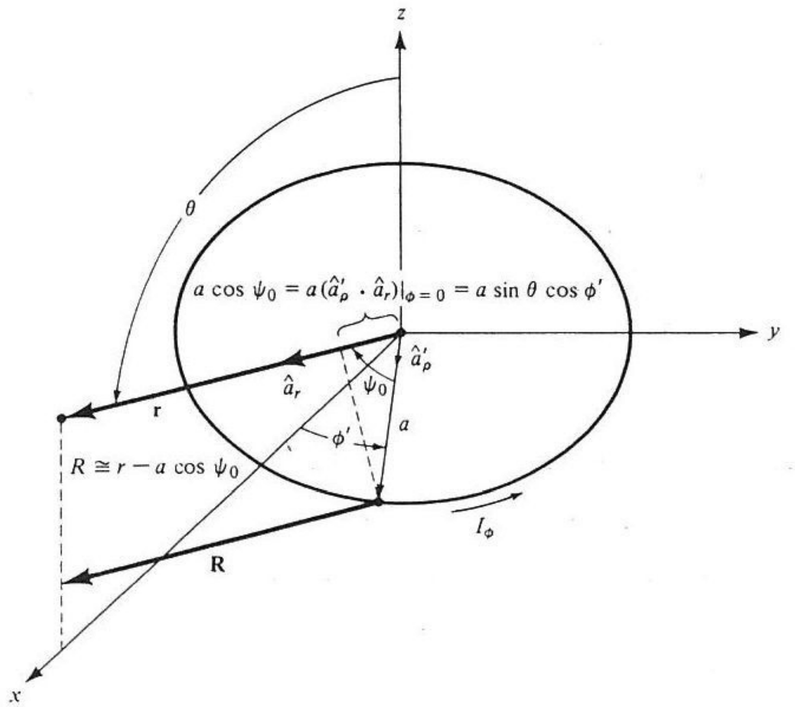
Therefore $$V_{oc} = -j\omega S B_z^i$$

- induced voltage proportional to magnetic flux density
- Loop antenna often used as a probe to measure magnetic flux density

Circular loop of constant current

Vector potential **A** can be given b

$$\mathbf{A} = \frac{\mu}{4\pi}\int_0^{2\pi} \hat{\phi}' I_0 \frac{e^{-jkR}}{R} a\,d\phi'$$



Electric field **E** can be given by

$$\mathbf{E}(\vec{r}) \approx \frac{j\omega\mu}{4\pi}\frac{e^{-jkr}}{r}\int [\hat{r}\times\hat{r}\times\vec{J}(\vec{r}\,')]e^{jk\hat{r}\cdot\vec{r}\,'}$$

$$\hat{r}\cdot\bar{r}' = \hat{r}\cdot\hat{r}'a = a(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)\cdot(\hat{x}\cos\phi' + \hat{y}\sin\phi')$$

$$= a(\sin\theta\cos\phi\cos\phi' + \sin\theta\sin\phi\sin\phi') = a\sin\theta\cos(\phi-\phi')$$

Circular loop of constant current (2)

$$\mathbf{E}(\vec{r}) \approx \frac{j\omega\mu I_0}{4\pi}\frac{e^{-jkr}}{r}\int_0^{2\pi}[\hat{r}\times\hat{r}\times\hat{\phi}']e^{jka\sin\theta\cos(\phi-\phi')}ad\phi'$$

$$= \frac{j\omega\mu I_0}{4\pi}\frac{e^{-jkr}}{r}\int_0^{2\pi}[\hat{r}\times\hat{r}\times(-\hat{x}\sin\phi'+\hat{y}\cos\phi')]e^{jka\sin\theta\cos(\phi-\phi')}ad\phi'$$

Using

$$\int_0^{2\pi}\begin{Bmatrix}\sin\phi' \\ \cos\phi'\end{Bmatrix}e^{jka\sin\theta\cos(\phi-\phi')}d\phi' = \begin{Bmatrix}\sin\phi \\ \cos\phi\end{Bmatrix}j2\pi J_1(ka\sin\theta)$$

where $J_1(x)$ is the Bessel function of the first order, yields

$$\mathbf{E}(\vec{r}) = \frac{j\omega\mu I_0 a}{4\pi}\frac{e^{-jkr}}{r}\hat{r}\times\hat{r}\times(-\hat{x}\sin\phi+\hat{y}\cos\phi)\,j2\pi J_1(ka\sin\theta)$$

Circular loop of constant current (3)

It can be shown that

$$\hat{r}\times\hat{r}\times(-\hat{x}\sin\phi+\hat{y}\cos\phi)=-\hat{\phi}$$

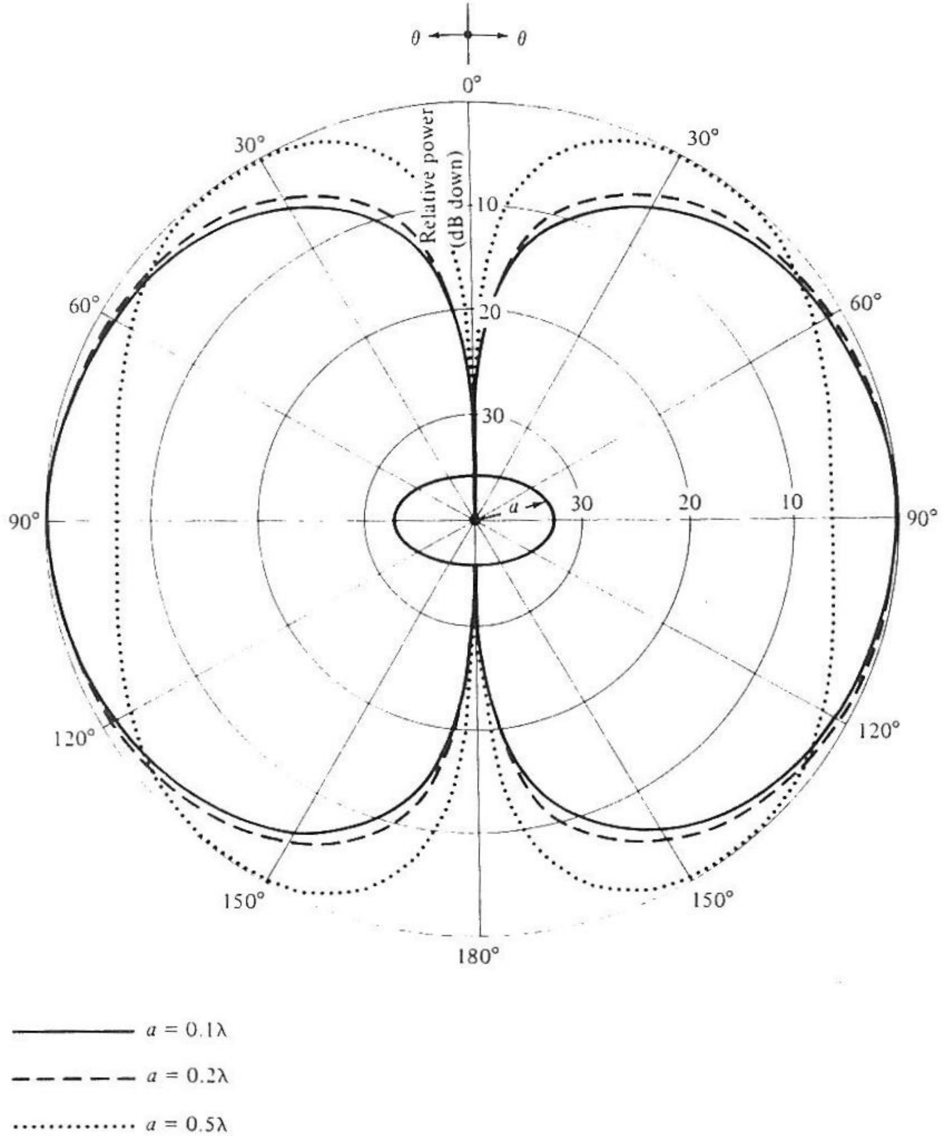
thus

$$\mathbf{E}(\vec{r})=\hat{\phi}\frac{\omega\mu I_0 a e^{-jkr}}{2r}J_1(ka\sin\theta)$$

Likewise

$$\mathbf{H}(\vec{r})=-\hat{\theta}\frac{kI_0 a e^{-jkr}}{2r}J_1(ka\sin\theta)$$

Radiation Pattern



Power Density and Radiation Intensity

Power Density

$$\mathbf{W}_{av} = \frac{1}{2}\text{Re}[\mathbf{E}\times\mathbf{H}^*] = \hat{r}\frac{|E_\phi|^2}{2\eta}$$

$$= \hat{r}\frac{(a\omega\mu)^2|I_0|^2}{8\eta r^2} J_1^2(ka\sin\theta)$$

Radiation Intensity

$$U = r^2 W_r = \frac{(a\omega\mu)^2|I_0|^2}{8\eta} J_1^2(ka\sin\theta)$$

Radiated Power

$$P_{rad} = \oiint_S \mathbf{W}\cdot d\mathbf{s} = \frac{\pi(a\omega\mu)^2|I_0|^2}{4\eta}\int_0^\pi J_1^2(ka\sin\theta)\sin\theta d\theta$$

Circular loop

- **So far, the analysis was done assuming a constant current, regardless of the loop radius.**
- **The "constant current" assumption is valid only when $a < 0.016\lambda$.**
- **For $a > 0.016\lambda$, the actual current can be written in terms of a Fourier series:**

$$I(\phi') = I_0 + 2\sum_{n=1}^{N} I_n \cos(n\phi')$$

where ϕ' is measured from the feed point.

Small circular loop approximation

If $ka << 1$, then $J_1(ka \sin\theta) \sim ka \sin\theta/2$. Then

$$\mathbf{E}(\vec{r}) = \hat{\phi}\frac{\eta(ka)^2 I_0 a e^{-jkr}}{4r}\sin\theta$$

$$\mathbf{H}(\vec{r}) = -\hat{\theta}\frac{(ka)^2 I_0 e^{-jkr}}{4r}\sin\theta$$

which is identical to the fields due to an infinitesimal circular loop antenna.

Small loop over ground plane

- **Using the infinitesimal magnetic dipole equivalence and following the same approach used for infinitesimal electric dipole over ground plane, the total electric field can be obtained.**