Chapter 7 : Antenna Synthesis

- Continuous sources vs. Discrete sources
- Schelkunoff polynomial method
- Fourier transform method
- Woodward-Lawson method
- Triangular, cosine and cosine-squared amplitude distributions

Continuous sources

Recall the array factor

$$\mathbf{AF} = \sum_{n=1}^{N} a_n e^{j(n-1)\psi} ; \psi = kd\cos\theta + \beta$$

If the number of elements increases in a fixed-length array, the source approaches a continuous distribution.

In the limit, the array factor becomes the space factor, i.e.,

$$\mathbf{SF} = \int_{-l/2}^{l/2} I_n(z')e^{j[kz'\cos\theta + \phi_n(z')]}dz'$$

The radiation characteristics of continuous sources can be approximated by discrete-element arrays, i.e.,

$$a_n e^{j(n-1)\beta} = I_n(z')e^{j\phi_n(z')}$$

Schelkunoff polynomial method

The array factor for an *N*-element, equally spaced, non-uniform amplitude, and progressive phase excitation is given by

$$\mathbf{AF} = \sum_{n=1}^{N} a_n e^{j(n-1)\psi} ; \psi = kd\cos\theta + \beta$$

Let $$z = x + jy = e^{j\psi} = e^{j(kd\cos\theta+\beta)}$$

$$\mathbf{AF} = \sum_{n=1}^{N} a_n z^{n-1} = a_1 + a_2 z + \cdots + a_N z^{N-1}$$

which is a polynomial of degree (*N*-1).

Schelkunoff polynomial method (2)

Thus

$$\mathbf{AF} = a_N (z - z_1)(z - z_2)\cdots(z - z_{N-1})$$

where $z_1, z_2, \ldots, z_{N-1}$ are the roots. The magnitude then becomes

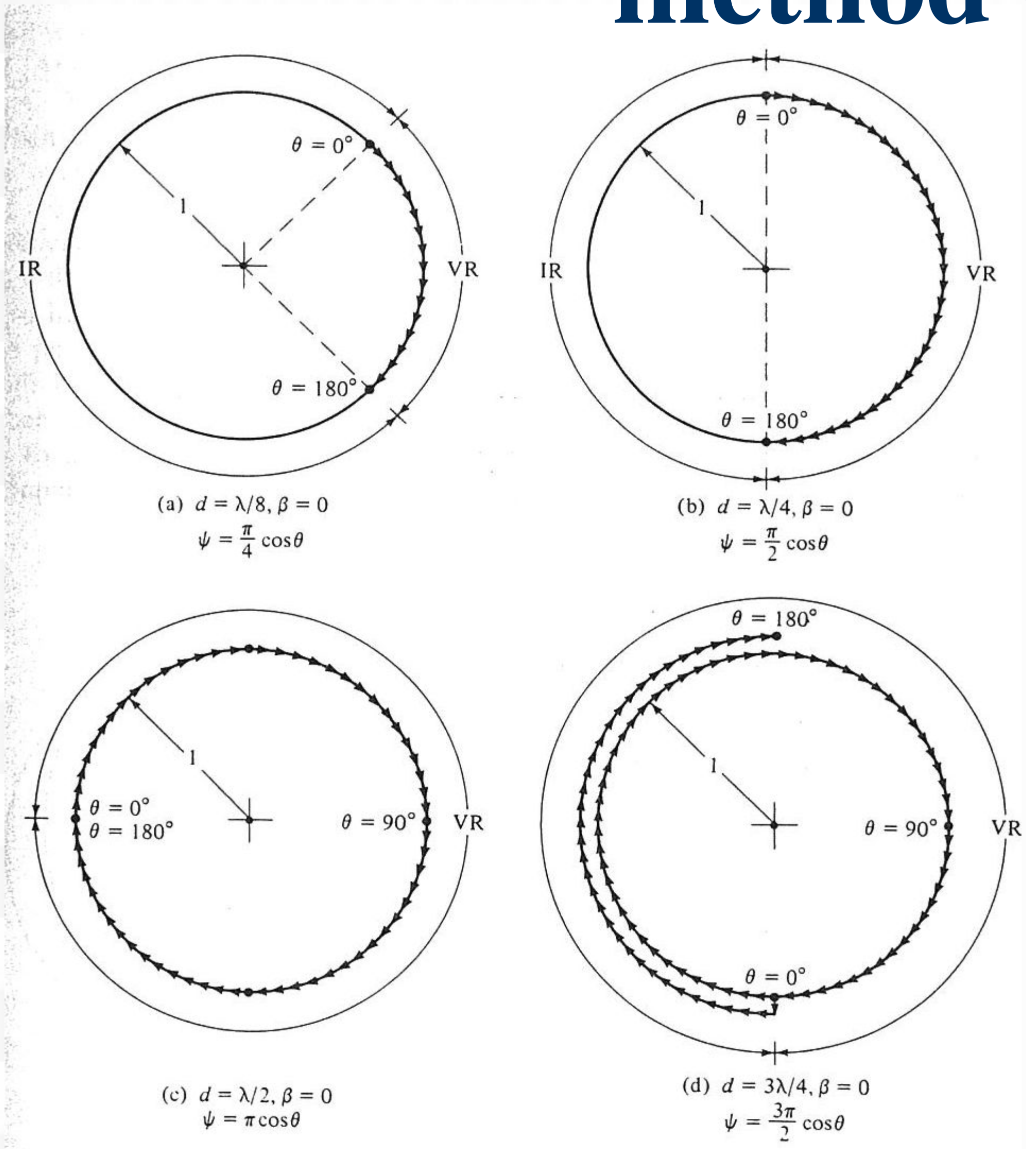
$$|\mathbf{AF}| = |a_N||z - z_1||z - z_2|\cdots|z - z_{N-1}|$$

Note that

$$z = |z| e^{j\psi} = |z| \angle \psi = 1 \angle \psi$$
$$\psi = kd\cos\theta + \beta = \frac{2\pi}{\lambda} d\cos\theta + \beta$$

z is on a unit circle.

Schelkunoff polynomial method (3)

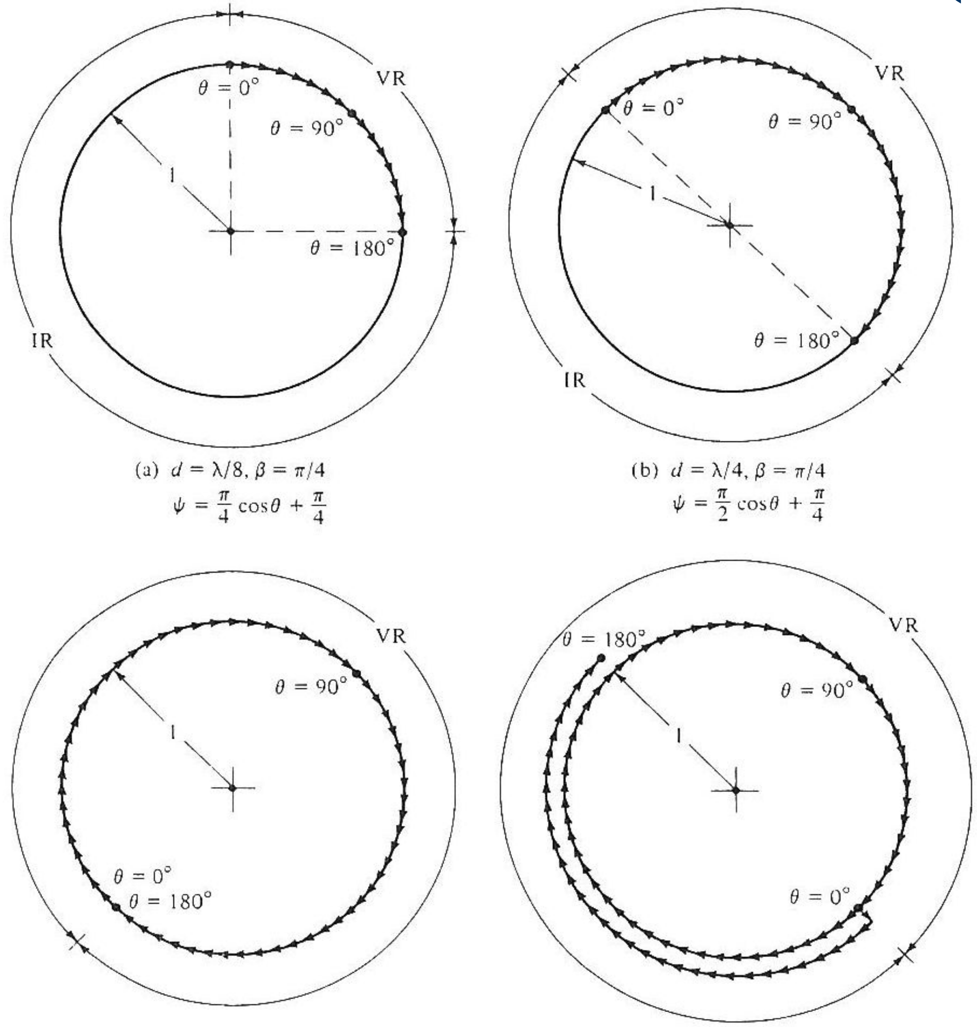


VR=Visible Region

IR=Invisible Region

$\beta = 0$

Schelkunoff polynomial method (4)



(a) $d = \lambda/8,\ \beta = \pi/4$, $\psi = \frac{\pi}{4}\cos\theta + \frac{\pi}{4}$

(b) $d = \lambda/4,\ \beta = \pi/4$, $\psi = \frac{\pi}{2}\cos\theta + \frac{\pi}{4}$

(c) $d = \lambda/2,\ \beta = \pi/4$, $\psi = \pi\cos\theta + \frac{\pi}{4}$

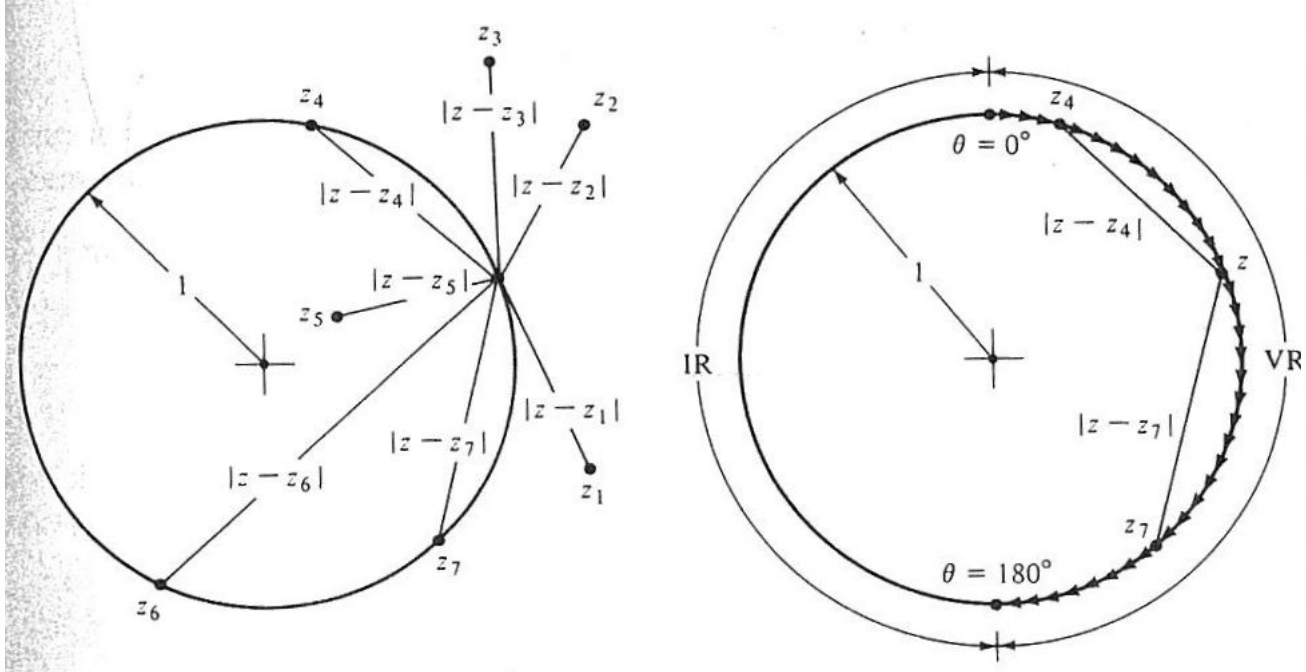
(d) $d = 3\lambda/4,\ \beta = \pi/4$, $\psi = \frac{3\pi}{2}\cos\theta + \frac{\pi}{4}$

VR=Visible Region

IR=Invisible Region

$\beta = \pi/4$

Schelkunoff polynomial method (5)



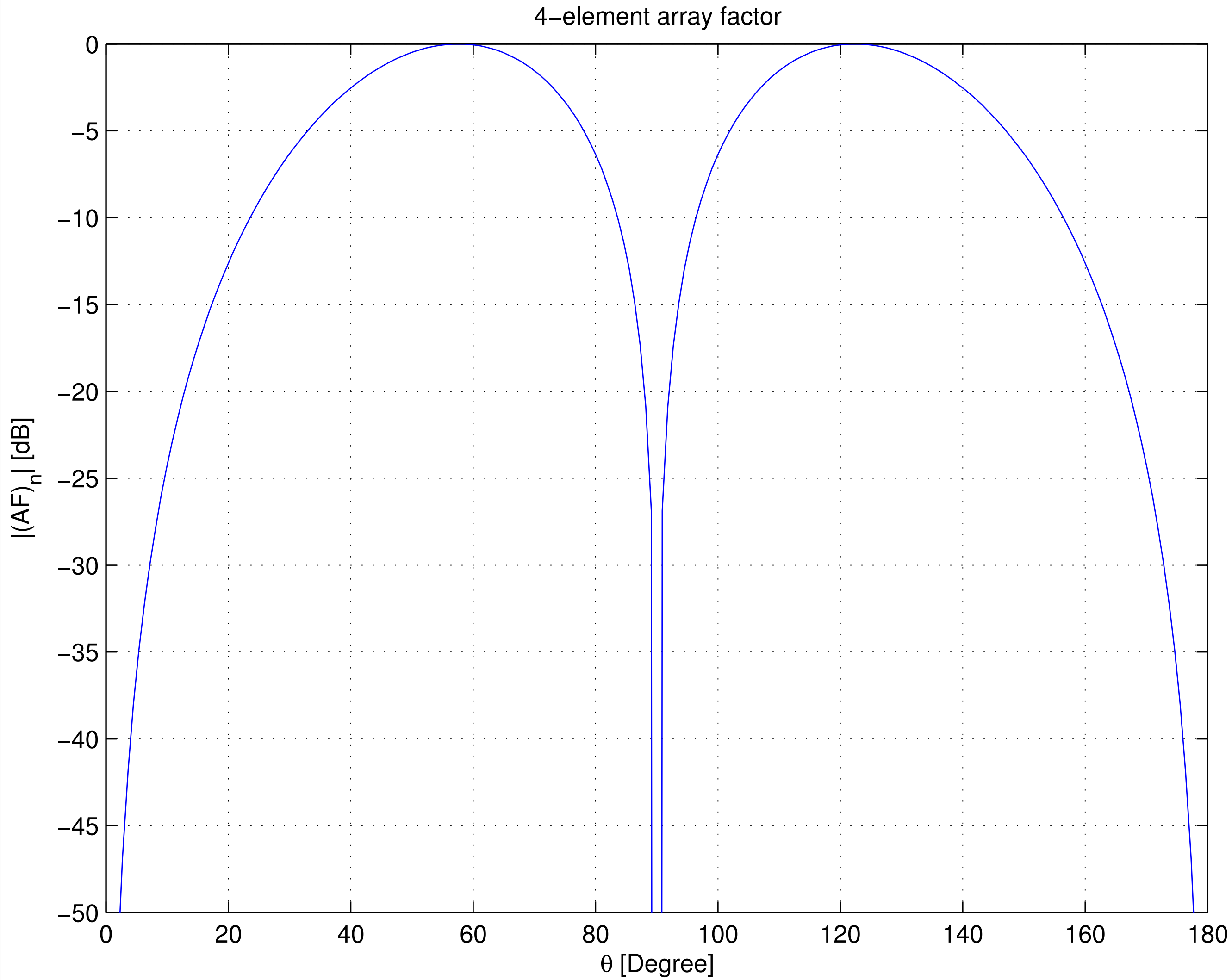
(a) Roots of array factor

(b) Roots of array factor on unit circle and within visible region

Schelkunoff polynomial method : Example

Design a linear array with a spacing between the elements of $d=\lambda/4$ such that it has zeros at $\theta=0,\pi/2,\pi$. Determine the number of elements, their excitation, and plot the derived pattern.

Schelkunoff polynomial method : Example pattern



Fourier Transform Method

The normalized space factor for a continuous line-source distribution of length *l* can be given by

$$\mathbf{SF}(\theta) = \int_{-l/2}^{l/2} I(z')e^{j(k\cos\theta - k_z)z'}dz' = \int_{-l/2}^{l/2} I(z')e^{j\xi z'}dz'$$

$$\xi = k\cos\theta - k_z \rightarrow \theta = \cos^{-1}\left(\frac{\xi + k_z}{k}\right)$$

where k_z is the excitation phase constant of the source. If $I(z') = I_0/l$,

$$\mathbf{SF}(\theta) = I_0 \frac{\sin\left[\frac{kl}{2}\left(\cos\theta - \frac{k_z}{k}\right)\right]}{\frac{kl}{2}\left(\cos\theta - \frac{k_z}{k}\right)}$$

Fourier Transform Method (2)

Since the current distribution extends only over $-l/2 \leq z' \leq l/2$,

$$\mathbf{SF}(\theta) = \mathbf{SF}(\xi) = \int_{-\infty}^{\infty} I(z') e^{j\xi z'} dz'$$

The current distribution can then be given by

$$I(z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{SF}(\xi) e^{-j\xi z'} d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{SF}(\theta) e^{-j\xi z'} d\xi$$

The approximate source distribution I_a (z') is given by

$$I_a(z') = \begin{cases} I(z') = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \mathbf{SF}(\xi) e^{-j\xi z'} d\xi & -l/2 \leq z' \leq l/2 \\ 0 & \text{elsewhere} \end{cases}$$

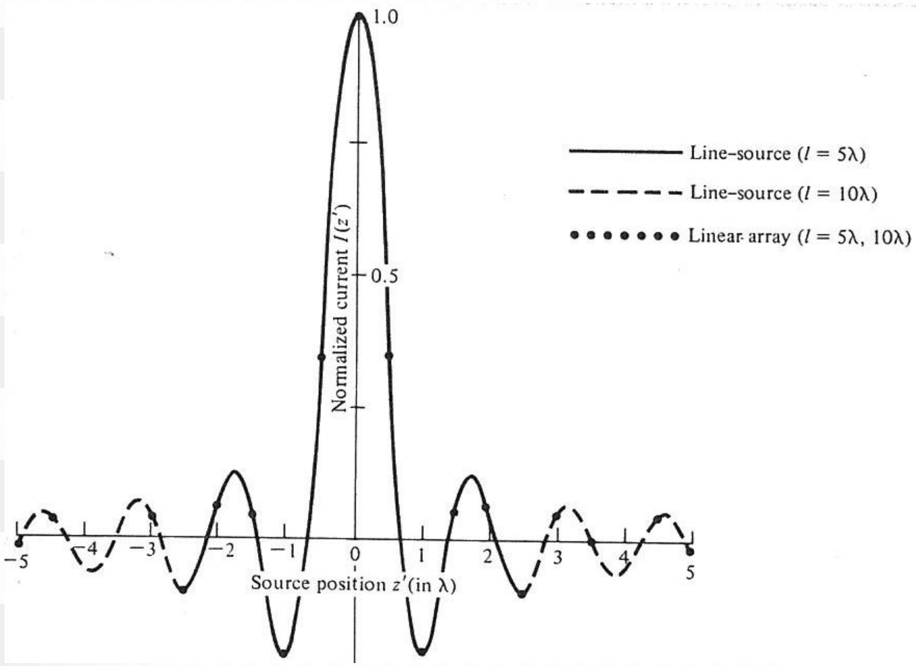
Thus $$\mathbf{SF}(\theta)_a = \mathbf{SF}(\xi)_a = \int_{-l/2}^{l/2} I_a(z') e^{j\xi z'} dz'$$

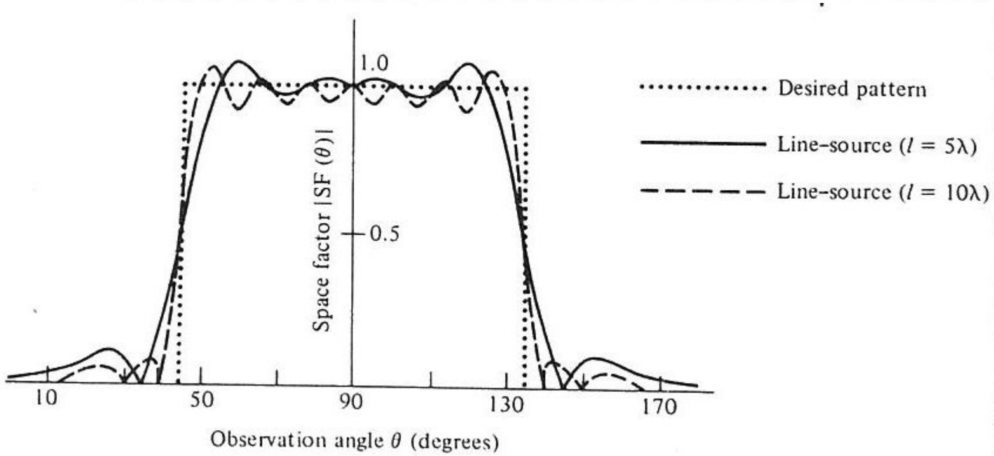
Fourier Transform Method : Example 7.2

Determine the current distribution and the approximate radiation pattern of a line source placed along the *z*-axis whose desired radiation pattern is symmetrical about $\theta=\pi/2$, and it is given by

$$\mathbf{SF}(\theta) = \begin{cases} 1 & \pi/4 \le \theta \le 3\pi/4 \\ 0 & elsewhere \end{cases}$$

Fourier Transform Method : Example





Fourier Transform Method : Linear Array

For an odd number of elements, the array factor is given by

$$\mathbf{AF}(\theta) = \mathbf{AF}(\psi) = \sum_{m=-M}^{M} a_m e^{jm\psi}$$

For an even number of elements,

$$\mathbf{AF}(\theta) = \mathbf{AF}(\psi) = \sum_{m=-M}^{-1} a_m e^{j[(2m+1)/2]\psi} + \sum_{m=1}^{M} a_m e^{j[(2m-1)/2]\psi}$$

where $\psi = kd\cos\theta + \beta$

Elements' locations

Odd-number

$$z_m' = md, \quad m = 0, \pm 1, \pm 2, \ldots, \pm M$$

Even-number

$$z_m' = \begin{cases} \dfrac{2m-1}{2} d & 1 \leq m \leq M \\ \dfrac{2m+1}{2} d & -M \leq m \leq -1 \end{cases}$$

Fourier Transform Method : Linear Array (2)

For an odd number of elements, the excitation coefficients can be obtained by

$$a_m = \frac{1}{T}\int_{-T/2}^{T/2} \mathbf{AF}(\psi) e^{-jm\psi} d\psi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{AF}(\psi) e^{-jm\psi} d\psi$$

$$-M \le m \le M$$

For an even number of elements,

$$a_m = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathbf{AF}(\psi) e^{-j[(2m+1)/2]\psi} d\psi & -M \le m \le -1 \\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathbf{AF}(\psi) e^{-j[(2m-1)/2]\psi} d\psi & 1 \le m \le M \end{cases}$$

where $\quad \psi = kd\cos\theta + \beta$

Fourier Transform Method : Example

Same as Example 7.2 with $d = \lambda/2$; non-zero only

$$\pi/4 \le \theta \le 3\pi/4$$

thus $-\pi/\sqrt{2} \le \psi = kd\cos\theta + \beta \le \pi/\sqrt{2}$

therefore

$$a_m = \frac{1}{2\pi}\int_{-\pi/\sqrt{2}}^{\pi/\sqrt{2}} e^{-jm\psi}\,d\psi = \frac{1}{\sqrt{2}}\frac{\sin\left(\frac{m\pi}{\sqrt{2}}\right)}{\frac{m\pi}{\sqrt{2}}}$$

Result

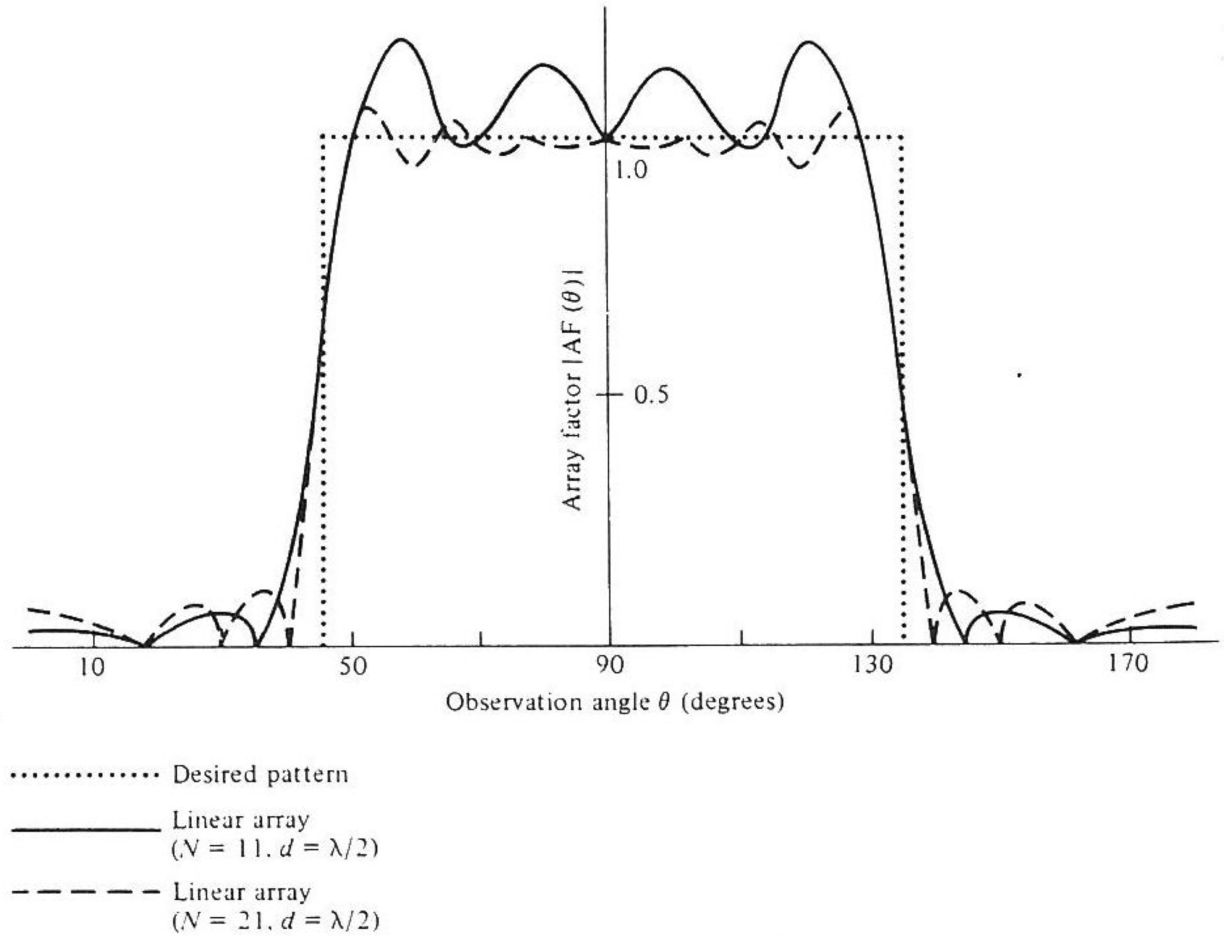
$a_0 = 1.0$ $\quad a_{\pm 4} = 0.0578$ $\quad a_{\pm 8} = -0.0496$

$a_{\pm 1} = 0.3582$ $\quad a_{\pm 5} = -0.0895$ $\quad a_{\pm 9} = 0.0455$

$a_{\pm 2} = -0.2170$ $\quad a_{\pm 6} = 0.0518$ $\quad a_{\pm 10} = -0.010$

$a_{\pm 3} = 0.0588$ $\quad a_{\pm 7} = 0.0101$

Fourier Transform Method : Example



Quiz

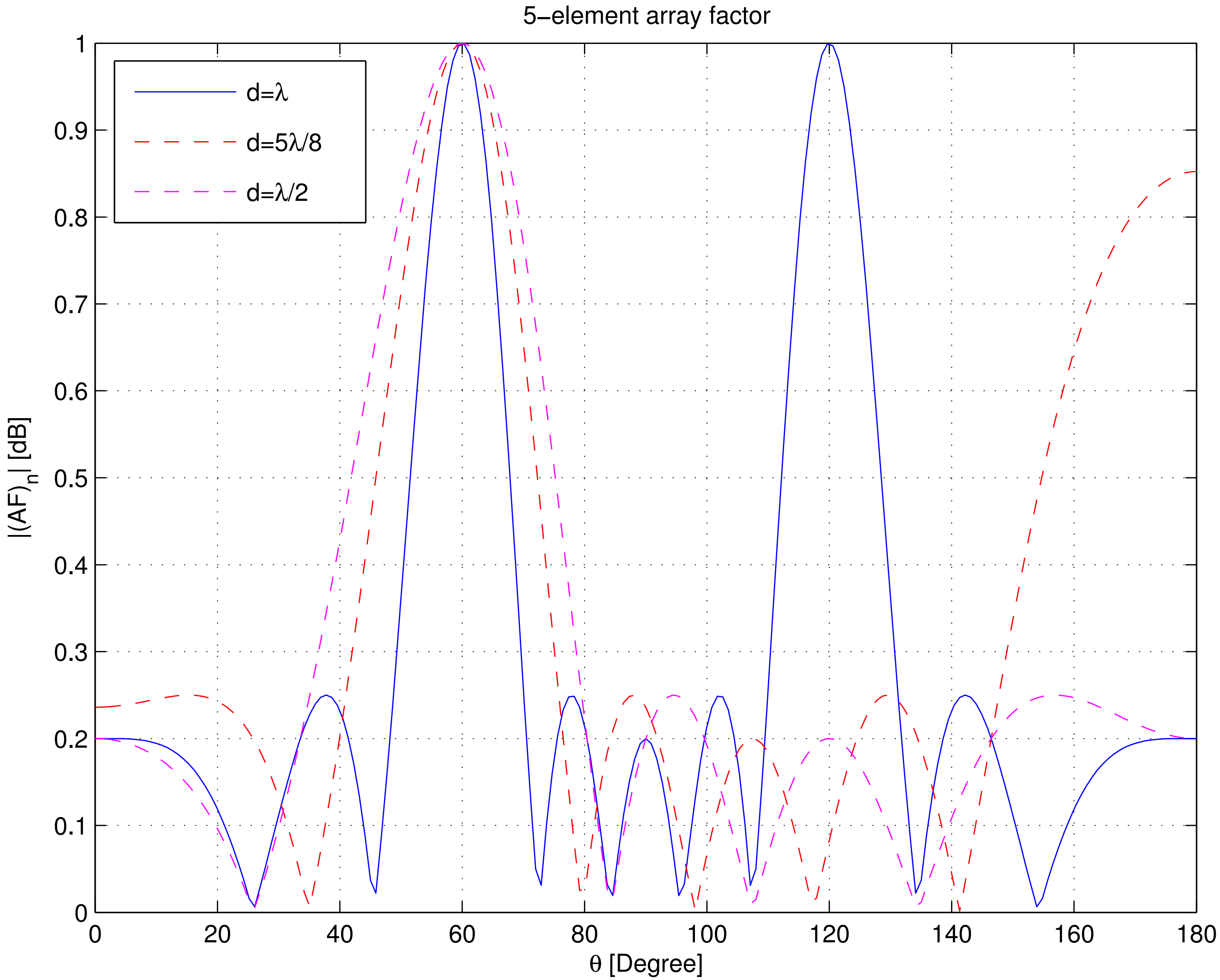
A 5-element uniform linear array with a spacing of λ between elements is designed to scan at $\theta=\pi/3$. Assume that the array is aligned along the z-axis.

a) Find the array factor

b) Find the angle of the grating lobe.

c) Find the condition such that there exists no grating lobe.

Quiz solution



Woodward-Lawson Method

- **Sampling the desired pattern at various discrete locations.**
- **Use *composing function* of the forms:**

$$b_m \sin(\psi_m)/\psi_m \text{ or } b_m \sin(N\phi_m)/N\sin\phi_m$$

as the field of each pattern sample

- **The synthesized pattern is represented by a finite sum of composing functions.**
- **The total excitation is a sum of space harmonics.**

Woodward-Lawson Method: Line-source

Let the source be represented by a sum of the following constant current source of length l.

$$i_m(z') = \frac{b_m}{l} e^{-jkz'\cos\theta_m} \qquad -l/2 \le z' \le l/2$$

Then the current source can be given by

$$I(z') = \frac{1}{l}\sum_{m=-M}^{M} b_m e^{-jkz'\cos\theta_m}$$

where $m = \pm1, \pm2, \ldots, \pm M$ (for $2M$ even number)

$m = 0, \pm1, \pm2, \ldots, \pm M$ (for $2M+1$ odd number)

The field pattern of each current source is given by

$$s_m(\theta) = b_m \frac{\sin\left[\frac{kl}{2}(\cos\theta - \cos\theta_m)\right]}{\frac{kl}{2}(\cos\theta - \cos\theta_m)}$$

Composing function

Woodward-Lawson Method: Line-source (2)

For an odd number samples, the total pattern becomes

$$\mathrm{SF}(\theta)=\sum_{m=-M}^{M} b_m \frac{\sin\left[\frac{kl}{2}\left(\cos\theta-\cos\theta_m\right)\right]}{\frac{kl}{2}\left(\cos\theta-\cos\theta_m\right)}$$

b_{m} can be obtained from the value at the sample points θ_{m}, i.e.,

$$b_m=\mathrm{SF}(\theta=\theta_m)_d$$

In order to satisfy the periodicity of 2π and faithfully reconstruct the desired pattern,

$$kz'\Delta\Big|_{|z'|=l}=2\pi \Rightarrow \Delta=\frac{\lambda}{l}$$

Woodward-Lawson Method: Line-source (3)

Thus the location of each sample is given by

$$\cos\theta_m = m\Delta = m\left(\frac{\lambda}{l}\right), \quad m = 0, \pm1, \pm2, \ldots \quad \text{for odd samples}$$

$$\cos\theta_m = \begin{cases} \dfrac{2m-1}{2}\Delta = \dfrac{2m-1}{2}\left(\dfrac{\lambda}{l}\right), & m = 1, 2, \ldots \\ \dfrac{2m+1}{2}\Delta = \dfrac{2m+1}{2}\left(\dfrac{\lambda}{l}\right), & m = -1, -2, \ldots \end{cases} \quad \text{for even samples}$$

Therefore, M should be the closest integer to $M = l/\lambda$.

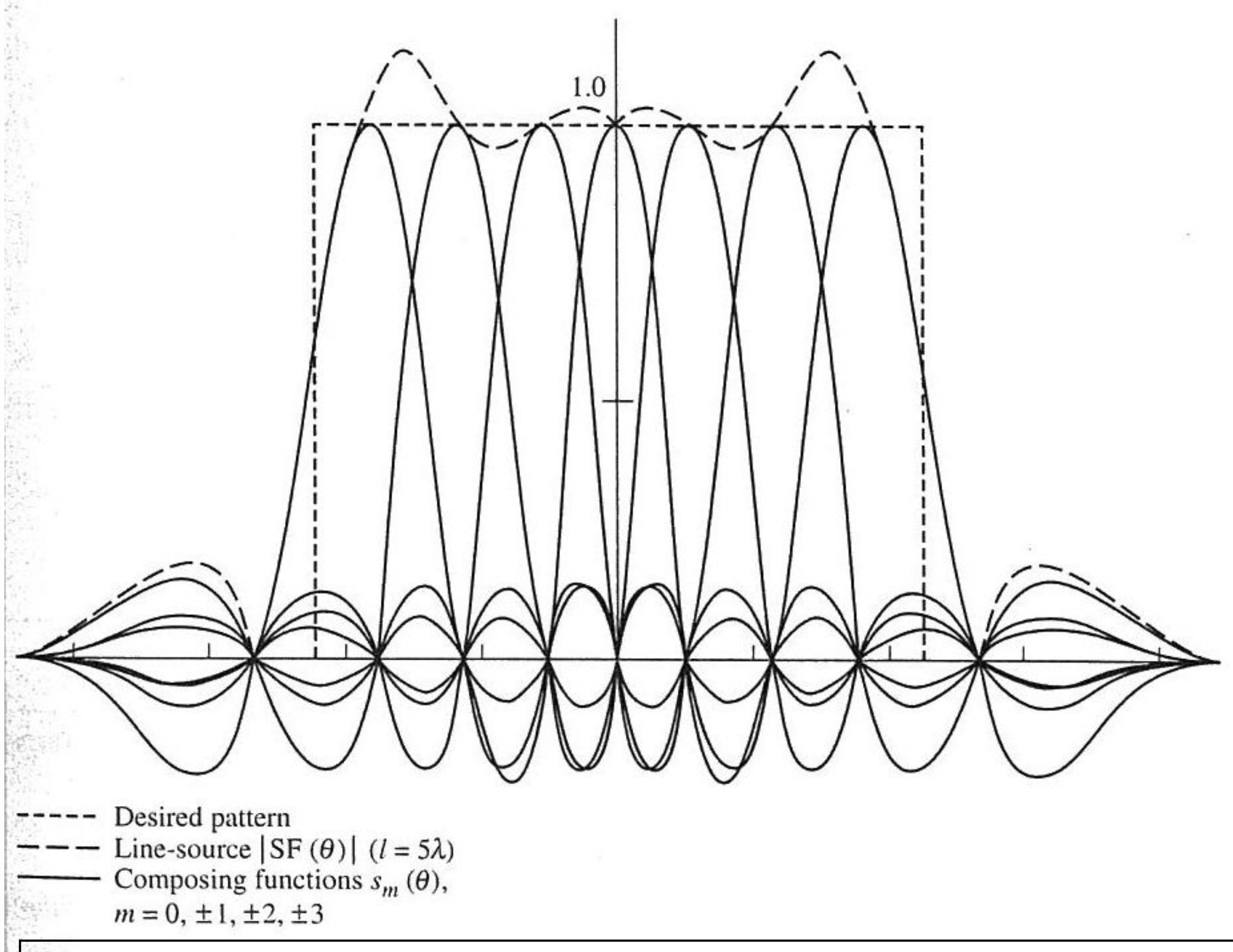
Woodward-Lawson Method: Example

Same as Example 7.2; for $l = 5\lambda$.
Since $l = 5\lambda$, $M = 5$ and $\Delta = 0.2$.

$$\theta_m = \cos^{-1}(m\Delta) = \cos^{-1}(0.2m),\ m = 0, \pm1, \pm2, \ldots, \pm5$$

m	θ_m	b_m	m	θ_m	b_m
0	90	1			
1	78.46	1	-1	101.54	1
2	66.42	1	-2	113.58	1
3	53.13	1	-3	126.87	1
4	36.87	0	-4	143.13	0
5	0	0	-5	180	0

Woodward-Lawson Method: Example (2)



Composing functions for line-source ($l = 5\lambda$)

Woodward-Lawson Method: Linear array

The pattern of each sample (uniform array) can be written as (assuming $l = Nd$)

$$f_m(\theta) = b_m \frac{\sin\left[\frac{N}{2}kd\left(\cos\theta - \cos\theta_m\right)\right]}{N\sin\frac{kd}{2}\left(\cos\theta - \cos\theta_m\right)}$$

Composing function

For an odd number elements, the array factor becomes

$$\mathrm{AF}(\theta) = \sum_{m=-M}^{M} b_m \frac{\sin\left[\frac{N}{2}kd\left(\cos\theta - \cos\theta_m\right)\right]}{N\sin\frac{kd}{2}\left(\cos\theta - \cos\theta_m\right)}$$

b_m can be obtained from the value at the sample points θ_m, i.e.,

$$b_m = \mathrm{AF}(\theta = \theta_m)_d$$

Woodward-Lawson Method: Linear array (2)

The location of each sample is given by

$$\cos\theta_m = m\Delta = m\left(\frac{\lambda}{Nd}\right), \qquad m = 0, \pm1, \pm2, \ldots \text{ for odd samples}$$

$$\cos\theta_m = \begin{cases} \dfrac{2m-1}{2}\Delta = \dfrac{2m-1}{2}\left(\dfrac{\lambda}{Nd}\right), & m = 1, 2, \ldots \\ \dfrac{2m+1}{2}\Delta = \dfrac{2m+1}{2}\left(\dfrac{\lambda}{Nd}\right), & m = -1, -2, \ldots \end{cases} \quad \text{for even samples}$$

The normalized excitation coefficient of each element is given by

$$a_n(z') = \frac{1}{N}\sum_{m=-M}^{M} b_m e^{-jkz'_n \cos\theta_m}$$

Woodward-Lawson Method: Example

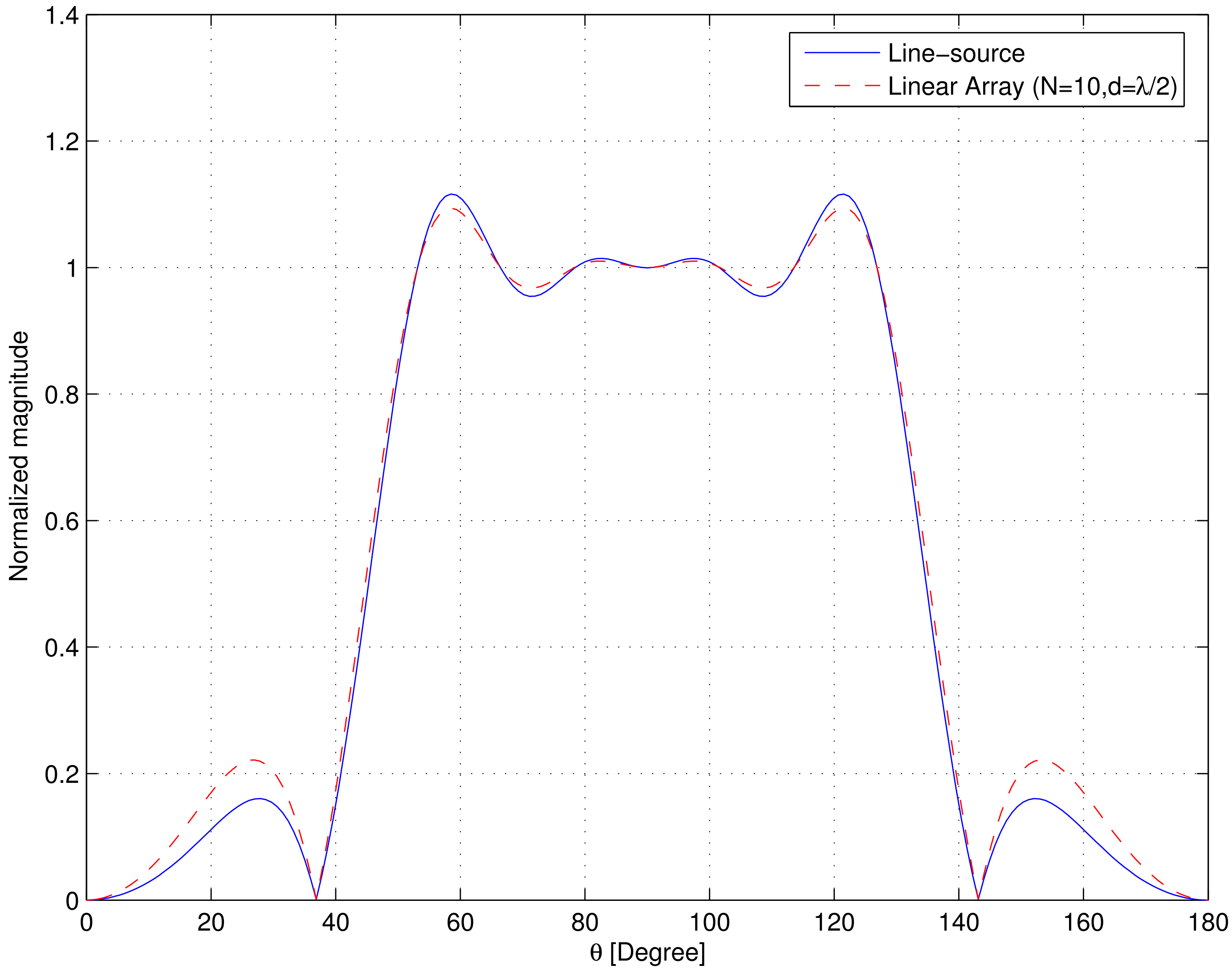
Same as Example 7.2; for N=10 and $d = \lambda/2$.

The coefficients can be found to be

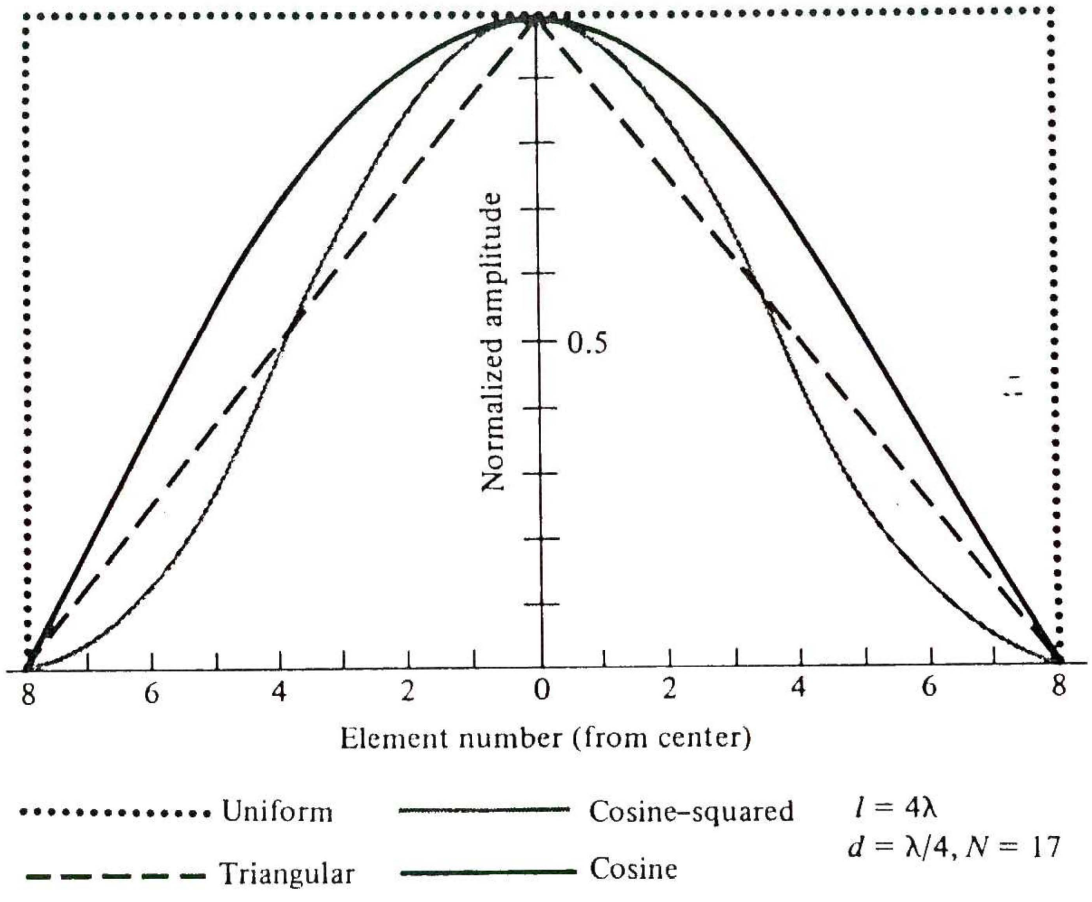
Element number n	Position z'_n	Coefficient a_n
±1	$\pm 0.25\lambda$	0.5696
±2	$\pm 0.75\lambda$	-0.0345
±3	$\pm 1.25\lambda$	-0.1001
±4	$\pm 1.75\lambda$	0.1108
±5	$\pm 2.25\lambda$	-0.0460

To obtain the normalized amplitude pattern of unity at $\theta=\pi/2$, the array factor has been divided by $\sum a_n = 0.4998$

Woodward-Lawson Method: Example



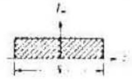
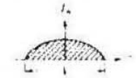
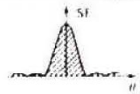
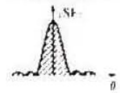
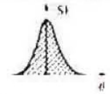
Triangular, cosine, cosine squared distributions



(b) Amplitude distribution of uniform, triangular, cosine, and cosine squared discrete-element arrays

Fig. 7.12b

Table 7.1 RADIATION CHARACTERISTICS FOR LINE SOURCES AND LINEAR ARRAYS WITH UNIFORM, TRIANGULAR, COSINE, AND COSINE-SQUARED DISTRIBUTIONS

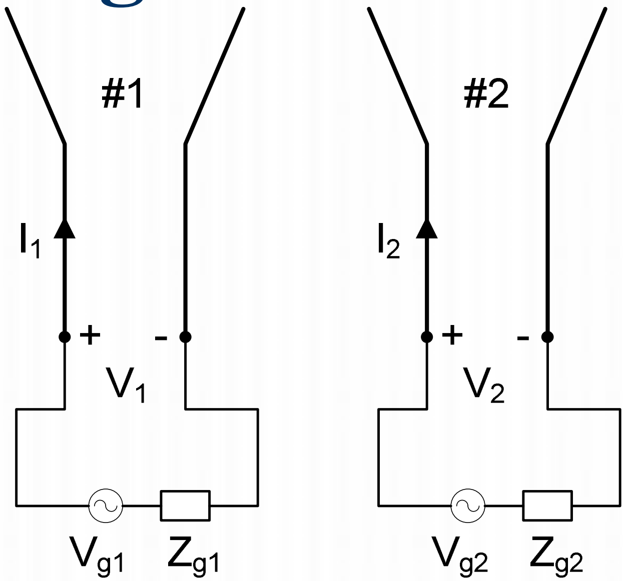
Distribution	Uniform	Triangular	Cosine	Cosine-Squared
Distribution I_n (analytical)	I_0	$I_1\left(1-\frac{2}{l}\lvert z'\rvert\right)$	$I_2\cos\left(\frac{\pi}{l}z'\right)$	$I_3\cos^2\left(\frac{\pi}{l}z'\right)$
Distribution (graphical)				
Space factor (SF) $u=\left(\frac{\pi l}{\lambda}\right)\sin\theta$	$I_0 l\frac{\sin(u)}{u}$	$I_1\frac{l}{2}\left[\frac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}}\right]^2$	$I_2 l\frac{\pi}{2}\frac{\cos(u)}{(\pi/2)^2-u^2}$	$I_3\frac{l}{2}\frac{\sin(u)}{u}\left[\frac{\pi^2}{\pi^2-u^2}\right]$
Space factor $\lvert SF\rvert$				
Half-power beamwidth (degrees) $l \gg \lambda$	$\frac{50.6}{(l/\lambda)}$	$\frac{73.4}{(l/\lambda)}$	$\frac{68.8}{(l/\lambda)}$	$\frac{83.2}{(l/\lambda)}$
First null beamwidth (degrees) $l \gg \lambda$	$\frac{114.6}{(l/\lambda)}$	$\frac{229.2}{(l/\lambda)}$	$\frac{171.9}{(l/\lambda)}$	$\frac{229.2}{(l/\lambda)}$
First side lobe max. (to main max.) (dB)	−13.2	−26.4	−23.2	−31.5
Directivity factor (l large)	$2\left(\frac{l}{\lambda}\right)$	$0.75\left[2\left(\frac{l}{\lambda}\right)\right]$	$0.810\left[2\left(\frac{l}{\lambda}\right)\right]$	$0.667\left[2\left(\frac{l}{\lambda}\right)\right]$

Mutual Coupling

- **Consider two antennas**

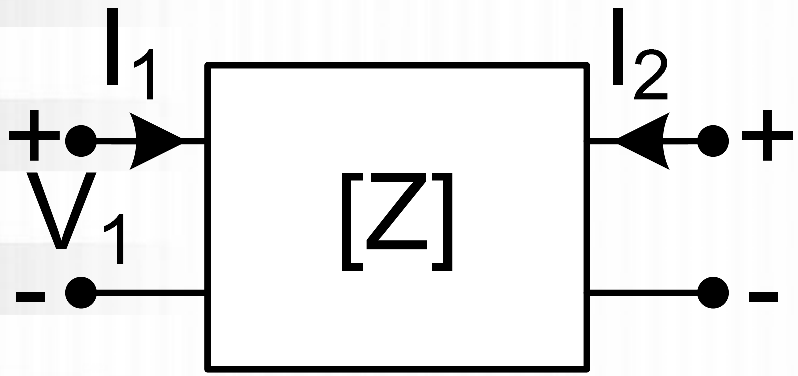
$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

$$\Rightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

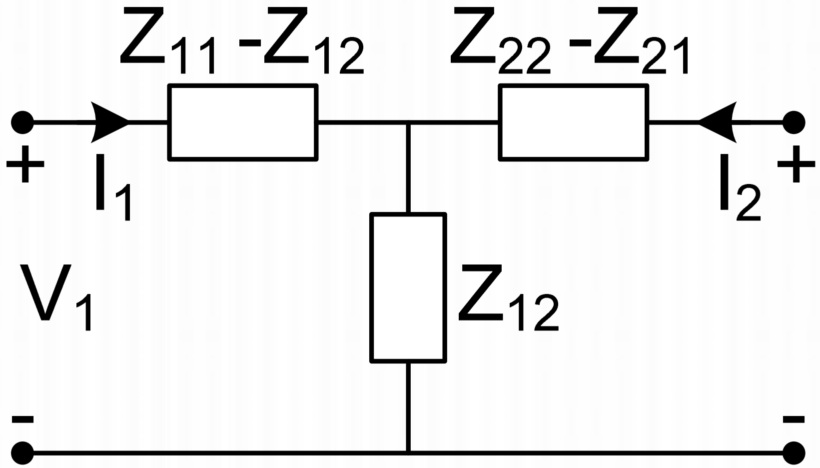


If reciprocal, $Z_{12} = Z_{21}$

Equivalent Circuit



Two-port Network



T-Network Equivalent Circuit

Mutual Coupling: 2 antennas

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

$$Z_{12} = Z_{21}$$

for reciprocal networks

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \quad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

$$Z_{11}, Z_{22}:$$

Input impedance

$$Z_{\substack{1d\\2d}} = \begin{cases}\text{active}\\\text{driving}\end{cases}\text{point impedance}$$

$$Z_{1d} = \frac{V_1}{I_1} = Z_{11} + Z_{12}\frac{I_2}{I_1} \qquad Z_{2d} = \frac{V_2}{I_2} = Z_{22} + Z_{21}\frac{I_1}{I_2}$$

$$\text{Note}: Z_{1d} \text{ depends on } \frac{I_2}{I_1}; Z_{2d} \text{ depends on } \frac{I_1}{I_2}$$

Mutual Coupling: 2 antennas (2)

- As I_1 and I_2 change, the driving point impedance changes.
- In a uniform array, the phase of I_1 and I_2 is changed to scan the beam.
- As the beam is scanned, the driving port impedance in each antenna <u>changes.</u>
- In general, $Z_{11}, Z_{12}=Z_{21}, Z_{22}$ can be calculated using numerical techniques.
- For some special cases, they can be calculated analytically.

Mutual Coupling: N antennas

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{2N} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

⇨

$$\underline{\mathbf{V}} = \mathbf{Z}\underline{\mathbf{I}}$$

$$Z_{ij} = \left.\frac{V_i}{I_j}\right|_{I_k=0, k\neq j}$$

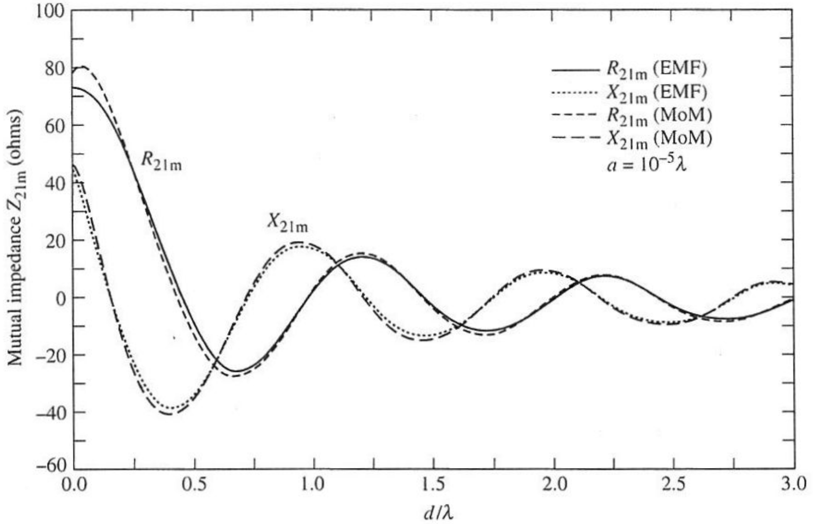
In an N-element uniform array $I_n = I_0 e^{j(n-1)\beta}$

$$Z_{1d} = \frac{V_1}{I_1} = Z_{11} + Z_{12}e^{j\beta} + \cdots + Z_{1N}e^{j(N-1)\beta}$$

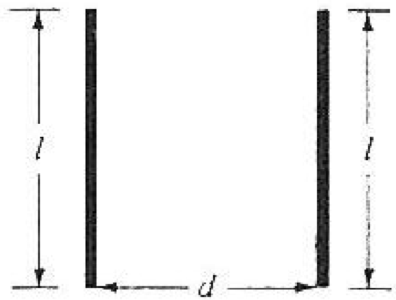
$$= \sum_{n=1}^{N} Z_{1n} e^{j(n-1)\beta}$$

$$\text{Note}: Z_{1d} \text{ changes as } \beta \text{ changes.}$$

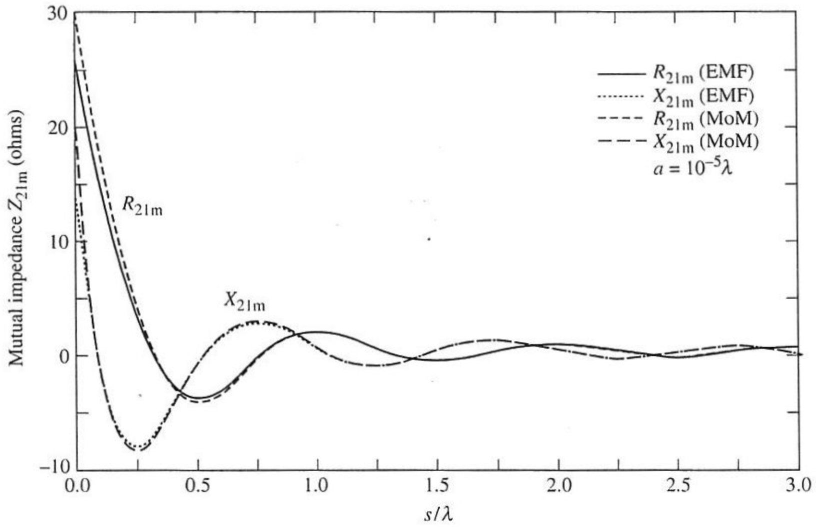
Mutual Coupling: 2 dipoles



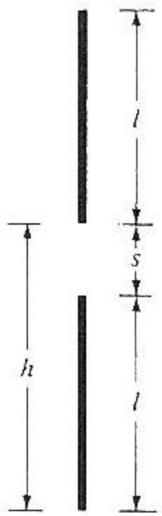
(a) Side-by-side



(a) Side-by-side

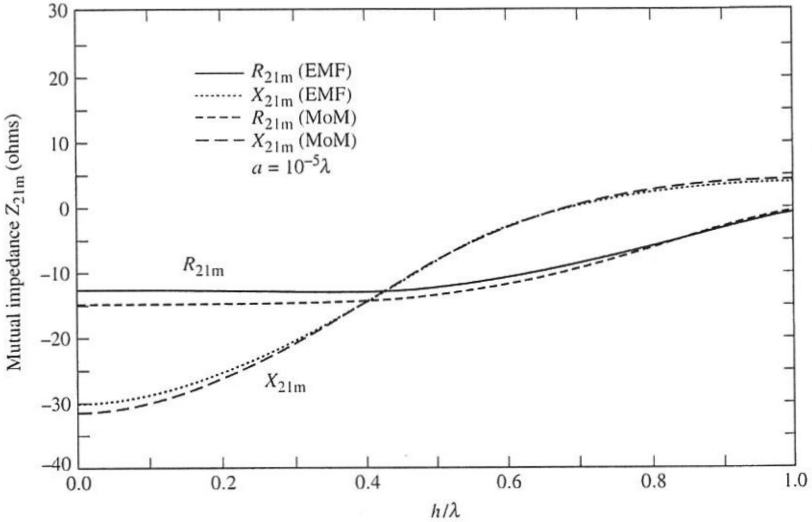


(b) Collinear

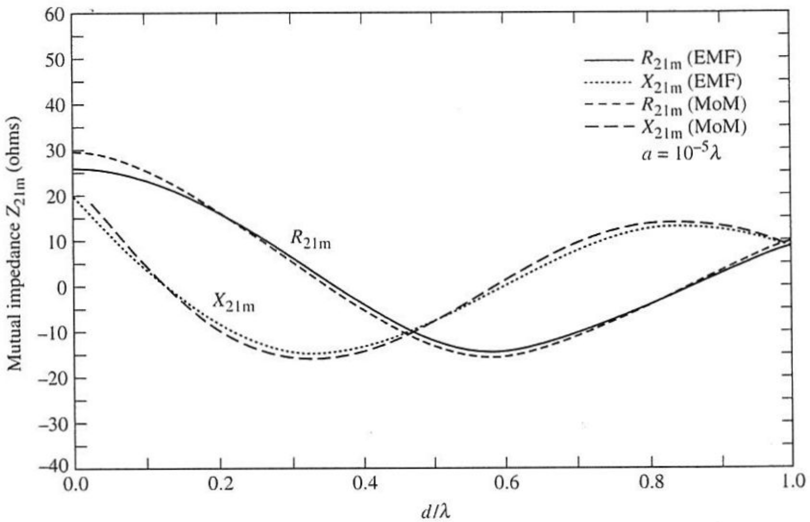


(b) Collinear

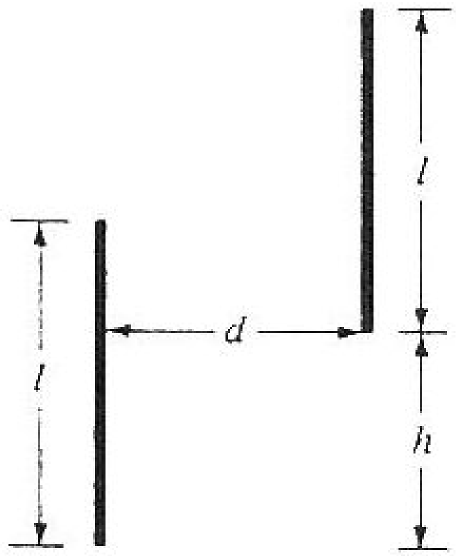
Mutual Coupling: 2 dipoles (2)



(a) $d = 0.5\lambda$



(b) $h = 0.5\lambda$



(c) Parallel-in-echelon