# Chapter 6 : Antenna Arrays

- Introduction
- Two-Element Array
- *N*-element Linear Array: Uniform Amplitude and Spacing
- *N*-element Linear Array: Directivity
- *N*-element Linear Array: Uniform spacing, Non-uniform Amplitude
- Planar Array

# Antenna Array: Introduction

- **Array is an assembly of antenna elements arranged in an orderly fashion. The elements are usually identical.**
- **Why array? When high gain and/or narrow beam are required:**
  – Single element -> Wide beam (low directivity)
  – Increasing size -> difficult to build and expensive
  – Useful especially when the element gain is low.

# Antenna Array: Introduction (2)

- **Advantages**
  - Higher directivity
  - Narrower beam
  - Lower sidelobes
  - Electronic steerable beam
- **Types**
  - Fix direction
  - Steerable : Mechanical or Electronic (phased arrays)

# Antenna Array: Introduction (3)

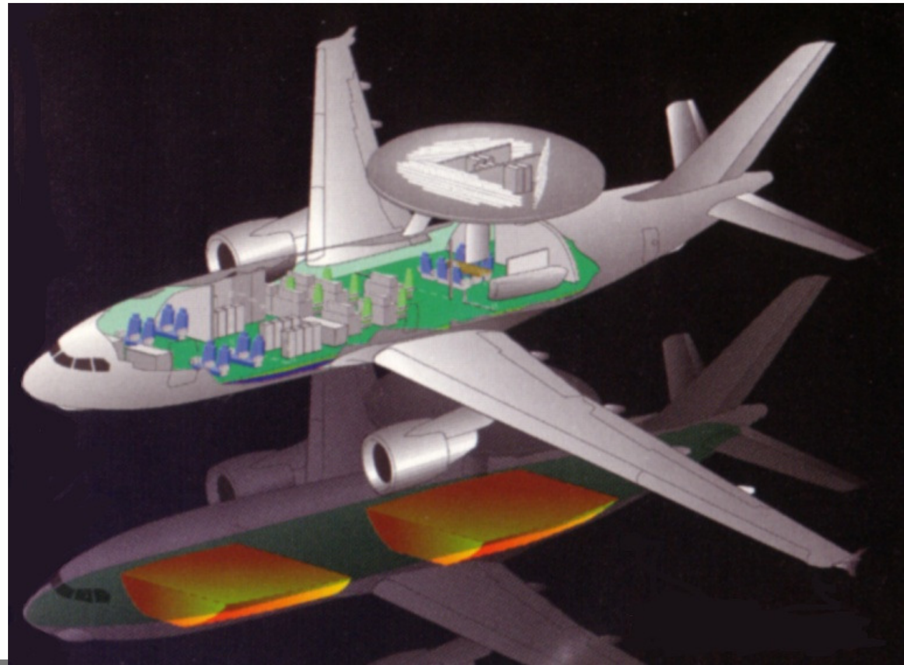
- **In an array of identical elements, there are in general five controls that can be used to shape the overall pattern of the antenna:**

1. Geometrical configuration (linear, circular, etc.)
2. Relative displacement between elements
3. Excitation amplitude of individual elements
4. Excitation phase of individual elements
5. Relative pattern of individual elements

# Examples

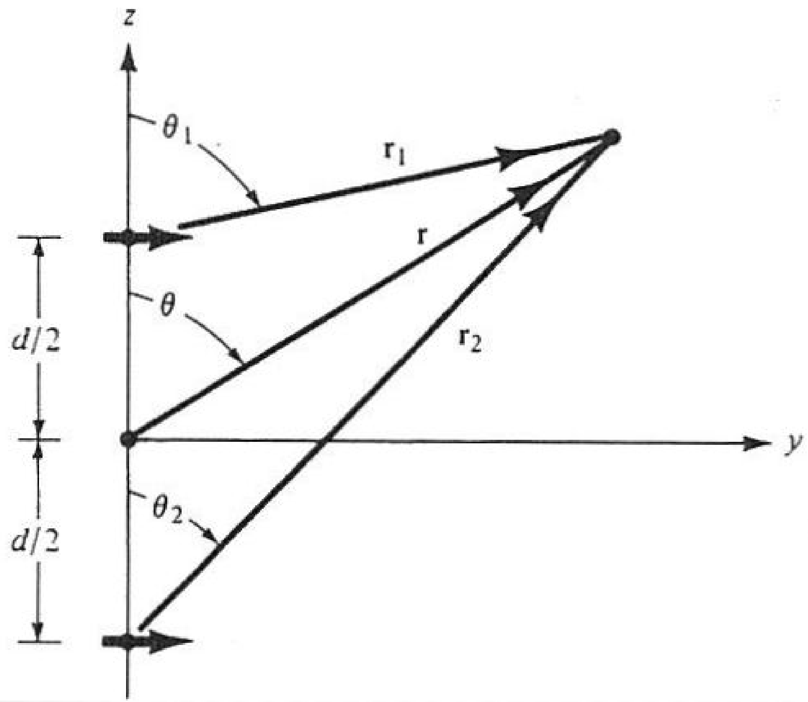
Very Large Antenna (VLA)

Airborne Warning and Control System (AWACS)

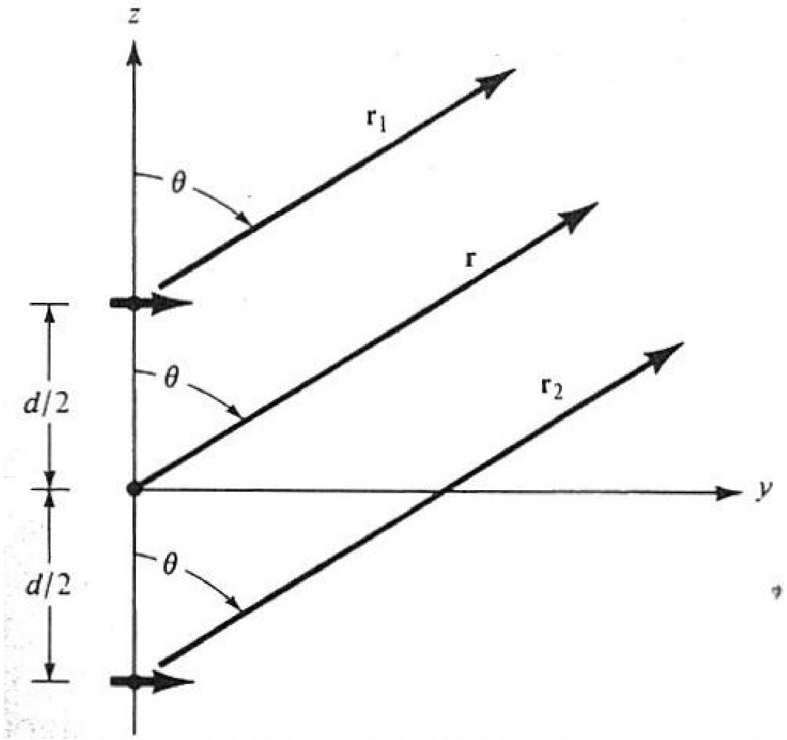


# Two-element Array

- **Consider two-element array of horizontal infinitesimal dipoles (assume no coupling between elements)**




Two infinitesimal dipoles




Far-field observation

# Two-element Array (2)

Recall the far-zone electric field of horizontal infinitesimal dipole in the y-z plane

$$\mathbf{E} = \hat{\theta} jk\eta I_0 l \frac{e^{-jkr}}{4\pi r}\cos\theta$$

Thus the total electric field becomes:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\theta} jk\eta l \left( I_1 \frac{e^{-jkr_1}}{4\pi r_1}\cos\theta_1 + I_2 \frac{e^{-jkr_2}}{4\pi r_2}\cos\theta_2 \right)$$

# Two-Element Array (3)

Using the far-field approximation

$$r_1 \cong r_2 \cong r \qquad r_1 \cong r - \frac{d}{2}\cos\theta \qquad r_2 \cong r + \frac{d}{2}\cos\theta$$

The total field becomes:

$$\mathbf{E} = \hat{\theta} jk\eta l \cos\theta \frac{e^{-jkr}}{4\pi r}\left(I_1 e^{jkd\cos\theta/2} + I_2 e^{-jkd\cos\theta/2}\right)$$

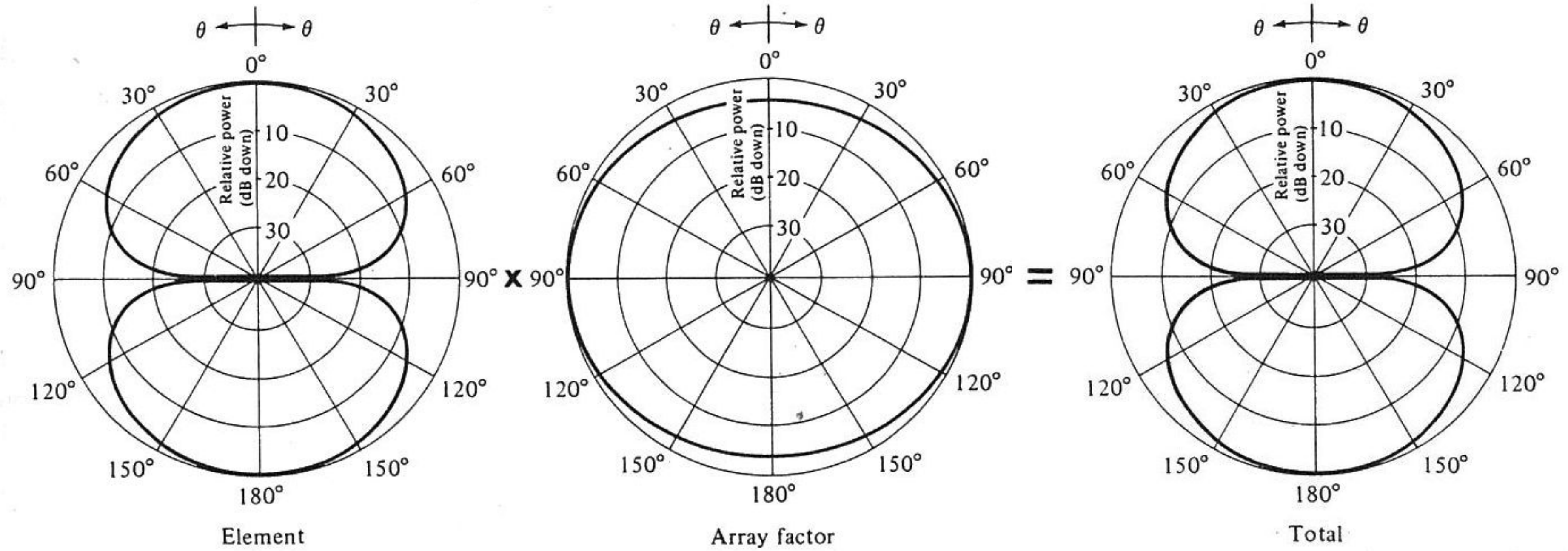
If $I_1 = I_0 e^{j\beta/2}; I_2 = I_0 e^{-j\beta/2}, \beta:$ phase difference

$$\mathbf{E} = \hat{\theta} jk\eta I_0 l \cos\theta \frac{e^{-jkr}}{4\pi r} 2\cos(k\frac{d}{2}\cos\theta + \frac{\beta}{2})$$

$$\boxed{\text{total field} = (\text{element factor}) \times (\text{array factor(AF)})}$$

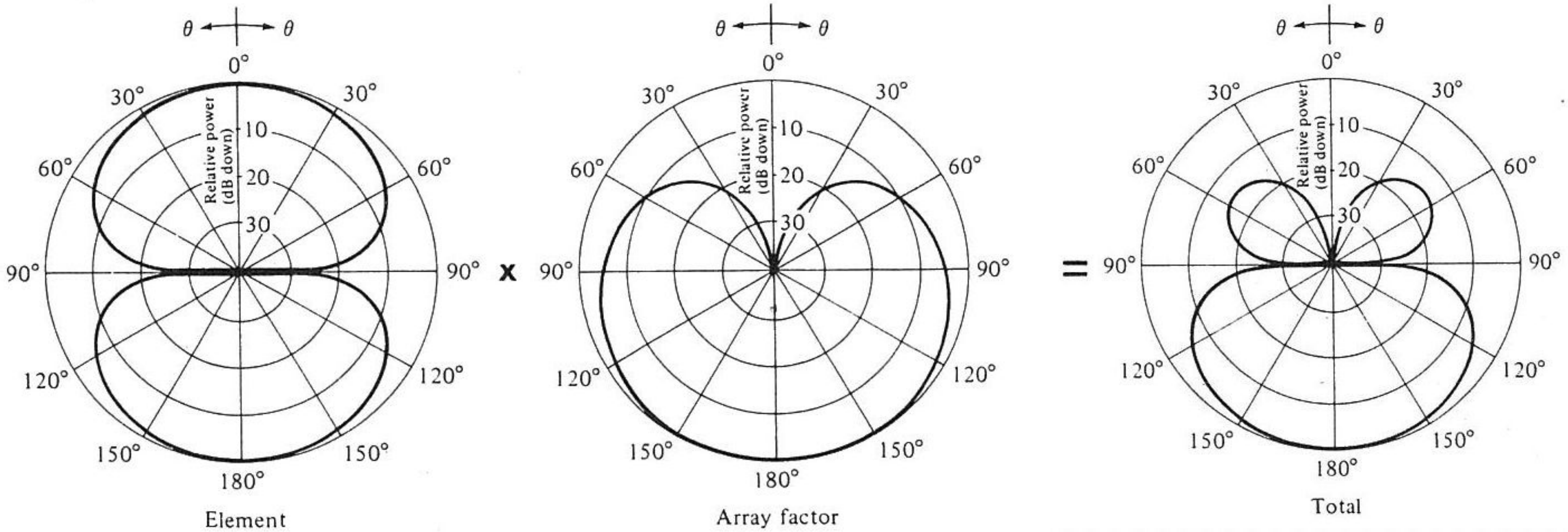
# Electric Field Pattern

$$\beta = 0, d = \lambda / 4$$



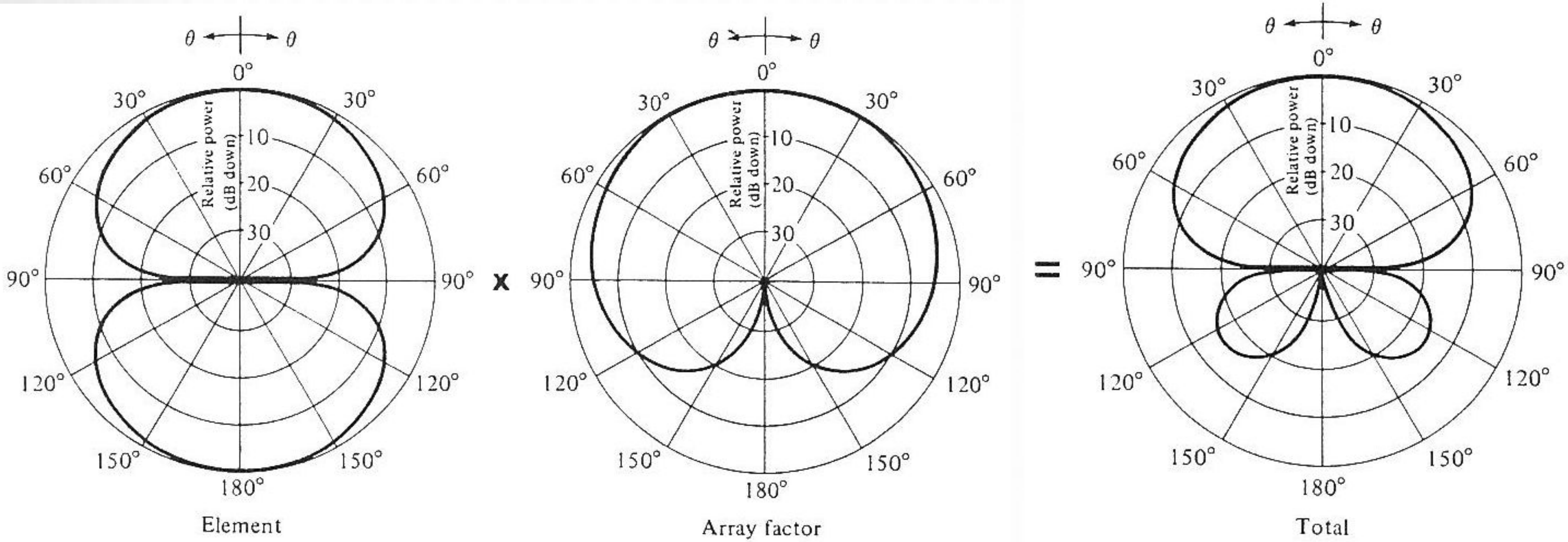
# Electric Field Pattern (2)

$$\beta = \pi / 2, d = \lambda / 4$$



# Electric Field Pattern (2)

$$\beta = -\pi / 2, d = \lambda / 4$$



# Quiz

- **Find the far-zone electric field of a two-element array of infinitesimal circular loops. Assume that the loops are parallel to the x-y plane and the two elements are aligned along the z axis.**
- **(i)** $I_1=I_0,\ I_2=I_0,\ d = \lambda/4$
- **(ii)** $I_1=I_0,\ I_2=I_0,\ d = \lambda/2$
- **(iii)** $I_1=I_0,\ I_2=-I_0,\ d = \lambda/2$
- **(iv)** $I_1=I_0,\ I_2=jI_0,\ d = \lambda/2$

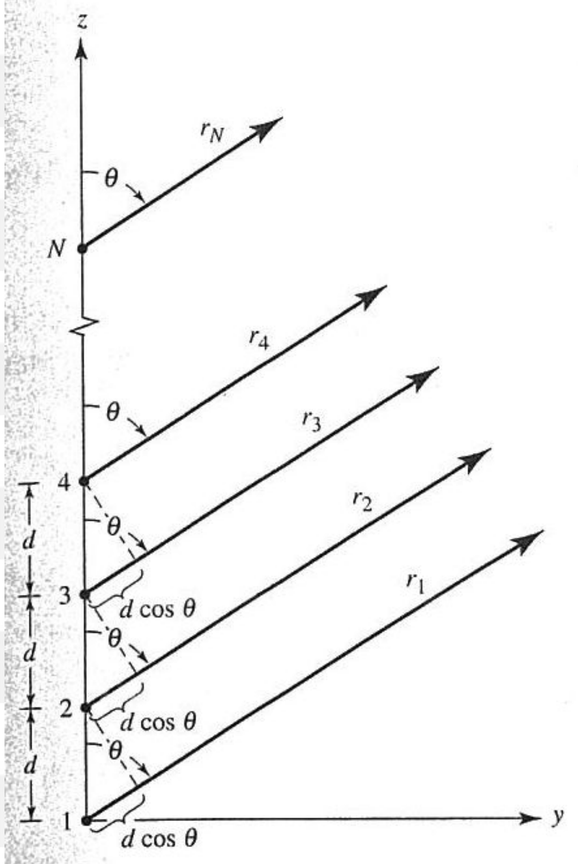
# $N$-element Linear Array: Uniform amplitude & spacing

If the amplitude and spacing are both uniform, the array factor becomes

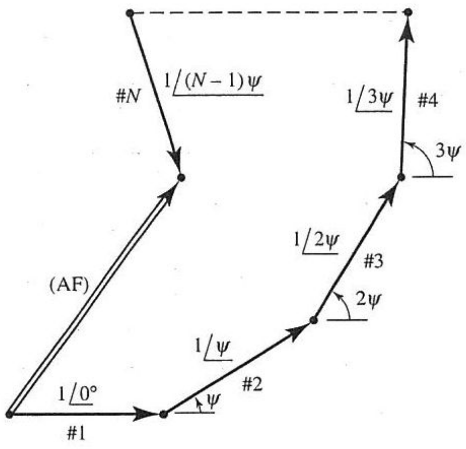
$$\mathrm{AF} = 1 + e^{j(kd\cos\theta+\beta)} + e^{j2(kd\cos\theta+\beta)} + \cdots + e^{j(N-1)(kd\cos\theta+\beta)}$$

$$= \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where $\psi = kd\cos\theta + \beta$




(a) Geometry




(b) Phasor diagram

# $N$-element Linear Array: Uniform amplitude & spacing (2)

thus

$$\mathrm{AF} = \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = e^{j\frac{N-1}{2}\psi} \frac{e^{j\frac{N}{2}\psi} - e^{-j\frac{N}{2}\psi}}{e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}}$$

$$= e^{j\frac{N-1}{2}\psi} \frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\frac{\psi}{2}}$$

If the reference point is the physical center of the array

$$\mathrm{AF} = \frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\frac{\psi}{2}} \underset{\psi:\text{small}}{\cong} \frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{\psi}{2}}$$

# $N$-element Linear Array: Uniform amplitude & spacing (3)

Normalized AF

$$(\mathrm{AF})_\mathrm{n} = \frac{\sin\left(\frac{N}{2}\psi\right)}{N\sin\frac{\psi}{2}} \underset{\psi:\text{small}}{\cong} \frac{\sin\left(\frac{N}{2}\psi\right)}{N\frac{\psi}{2}} = \mathrm{sinc}\left(\frac{N}{2}\psi\right)$$

Nulls

$$\sin\left(\frac{N}{2}\psi\right) = 0 \Rightarrow \frac{N}{2}\psi\Big|_{\theta=\theta_n} = \pm n\pi$$

$$\Rightarrow \theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n}{N}\pi\right)\right]$$

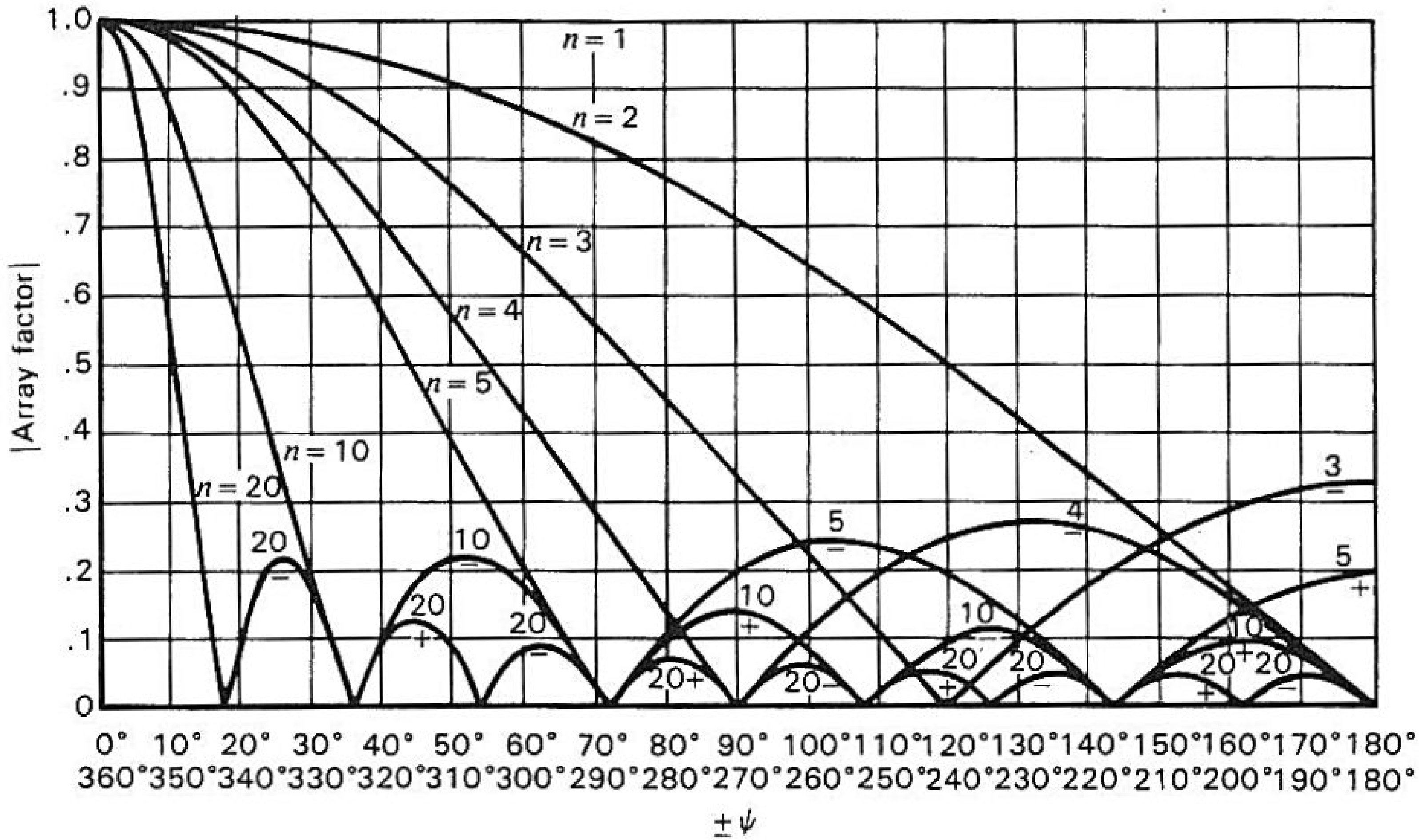
$$n = 1,2,3,\ldots; n \neq N, 2N, 3N, \ldots$$

Maxima

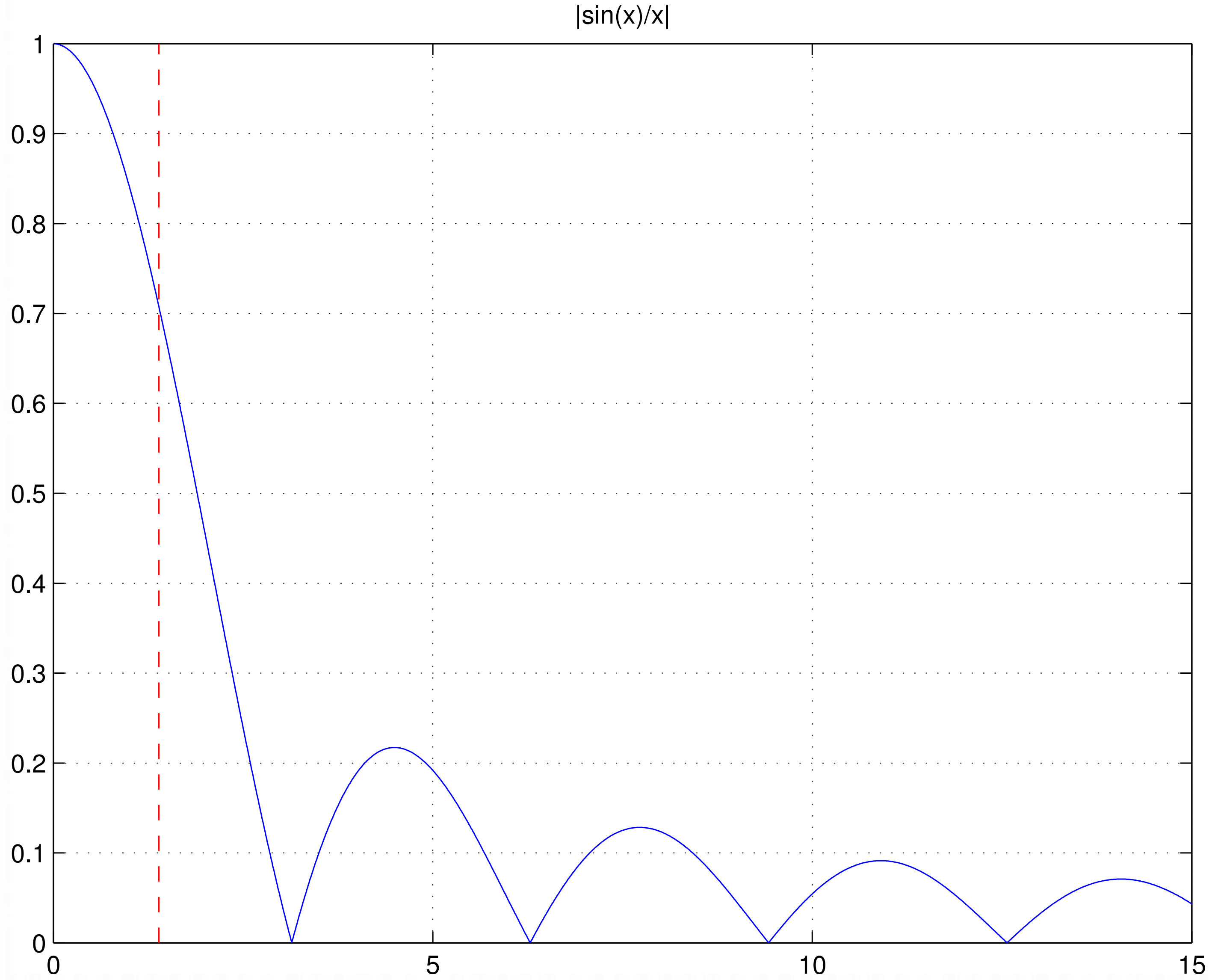
$$\frac{\psi}{2}\Big|_{\theta=\theta_m} = \pm m\pi \Rightarrow \theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm 2m\pi\right)\right]$$

$$m = 0,1,2,\ldots$$

# Normalized Array Factor



# Sinc function plot



# $N$-element Linear Array: Uniform amplitude & spacing (4)

3-dB point

$$\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\frac{\psi}{2}} = \frac{1}{\sqrt{2}} \Rightarrow \frac{N}{2}\psi\Big|_{\theta=\theta_h} = \pm 1.391$$

$$\Rightarrow \theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]$$

$$= \frac{\pi}{2} - \sin^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]$$

HPBW (symmetrical case) $\Theta_h = 2\left|\theta_m - \theta_h\right|$

Secondary Maxima

$$\sin\left(\frac{N}{2}\psi\right)\Big|_{\theta=\theta_s} \cong \pm 1 \Rightarrow \frac{N}{2}\psi\Big|_{\theta=\theta_s} \cong \pm\frac{2s+1}{2}\pi$$

$$\Rightarrow \theta_s \cong \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2s+1}{N}\pi\right)\right]; s = 1,2,3,\ldots$$

# $N$-element Linear Array: Uniform amplitude & spacing (5)

First sidelobe
$$\left.\frac{N}{2}\psi\right|_{\theta=\theta_s} \cong \pm\frac{3}{2}\pi$$

$$\Rightarrow \theta_s \cong \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{3}{N}\pi\right)\right]$$

First sidelobe level

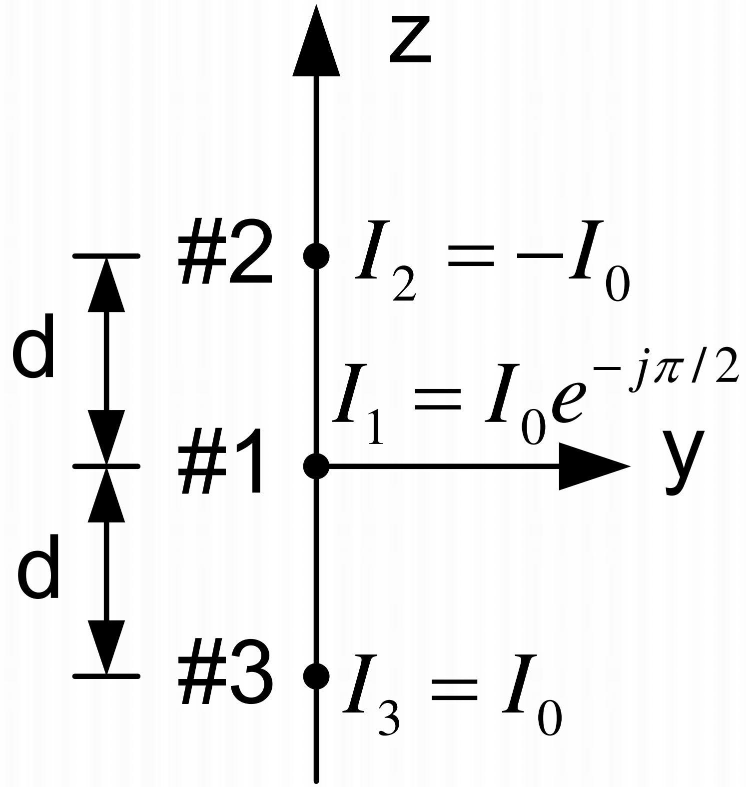
$$(\mathrm{AF})_\mathrm{n} \cong \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{N\frac{\psi}{2}}\right]_{\theta=\theta_s, s=1} = \frac{2}{3\pi} = 0.212 = -13.46\,\mathrm{dB}$$

# Example

A 3-element array of isotropic sources has the phase and amplitude relationships shown. The spacing between elements is $d=\lambda/2$.

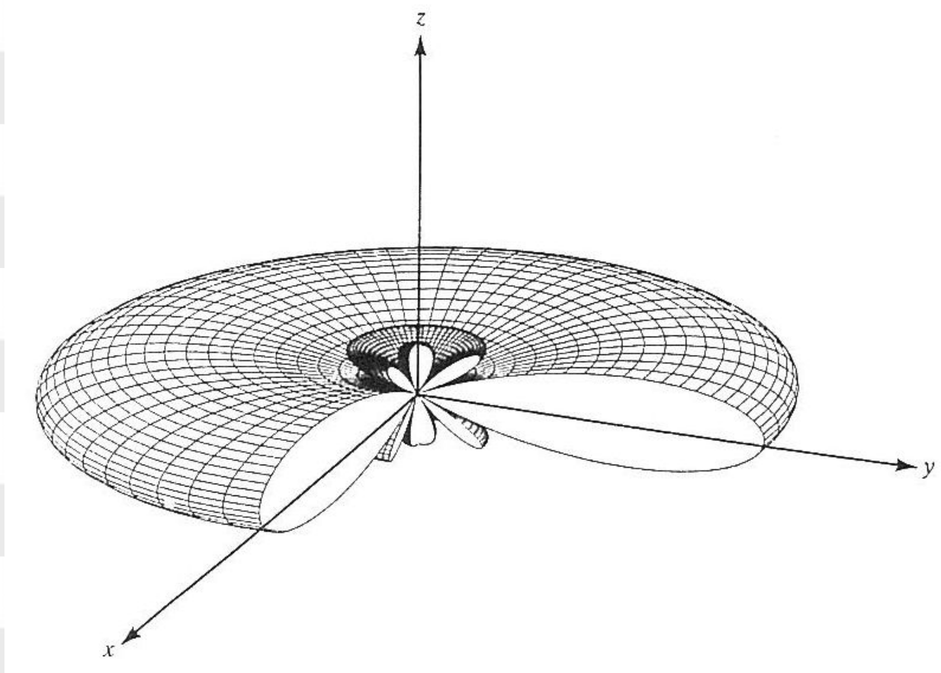
(a)Find the array factor.

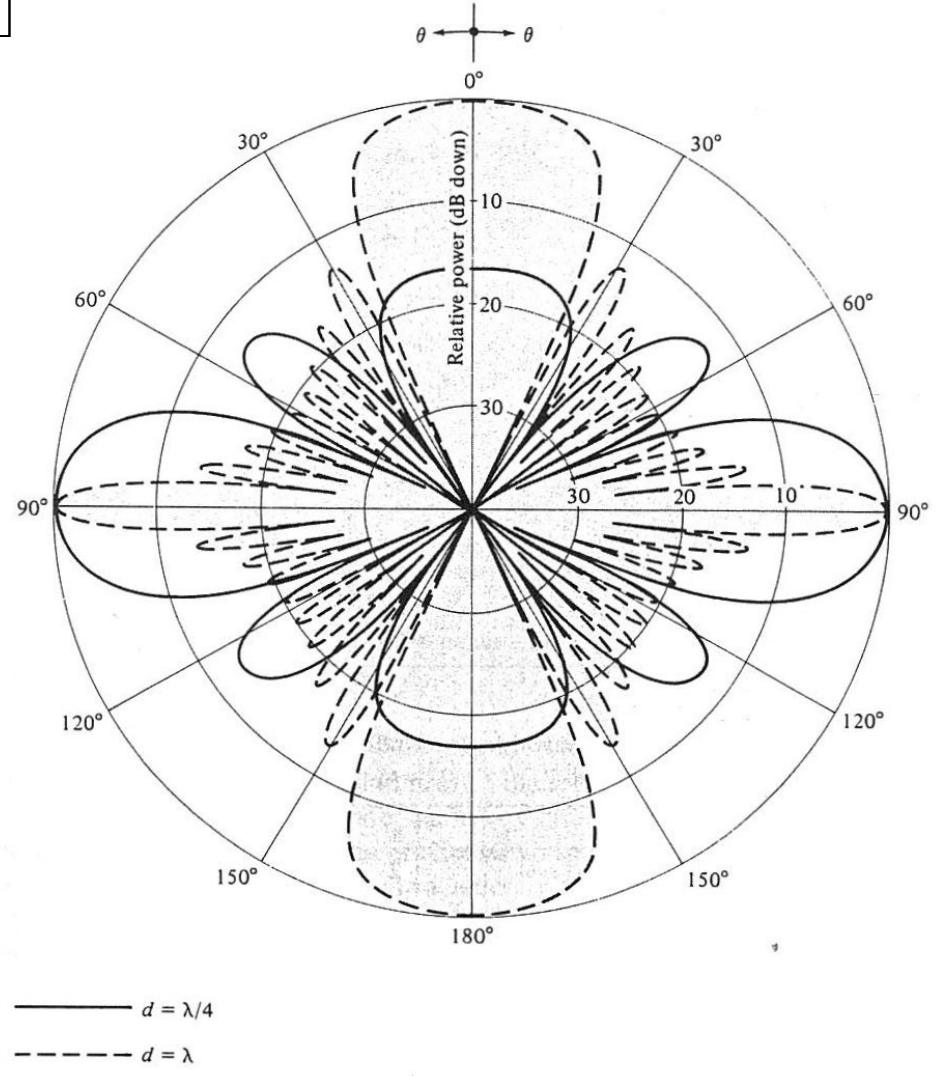
(b)Find all the nulls.



# Broadside Array

$$\psi = kd\cos\theta + \beta\big|_{\theta=\pi/2} = \beta = 0$$

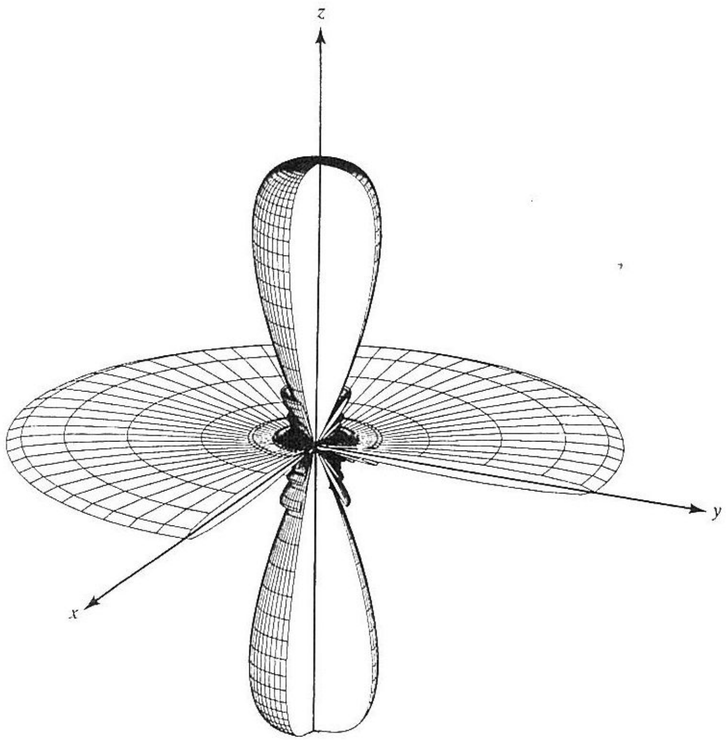



$$N = 10, d = \lambda / 4$$

# Grating Lobes

Should avoid $d = n\lambda$ because




$$\psi = kd\cos\theta + \beta \Big|_{\substack{d=n\lambda,\beta=0 \\ n=1,2,3,\ldots}}$$

$$= 2n\pi\cos\theta \Big|_{\theta=0,\pi} = \pm 2n\pi$$

No grating lobes

$$d_{\max} < \lambda$$

$N = 10, d = \lambda$

**Table 6.1** NULLS, MAXIMA, HALF-POWER POINTS, AND MINOR LOBE MAXIMA FOR UNIFORM AMPLITUDE BROADSIDE ARRAYS

| | |
|---|---|
| NULLS | $\theta_n = \cos^{-1}\left(\pm \frac{n}{N}\frac{\lambda}{d}\right)$ <br> $n = 1, 2, 3, \ldots$ <br> $n \neq N, 2N, 3N, \ldots$ |
| MAXIMA | $\theta_m = \cos^{-1}\left(\pm \frac{m\lambda}{d}\right)$ <br> $m = 0, 1, 2, \ldots$ |
| HALF-POWER POINTS | $\theta_h \simeq \cos^{-1}\left(\pm \frac{1.391\lambda}{\pi N d}\right)$ <br> $\pi d/\lambda \ll 1$ |
| MINOR LOBE MAXIMA | $\theta_s \simeq \cos^{-1}\left[\pm \frac{\lambda}{2d}\left(\frac{2s+1}{N}\right)\right]$ <br> $s = 1, 2, 3, \ldots$ <br> $\pi d/\lambda \ll 1$ |

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Table 6.2 BEAMWIDTHS FOR UNIFORM AMPLITUDE BROADSIDE ARRAYS

| | |
|---|---|
| FIRST NULL BEAMWIDTH (FNBW) | $\Theta_n = 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{\lambda}{Nd}\right)\right]$ |
| HALF-POWER BEAMWIDTH (HPBW) | $\Theta_h \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi Nd}\right)\right]$ <br> $\pi d/\lambda \ll 1$ |
| FIRST SIDE LOBE BEAMWIDTH (FSLBW) | $\Theta_s \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{3\lambda}{2dN}\right)\right]$ <br> $\pi d/\lambda \ll 1$ |

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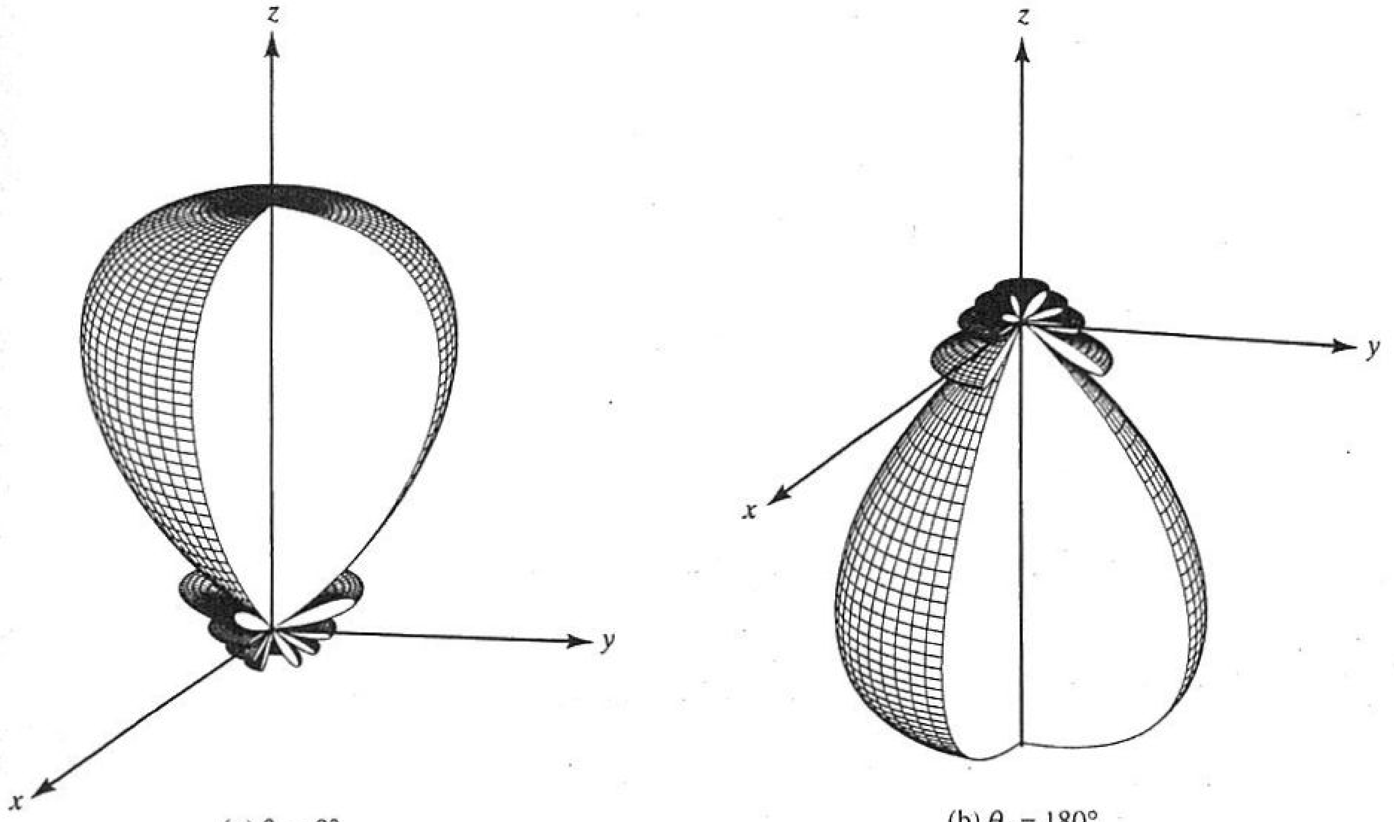
# Ordinary End-fire Array

$$\psi = kd\cos\theta + \beta\big|_{\theta=0} = kd + \beta \Rightarrow \beta = -kd$$

$$\psi = kd\cos\theta + \beta\big|_{\theta=\pi} = -kd + \beta \Rightarrow \beta = kd$$

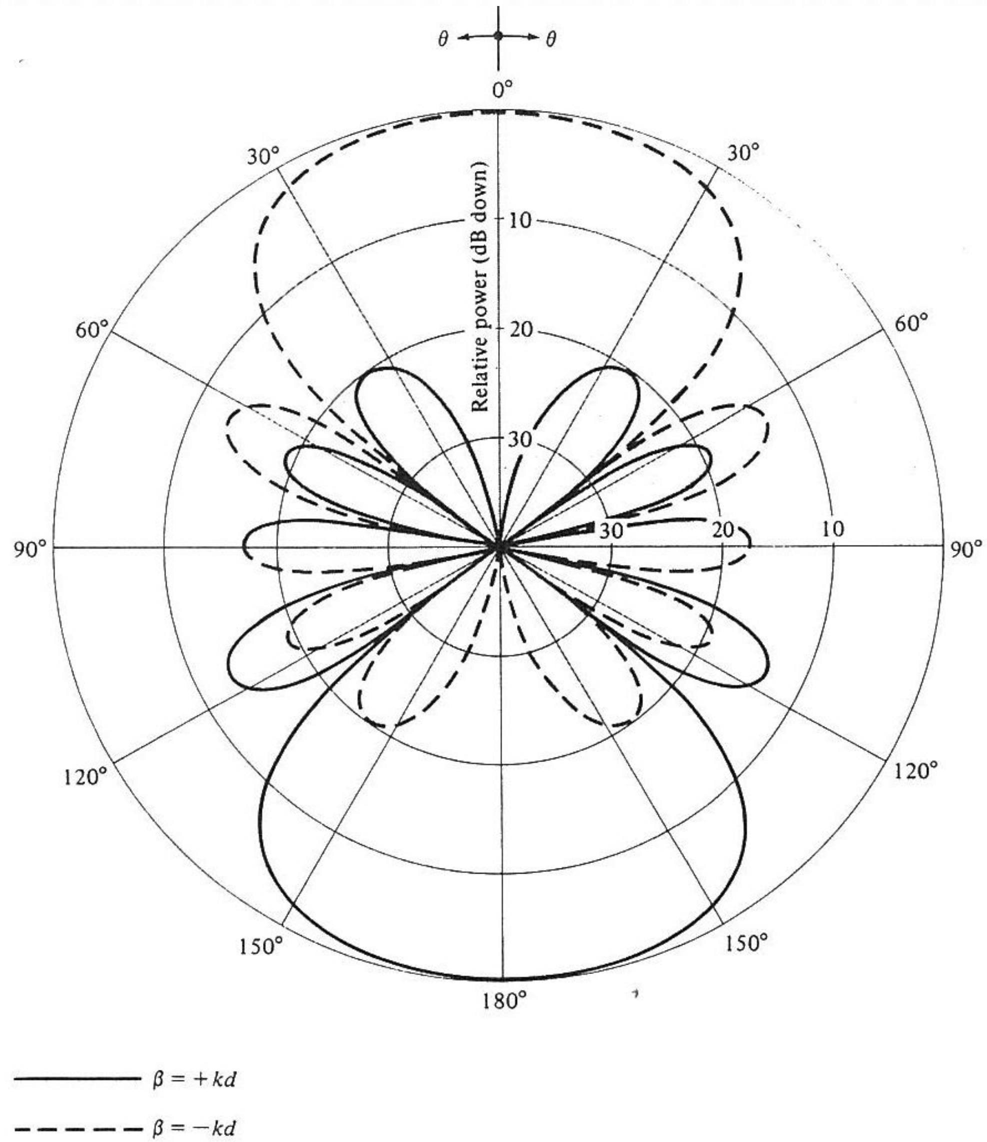



(a) $\theta_0 = 0°$ (b) $\theta_0 = 180°$

$N = 10$

$d = \lambda / 4$

# Ordinary End-fire Array (2)




$$N = 10, d = \lambda / 4$$

# Grating Lobes

- If $d=\lambda/2$, end-fire radiation exists simultaneously in both directions.
- If $d = n\lambda$, also broad-side radiation.
- To avoid grating lobes,

$$d_{\max} < \frac{\lambda}{2}$$

**Table 6.3** NULLS, MAXIMA, HALF-POWER POINTS, AND MINOR LOBE MAXIMA FOR UNIFORM AMPLITUDE ORDINARY END-FIRE ARRAYS

| | |
|---|---|
| NULLS | $\theta_n = \cos^{-1}\left(1 - \frac{n\lambda}{Nd}\right)$<br>$n = 1, 2, 3, \ldots$<br>$n \neq N, 2N, 3N, \ldots$ |
| MAXIMA | $\theta_m = \cos^{-1}\left(1 - \frac{m\lambda}{d}\right)$<br>$m = 0, 1, 2, \ldots$ |
| HALF-POWER POINTS | $\theta_h \simeq \cos^{-1}\left(1 - \frac{1.391\lambda}{\pi dN}\right)$<br>$\pi d/\lambda \ll 1$ |
| MINOR LOBE MAXIMA | $\theta_s \simeq \cos^{-1}\left[1 - \frac{(2s+1)\lambda}{2Nd}\right]$<br>$s = 1, 2, 3, \ldots$<br>$\pi d/\lambda \ll 1$ |

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**Table 6.4 BEAMWIDTHS FOR UNIFORM AMPLITUDE ORDINARY END-FIRE ARRAYS**

| | |
|---|---|
| FIRST NULL BEAMWIDTH (FNBW) | $\Theta_n = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right)$ |
| HALF-POWER BEAMWIDTH (HPBW) | $\Theta_h \simeq 2\cos^{-1}\left(1 - \frac{1.391\lambda}{\pi dN}\right)$ <br> $\pi d/\lambda \ll 1$ |
| FIRST SIDE LOBE BEAMWIDTH (FSLBW) | $\Theta_s \simeq 2\cos^{-1}\left(1 - \frac{3\lambda}{2Nd}\right)$ <br> $\pi d/\lambda \ll 1$ |

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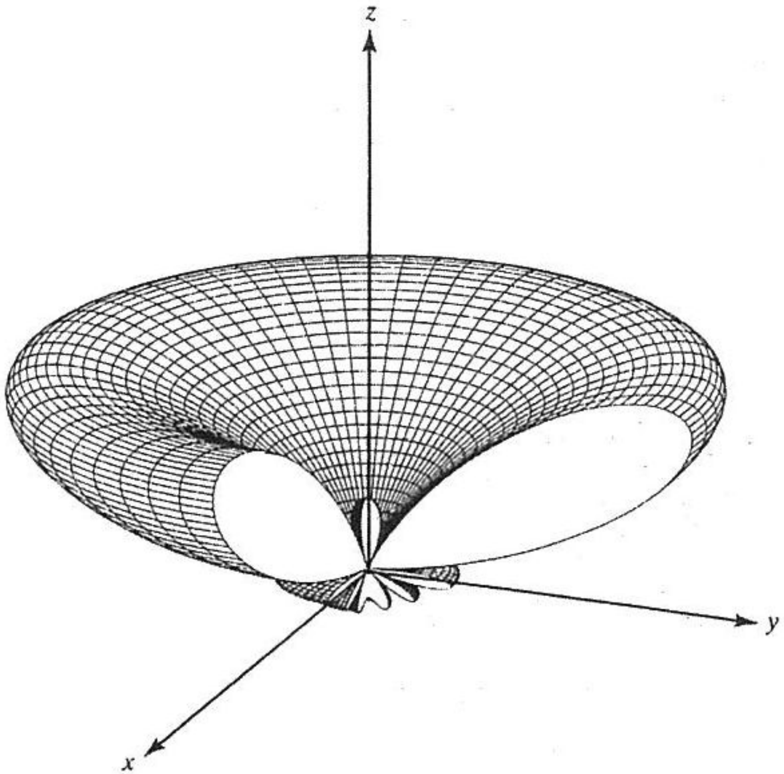
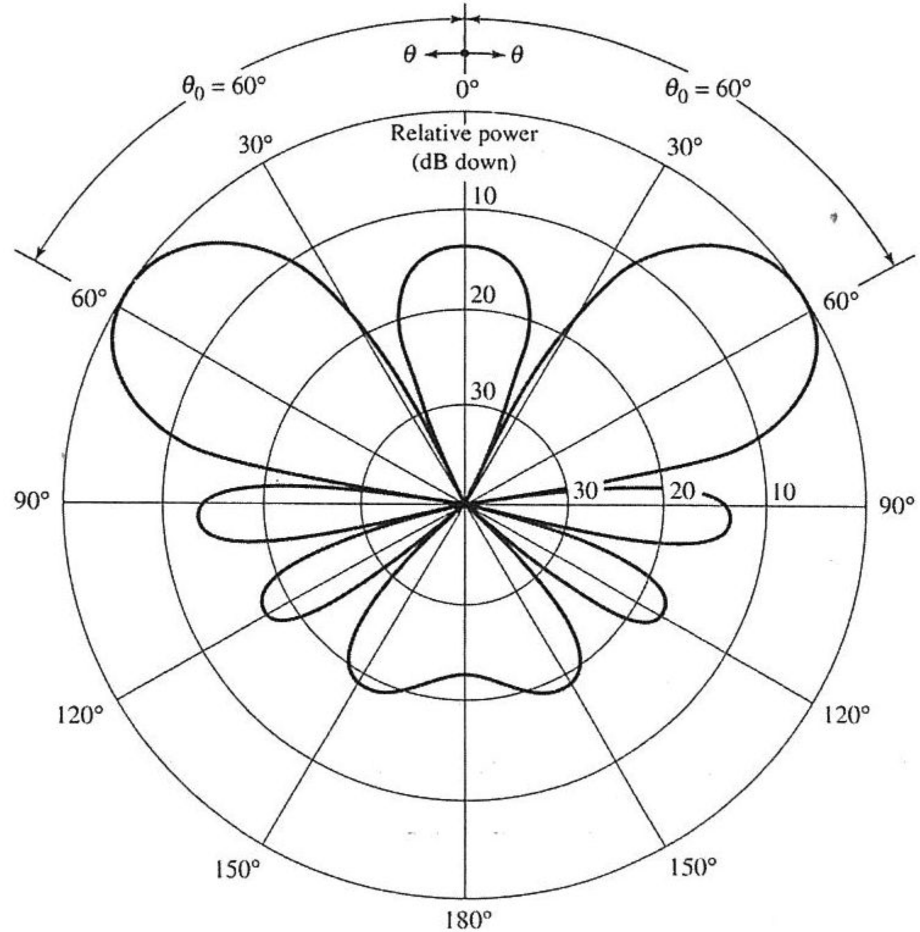
# Phased (Scanning) Array

$$\psi = kd\cos\theta + \beta\Big|_{\theta=\theta_0} = kd\cos\theta_0 + \beta \Rightarrow \beta = -kd\cos\theta_0$$




$$N = 10, d = \lambda/4$$

# Phased (Scanning) Array (2)

HPBW

$$\Theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(kd\cos\theta_0 - \frac{2.782}{N}\right)\right] - \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(kd\cos\theta_0 + \frac{2.782}{N}\right)\right]$$

$$= \cos^{-1}\left(\cos\theta_0 - \frac{2.782}{Nkd}\right) - \cos^{-1}\left(\cos\theta_0 + \frac{2.782}{Nkd}\right)$$

Since $N = (L+d)/d$,

$$\Theta_h = \cos^{-1}\left(\cos\theta_0 - 0.443\frac{\lambda}{L+d}\right) - \cos^{-1}\left(\cos\theta_0 + 0.443\frac{\lambda}{L+d}\right)$$

Not valid for end-fire arrays, i.e., $\theta_0 = 0, \pi$.

# Hansen-Woodyard End-fire Array

$$\psi = -\left(kd + \frac{2.92}{N}\right) \cong -\left(kd + \frac{\pi}{N}\right) \text{ for } \theta_0 = 0$$

$$\psi = \left(kd + \frac{2.92}{N}\right) \cong \left(kd + \frac{\pi}{N}\right) \text{ for } \theta_0 = \pi$$

Additional Conditions:

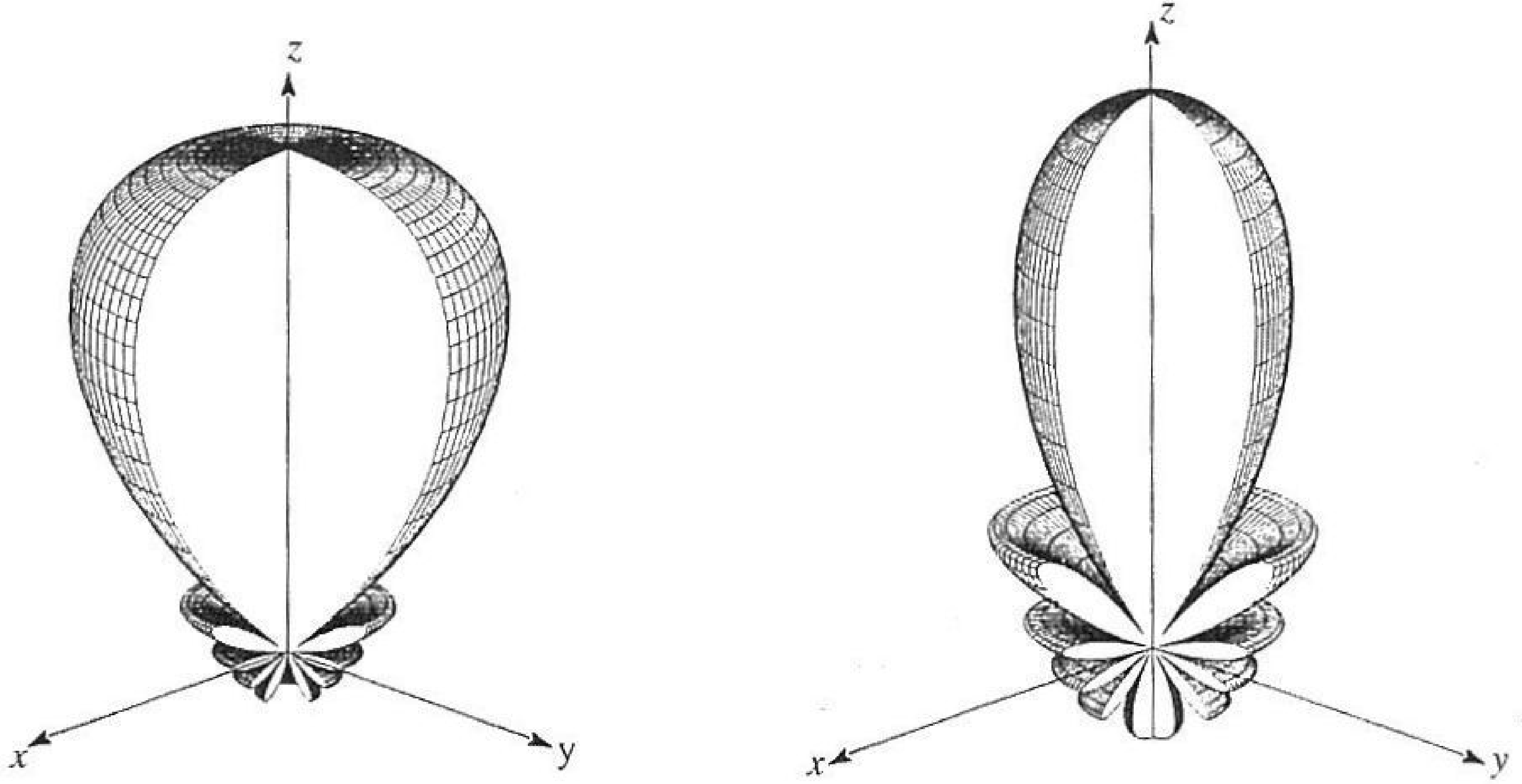
$$\text{For } \theta_0 = 0; |\psi| = |kd\cos\theta + \beta|_{\theta=0} = \frac{\pi}{N}; |\psi| = |kd\cos\theta + \beta|_{\theta=\pi} \cong \pi$$

$$\text{For } \theta_0 = \pi; |\psi| = |kd\cos\theta + \beta|_{\theta=\pi} = \frac{\pi}{N}; |\psi| = |kd\cos\theta + \beta|_{\theta=0} \cong \pi$$

which yields

$$d = \frac{N-1}{N}\frac{\lambda}{4} \underset{N:\text{large}}{\cong} \frac{\lambda}{4}$$

# Hansen-Woodyard End-fire Array (2)




(a) Ordinary (b) Hansen-Woodyard

# Hansen-Woodyard End-fire Array (3)

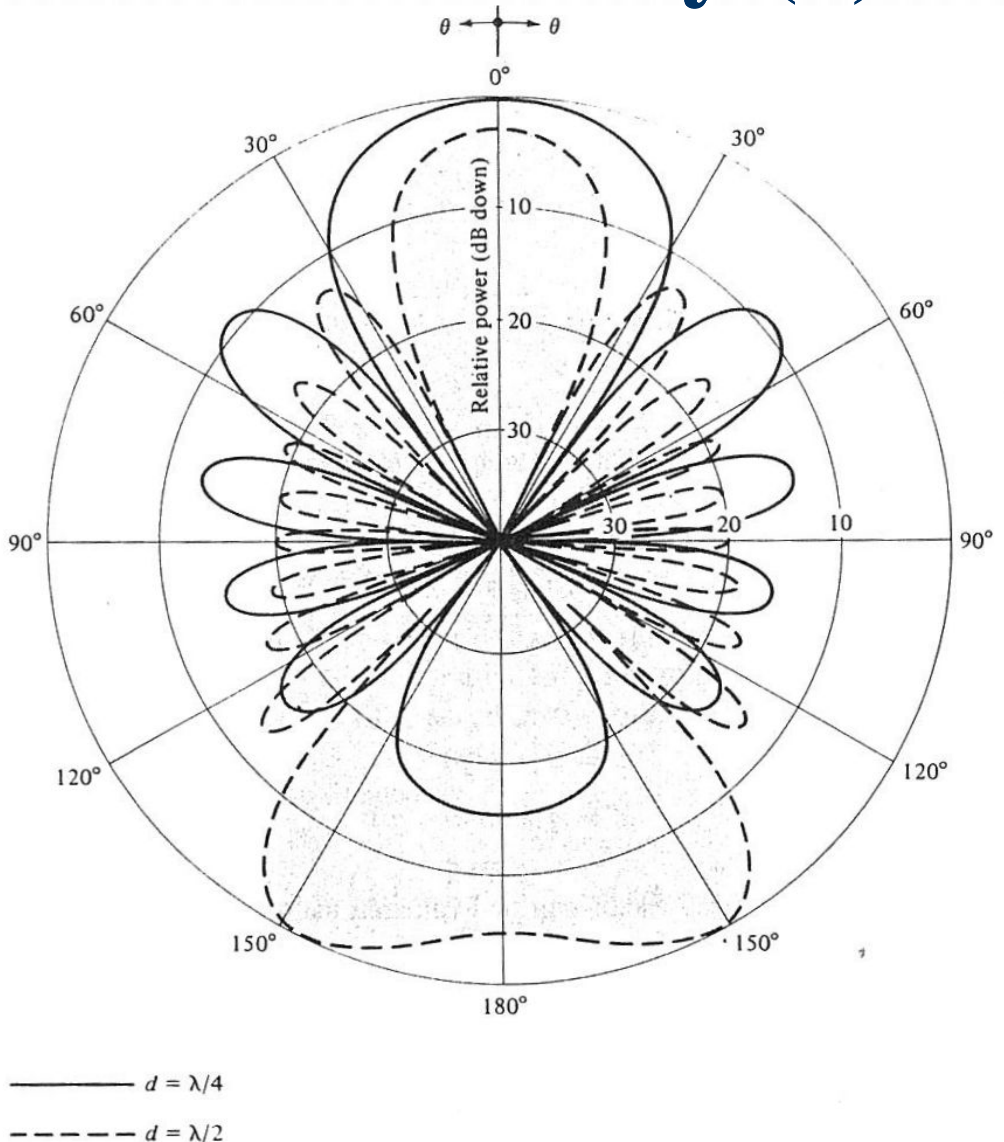


Table 6.5 NULLS, MAXIMA, HALF-POWER POINTS, AND MINOR LOBE MAXIMA FOR UNIFORM AMPLITUDE HANSEN-WOODYARD END-FIRE ARRAYS

| | |
|---|---|
| NULLS | $\theta_n = \cos^{-1}\left[1 + (1 - 2n)\frac{\lambda}{2dN}\right]$ <br> $n = 1, 2, 3, \ldots$ <br> $n \neq N, 2N, 3N, \ldots$ |
| SECONDARY MAXIMA | $\theta_m = \cos^{-1}\left\{1 + [1 - (2m + 1)]\frac{\lambda}{2Nd}\right\}$ <br> $m = 1, 2, 3, \ldots$ <br> $\pi d/\lambda \ll 1$ |
| HALF-POWER POINTS | $\theta_h = \cos^{-1}\left(1 - 0.1398\frac{\lambda}{Nd}\right)$ <br> $\pi d/\lambda \ll 1$ <br> $N$ large |
| MINOR LOBE MAXIMA | $\theta_s = \cos^{-1}\left(1 - \frac{s\lambda}{Nd}\right)$ <br> $s = 1, 2, 3, \ldots$ <br> $\pi d/\lambda \ll 1$ |

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**Table 6.6 BEAMWIDTHS FOR UNIFORM AMPLITUDE HANSEN-WOODYARD END-FIRE ARRAYS**

| | |
|---|---|
| FIRST NULL BEAMWIDTH (FNBW) | $\Theta_n = 2\cos^{-1}\left(1 - \dfrac{\lambda}{2dN}\right)$ |
| HALF-POWER BEAMWIDTH (HPBW) | $\Theta_h = 2\cos^{-1}\left(1 - 0.1398\dfrac{\lambda}{Nd}\right)$ <br> $\pi d/\lambda \ll 1$ <br> $N$ large |
| FIRST SIDE LOBE BEAMWIDTH (FSLBW) | $\Theta_s = 2\cos^{-1}\left(1 - \dfrac{\lambda}{Nd}\right)$ <br> $\pi d/\lambda \ll 1$ |

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# $N$-Element Array: Directivity

- **Broadside Array**

Recall that AF for broadside arrays is given by

$$(\mathrm{AF})_n = \frac{\sin\left(\frac{N}{2}\psi\right)}{N\sin\frac{\psi}{2}} \underset{\psi:\text{small}}{\cong} \frac{\sin\left(\frac{N}{2}\psi\right)}{N\frac{\psi}{2}};\psi = kd\cos\theta$$

The radiation intensity then becomes:

$$U(\theta,\phi) = \left[(\mathrm{AF})_n\right]^2 = \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{N\frac{kd\cos\theta}{2}}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2;Z = \frac{N}{2}kd\cos\theta$$

Clearly, the maximum $U_{max}$=1 at $\theta=\pi/2$

# $N$-Element Array: Directivity (2)

- **Broadside Array (cont'd)**

The "average" radiation intensity can be obtained from

$$U_0 = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi}\int U(\theta,\phi)d\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sin(Z)}{Z}\right]^2 \sin\theta d\theta d\phi$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{\sin(\frac{N}{2}kd\cos\theta)}{\frac{N}{2}kd\cos\theta}\right]^2 \sin\theta d\theta$$

Using $\quad Z = \frac{N}{2}kd\cos\theta; dZ = -\frac{N}{2}kd\sin\theta d\theta$

$$U_0 = -\frac{1}{Nkd}\int_{Nkd/2}^{-Nkd/2}\left[\frac{\sin(Z)}{Z}\right]^2 dZ = \frac{1}{Nkd}\int_{-Nkd/2}^{Nkd/2}\left[\frac{\sin(Z)}{Z}\right]^2 dZ$$

# *N*-Element Array: Directivity (3)

- **Broadside Array (cont'd)**

For a large array (*Nkd*/2 -> large),

$$U_0 = \frac{1}{Nkd}\int_{-Nkd/2}^{Nkd/2}\left[\frac{\sin(Z)}{Z}\right]^2 dZ \cong \frac{1}{Nkd}\int_{-\infty}^{\infty}\left[\frac{\sin(Z)}{Z}\right]^2 dZ$$

Since $\int_{-\infty}^{\infty}\left[\frac{\sin(Z)}{Z}\right]^2 dZ = \pi$

$$\boxed{U_0 \cong \frac{\pi}{Nkd}}$$

The directivity is then given by

$$D_0 = \frac{U_{max}}{U_0} \cong \frac{Nkd}{\pi} = 2N\frac{d}{\lambda}$$

Using $L=(N-1)d$

$$D_0 \cong 2N\frac{d}{\lambda} \underset{L=(N-1)d}{=} 2\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \underset{L>>d}{\cong} 2\frac{L}{\lambda}$$

# $N$-Element Array: Directivity (4)

- **Ordinary end-fire Array**

Recall that AF for ordinary end-fire arrays ($\theta$=0) is given by

$$(\mathrm{AF})_n = \frac{\sin\left(\frac{N}{2}\psi\right)}{N\sin\frac{\psi}{2}} \underset{\psi:\text{small}}{\cong} \frac{\sin\left(\frac{N}{2}\psi\right)}{N\frac{\psi}{2}};\psi = kd(\cos\theta - 1)$$

The radiation intensity then becomes:

$$U(\theta,\phi) = \left[(\mathrm{AF})_n\right]^2 = \left[\frac{\sin\left(\frac{N}{2}kd(\cos\theta - 1)\right)}{N\frac{kd(\cos\theta - 1)}{2}}\right]^2 = \left[\frac{\sin(Z)}{Z}\right]^2;Z = \frac{N}{2}kd(\cos\theta - 1)$$

Clearly, the maximum $U_{max}$=1 at θ=0

# *N*-Element Array: Directivity (5)

- **Ordinary end-fire Array (cont'd)**

The "average" radiation intensity can be obtained from

$$U_0 = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi}\int U(\theta,\phi)d\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sin(Z)}{Z}\right]^2 \sin\theta d\theta d\phi$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{\sin(\frac{N}{2}kd(\cos\theta-1))}{\frac{N}{2}kd(\cos\theta-1)}\right]^2 \sin\theta d\theta$$

Using $Z = \frac{N}{2}kd(\cos\theta-1); dZ = -\frac{N}{2}kd\sin\theta d\theta$

$$U_0 = -\frac{1}{Nkd}\int_0^{-Nkd}\left[\frac{\sin(Z)}{Z}\right]^2 dZ = \frac{1}{Nkd}\int_0^{Nkd}\left[\frac{\sin(Z)}{Z}\right]^2 dZ$$

# $N$-Element Array: Directivity (6)

- **Ordinary end-fire Array (cont'd)**

For a large array ($Nkd$ -> large),

$$U_0 = \frac{1}{Nkd}\int_0^{Nkd}\left[\frac{\sin(Z)}{Z}\right]^2 dZ \cong \frac{1}{Nkd}\int_0^{\infty}\left[\frac{\sin(Z)}{Z}\right]^2 dZ$$

Thus
$$\boxed{U_0 \cong \frac{\pi}{2Nkd}}$$

The directivity is then given by

$$D_0 = \frac{U_{\max}}{U_0} \cong \frac{2Nkd}{\pi} = 4N\frac{d}{\lambda}$$

Using $L=(N-1)d$
$$D_0 \cong 4N\frac{d}{\lambda} \underset{L=(N-1)d}{=} 4\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \underset{L>>d}{\cong} 4\frac{L}{\lambda}$$

# *N*-Element Array: Directivity (7)

- **Hansen-Woodyard end-fire Array**

For a large array (*Nkd* -> large),

$$U_0 = \frac{1}{Nkd}\left(\frac{\pi}{2}\right)^2\left[\frac{\pi}{2}+\frac{2}{\pi}-1.8515\right] = \frac{0.871}{Nkd} = 0.554\frac{\pi}{2Nkd}$$

The directivity is then given by

$$D_0 = \frac{U_{\max}}{U_0} = \frac{1}{0.554}\frac{2Nkd}{\pi} = 1.805\left(4N\frac{d}{\lambda}\right)$$

Using $L=(N-1)d$

$$D_0 \underset{L=(N-1)d}{=} 1.805\left[4\left(1+\frac{L}{d}\right)\frac{d}{\lambda}\right] \underset{L>>d}{\cong} 1.805\left[4\frac{L}{\lambda}\right]$$

**TABLE 6.8 Directivities for Broadside and End-Fire Arrays**

| Array | Directivity | |
|---|---|---|
| BROADSIDE | $D_0 = 2N\left(\frac{d}{\lambda}\right) = 2\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 2\left(\frac{L}{\lambda}\right)$<br>$N\pi d/\lambda \to \infty, L \gg d$ | |
| END-FIRE (ORDINARY) | $D_0 = 4N\left(\frac{d}{\lambda}\right) = 4\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 4\left(\frac{L}{\lambda}\right)$<br>$2N\pi d/\lambda \to \infty, L \gg d$ | Only one maximum ($\theta = 0°$ or $180°$) |
| | $D_0 = 2N\left(\frac{d}{\lambda}\right) = 2\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 2\left(\frac{L}{\lambda}\right)$ | Two maxima ($\theta = 0°$ and $180°$) |
| END-FIRE (HANSEN-WOODYARD) | $D_0 = 1.805\left[4N\left(\frac{d}{\lambda}\right)\right] = 1.805\left[4\left(1+\frac{L}{d}\right)\frac{d}{\lambda}\right] = 1.805\left[4\left(\frac{L}{\lambda}\right)\right]$<br>$2N\pi d/\lambda \to \infty, L \gg d$ | |

2.56dB

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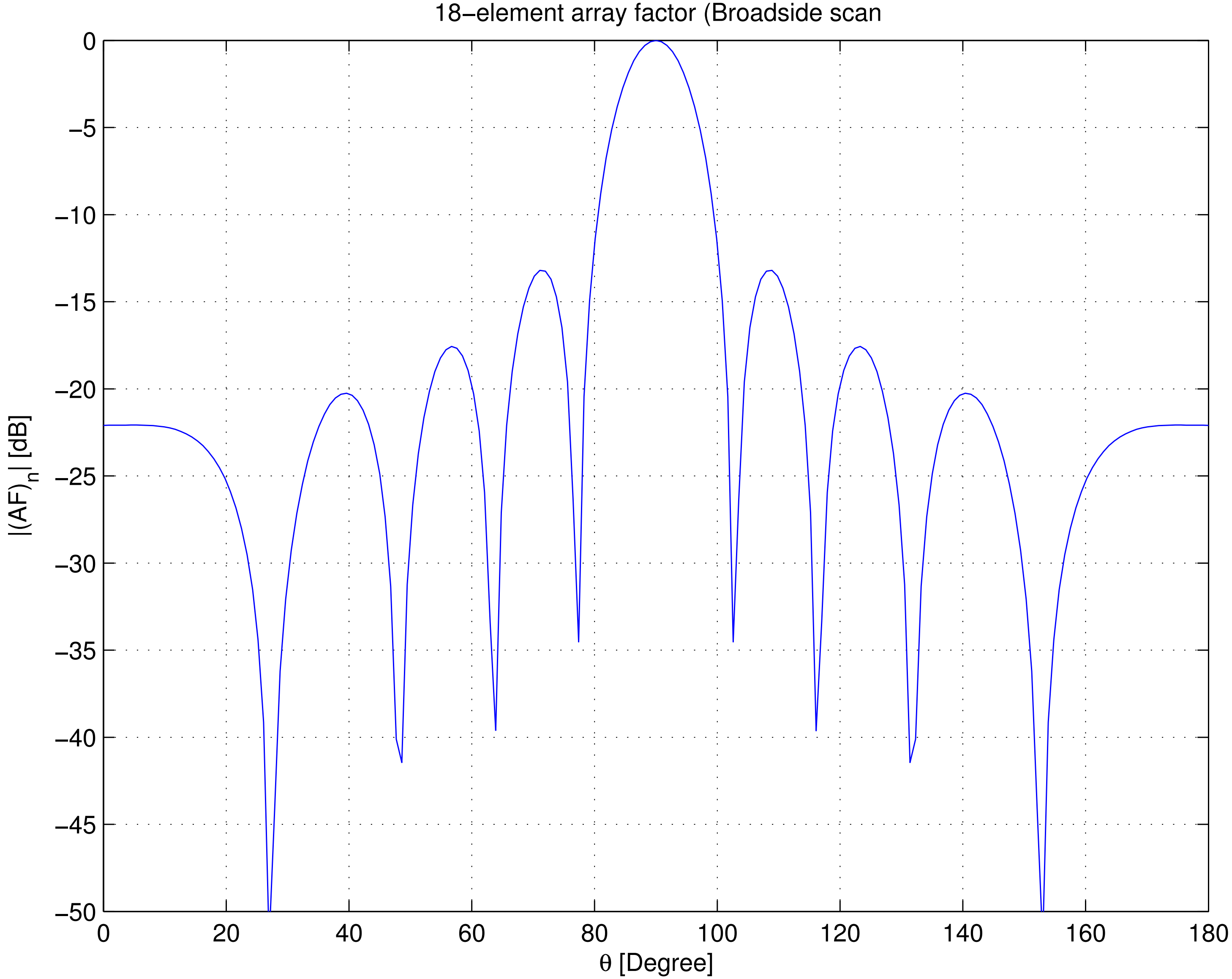
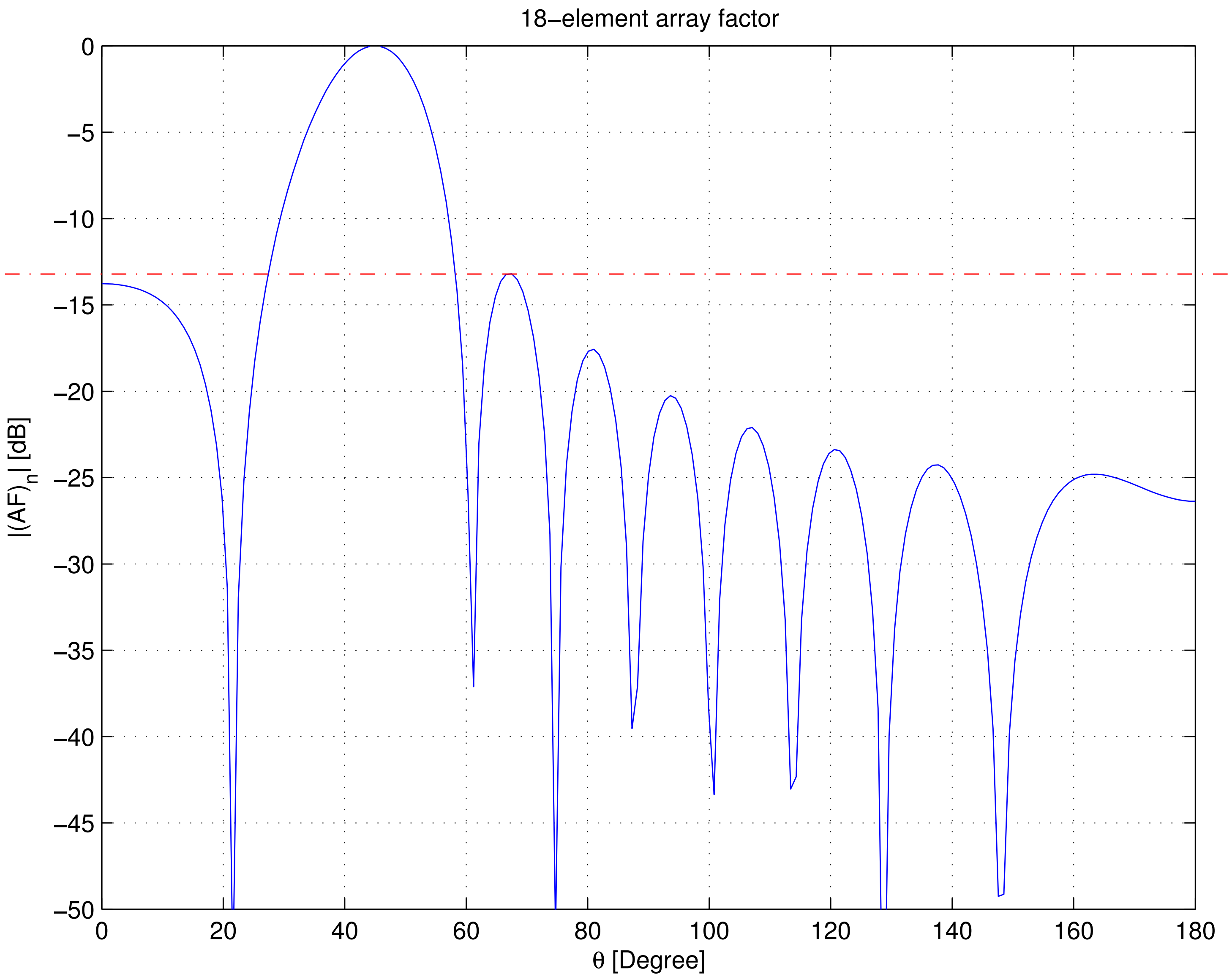
# Example

- **Design an 18-element uniform linear array with a spacing of $\lambda/4$ between elements. Assume that the array is aligned along the z-axis.**
  a) Find the array factor for the broadside array case.
  b) Find the first null and sidelobe locations of a).
  c) Find the phase shift such that the maximum of the array factor is at $\theta_0=45°$.
  d) Find the first null and sidelobe locations of c).

# 18-element AF (broadside array)

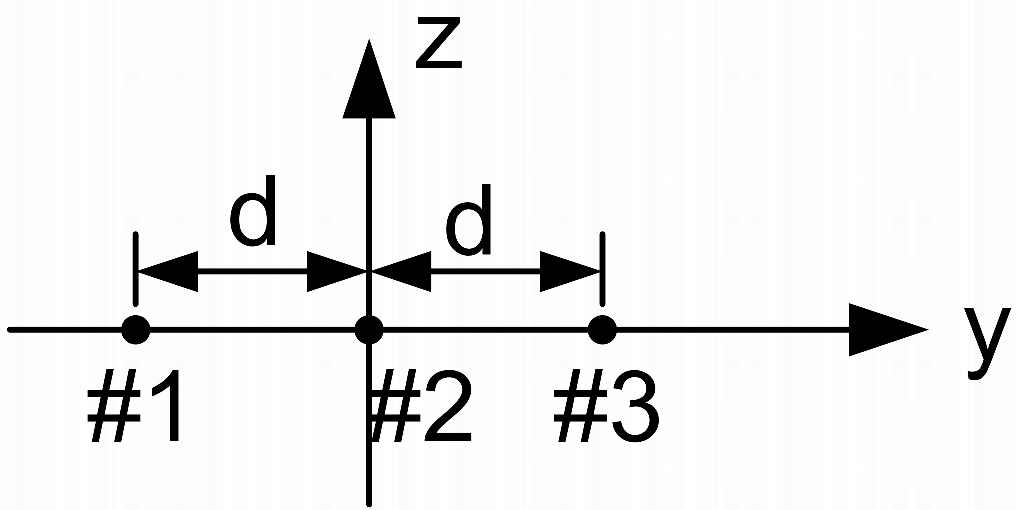


# 18-element AF (scan array)



# Quiz

Find the array factor of the 3-element array of isotropic sources shown below. The spacing between elements is $d=\lambda/4$ and $I_1 = 1$, $I_2 = -j2$, $I_3 = -1$.



# *N*-element Array: Non-uniform amplitude, uniform spacing

- **Uniform amplitude -> High sidelobe**
- **Two popular distributions:**
  - Binomial (maximally flat)
  - Tschebysheff (equiripple)
- **HPBW: Uniform<Tschebysheff<Binomial**
- **Sidelobe level: Binomial<Tschebysheff<Uniform**

# Nonuniform Amplitude Arrays of Even & Odd Number of Elements

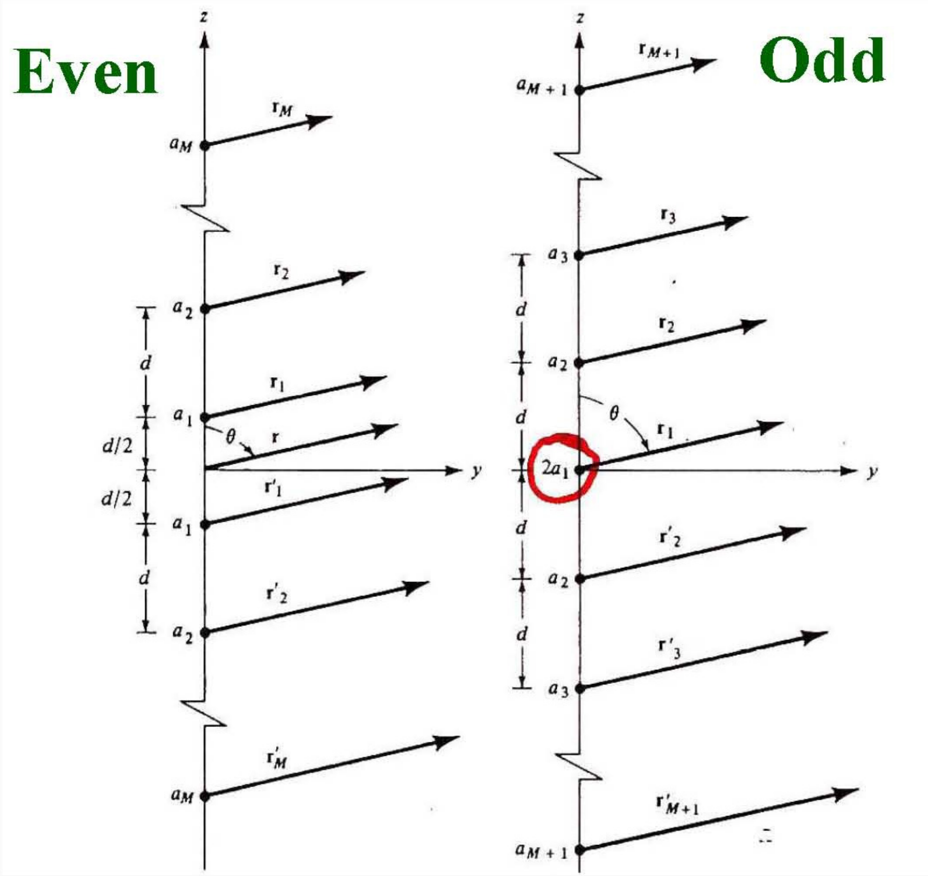



Fig. 6.19

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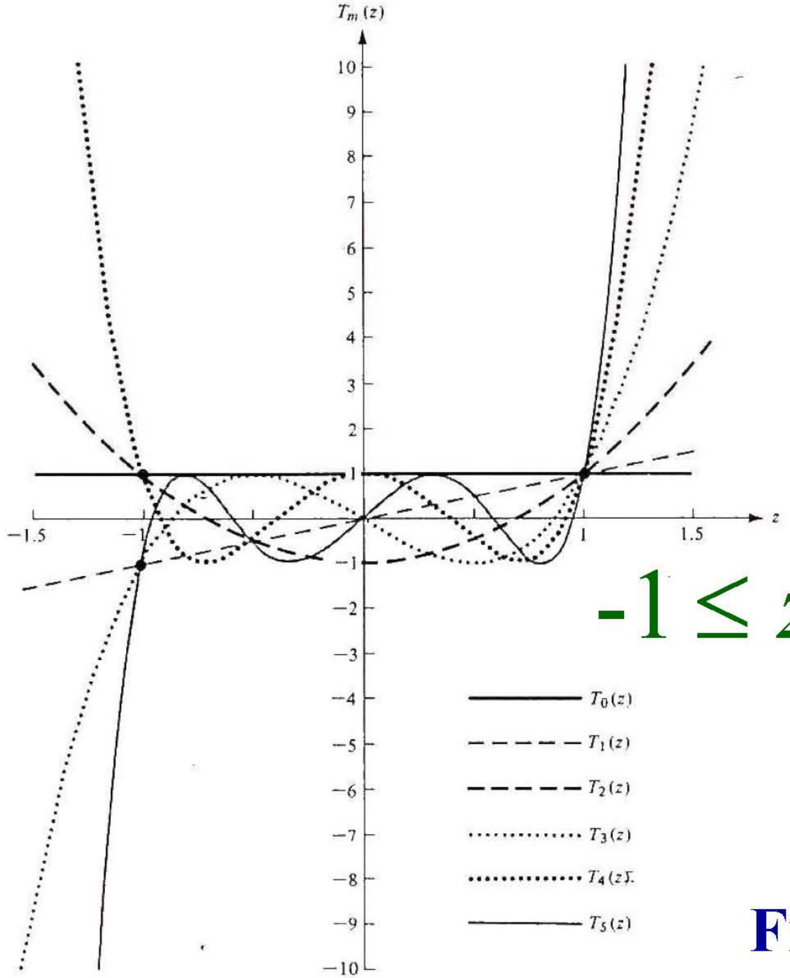
# Tschebyscheff Polynomials of Orders $n = 0 - 5$

$T_n(z)$




$-1 \leq z \leq +1$

**Fig. 6.21**

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**Chapter 6**
***Arrays: Linear, Planar, & Circular***

# Tschebyscheff Polynomial of $n = 9$

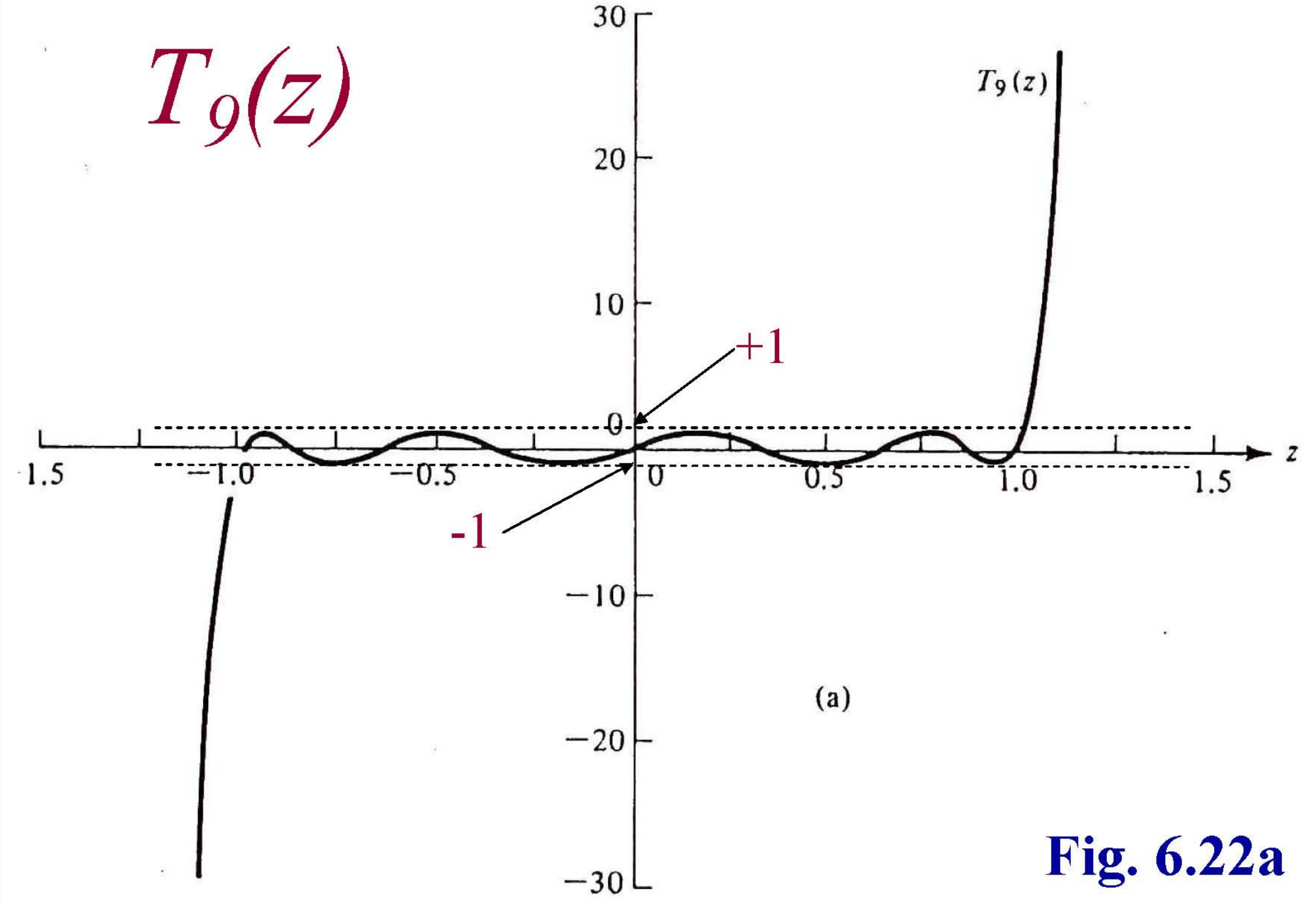


**Fig. 6.22a**

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# Broadside ($\beta = 0$) Dolph-Tschebyscheff Design Procedure

1. Specify
   A. No. Of Elements ($2M$ or $2M+1$)
   B. Sidelobe Level ($dB$)
2. Find
   A. Amplitude Coefficients $a_n$
   B. Spacing $d$

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# Design (Synthesis) Procedure

1. Calculate $R_{\text{ovr}} \equiv$ Sidelobe Voltage Ratio

$$R_{ovr} = 10^{R_0(dB)/20}$$

2. Calculate $p$

$$p = \text{No. of Elements -1} = N\text{-1}$$

3. Calculate $z_o$

$$z_o = \cosh\left[\frac{1}{p}\cosh^{-1}(R_{ovr})\right]$$

or

$$z_o = \frac{1}{2}\left[\left(R_{ovr} + \sqrt{R_{ovr}^2 - 1}\right)^{1/p} + \left(R_{ovr} - \sqrt{R_{ovr}^2 - 1}\right)^{1/p}\right] \quad (6\text{-}73)$$

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# 4. Find Amplitude Coefficients

## Even Number $(N=2M) \Rightarrow M=N/2$

$$a_n = \sum_{q=n}^{M} (-1)^{M-q} (z_o)^{2q-1} \frac{(q+M-2)!(2M-1)}{(q-n)!(q+n-1)!(M-q)!}$$

$n = 1,2,\ldots,M$ (6-77a)

## Odd Number $(N=2M+1) \Rightarrow M=(N-1)/2$

$$a_n = \sum_{q=n}^{M+1} (-1)^{M-q+1} (z_o)^{2(q-1)} \frac{(q+M-2)!2M}{\varepsilon_n (q-n)!(q+n-2)!(M-q+1)!}$$

$$n = 1,2,\ldots,M+1$$

(6-77b)

$$\varepsilon_n = \begin{cases} 2 & n=1 \\ 1 & n \neq 1 \end{cases}$$

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5. Select spacing $d$ so that

$$d_{\max} \le \frac{\lambda}{\pi}\cos^{-1}\left(-\frac{1}{z_o}\right) \qquad (6\text{-}76a)$$

(if you want all the minor lobes to be of the same level)

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6. The number of minor lobes, not necessarily complete, for the 3-D pattern on either side of the main maximum $\left(0 \le \theta \le 90^\circ\right)$ using the maximum permissible spacing, is equal to *N-1*.
7. The order of the Tschebyscheff polynomial is *N-1*

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# Array Factor Power Pattern of a 10-Element Broadside Dolph-Tschebyscheff Array

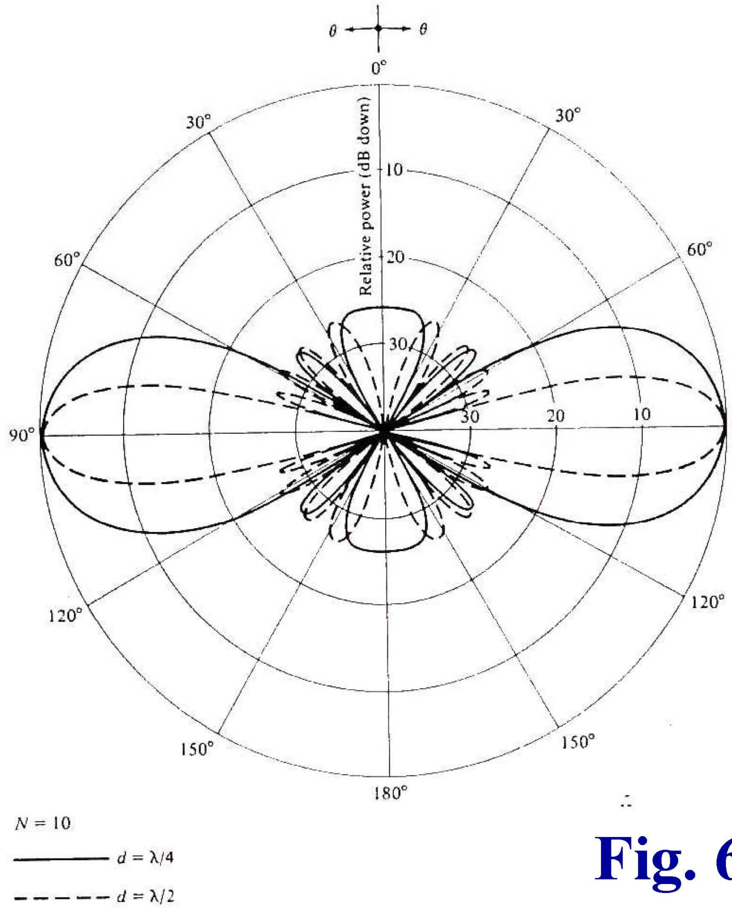


**Fig. 6.23**

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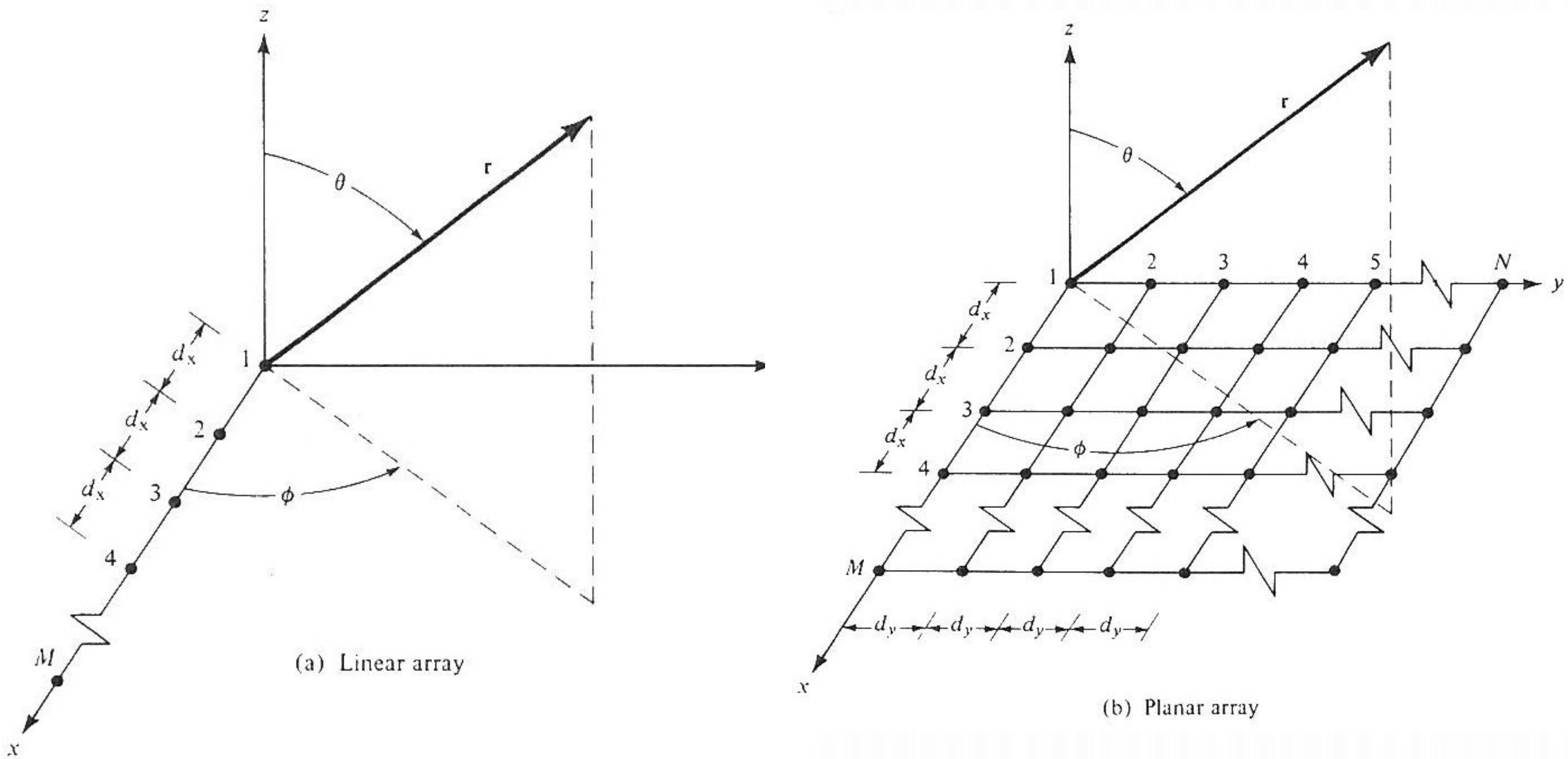
# Planar Array

- Linear Array = one-dimensional array, i.e., can scan the beam only in one plane.
- In order to be able to scan the beam in any direction, two-dimensional arrays are needed. Geometries can be planar, circle, cylindrical, spherical and so on.

# Planar Array (2)




(a) Linear array

(b) Planar array

# Array Factor

AF for each linear array along x-axis:

$$AF = \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd_x \sin\theta\cos\phi + \beta_x)}$$

AF for the entire planar array:

$$AF = \sum_{n=1}^{N} I_{1n} \left[ \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd_x \sin\theta\cos\phi + \beta_x)} \right] e^{j(n-1)(kd_y \sin\theta\sin\phi + \beta_y)}$$

$$= S_{xm} S_{yn}$$

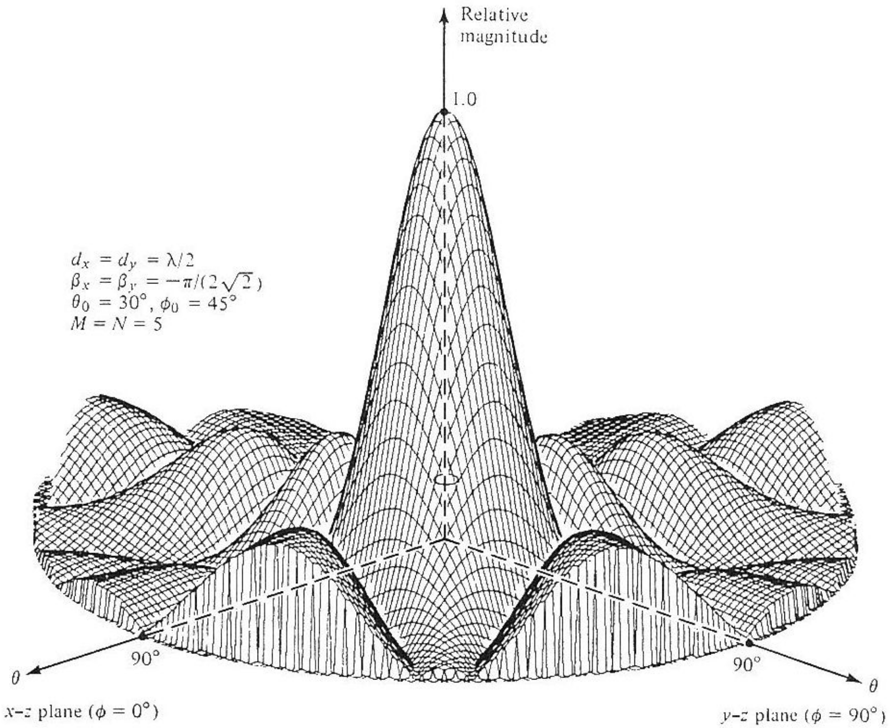
For uniform excitation, i.e., $|I_{m1} I_{1n}| = I_0$,

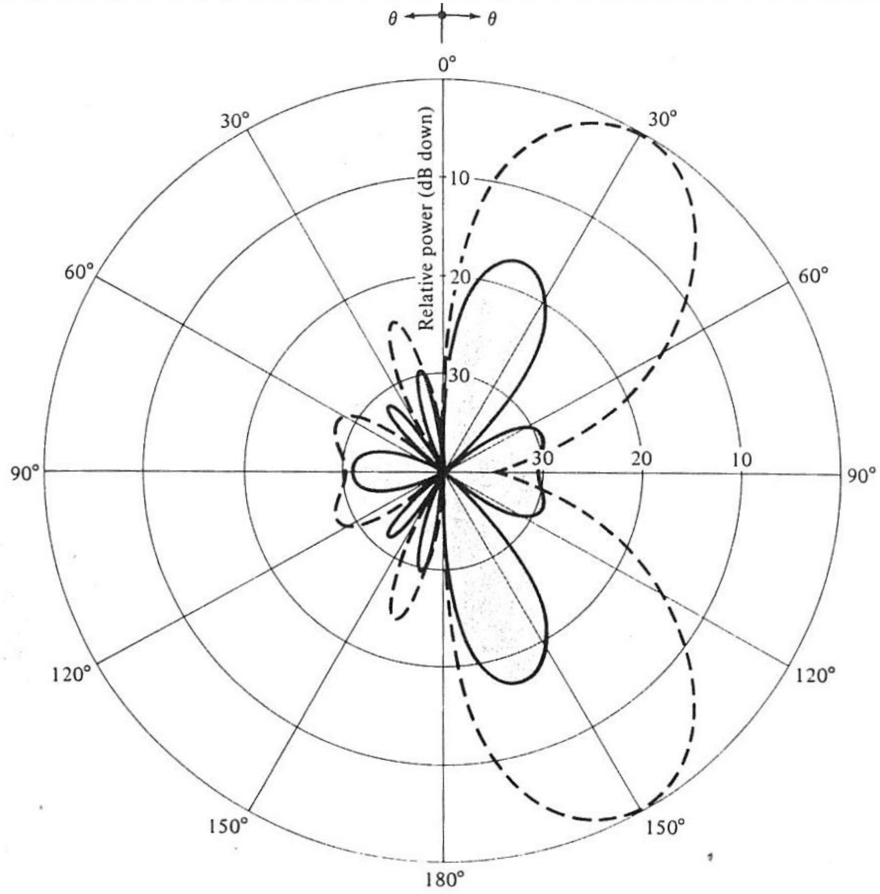
$$(AF)_n = \frac{\sin\left(\frac{M}{2}\psi_x\right)}{M \sin\frac{\psi_x}{2}} \frac{\sin\left(\frac{N}{2}\psi_y\right)}{N \sin\frac{\psi_y}{2}};$$

$$\psi_x = kd_x \sin\theta\cos\phi + \beta_x;$$

$$\psi_y = kd_y \sin\theta\sin\phi + \beta_y$$

# Planar Array Example




$$d_x = d_y = \lambda/2; \beta_x = \beta_y = -\pi/(2\sqrt{2});$$

$$\theta_0 = \pi/6, \phi_0 = \pi/4; M = N = 5$$

——— $\phi = 0°$ (x–z plane)
$\phi = 90°$ (y–z plane)
– – – $\phi = 45°$
$d_x = d_y = \lambda/2$
$\beta_x = \beta_y = -\pi/(2\sqrt{2})$
$\theta_0 = 30°, \phi_0 = 45°$
$M = N = 5$