Wave Equation and its Solution

Maxwell's equations in an isotropic homogeneous media are:

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} && (1) \\ \nabla \times \mathbf{H} &= \mathbf{J} + j\omega\epsilon\mathbf{E} && (2) \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon} && (3) \\ \nabla \cdot \mathbf{B} &= 0 && (4) \end{aligned}$$

Taking the curl of (2) yields

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \mathbf{J} + j\omega\epsilon(\nabla \times \mathbf{E}), \quad (5)$$

thus,

$$\nabla(\nabla \cdot \mathbf{H}) - \nabla^2\mathbf{H} = \nabla \times \mathbf{J} + j\omega\epsilon(-j\omega\mu\mathbf{H}), \quad (6)$$

or

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = -\nabla \times \mathbf{J}; \quad k = \omega\sqrt{\mu\epsilon}. \quad (7)$$

Likewise, taking the curl of (1) yields

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu(\nabla \times \mathbf{H}), \quad (8)$$

thus,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -j\omega\mu(\mathbf{J} + j\omega\epsilon\mathbf{E}), \quad (9)$$

or

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = j\omega\mu(\mathbf{J} + \frac{1}{k^2}\nabla(\nabla \cdot \mathbf{J}). \quad (10)$$

Since $\nabla \cdot \mathbf{J} = -j\omega\epsilon\nabla \cdot \mathbf{E}$ upon taking the divergence of (2). Clearly, (7) and (10) are relatively complicated to solve directly for $\mathbf{H}$ and $\mathbf{E}$, respectively. On the other hand, one can introduce a potential field $\mathbf{A}$ via (4)

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \quad (11)$$

because $\nabla \cdot \mathbf{B} = \nabla \cdot \nabla \times \mathbf{A} \equiv 0$ and (4) is thus satisfied . From (1) and (11):

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0, \quad (12)$$

or,

$$\mathbf{E} + j\omega\mathbf{A} = -\nabla V, \quad (13)$$

where V is a scalar potential because $\nabla \times \nabla V = 0$ and (12) is thus satisfied. Likewise, from (2) and (13)

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon(-j\omega\mathbf{A} - \nabla V), \quad (14)$$

or

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{A} \quad = \quad \mathbf{J} + j\omega\epsilon(-j\omega\mathbf{A} - \nabla V), \tag{15}$$

i.e.,

$$\nabla \times \nabla \times \mathbf{A} \quad = \quad \mu\mathbf{J} + k^2\mathbf{A} - j\omega\mu\epsilon\nabla V. \tag{16}$$

Therefore,

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \quad = \quad \mu\mathbf{J} + k^2\mathbf{A} - j\omega\mu\epsilon\nabla V, \tag{17}$$

or,

$$\nabla^2\mathbf{A} + k^2\mathbf{A} \quad = \quad -\mu\mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\mu\epsilon V). \tag{18}$$

In order to completely specify $\mathbf{A}$, it is necessary to specify $\nabla \times \mathbf{A}$ as well as $\nabla \cdot \mathbf{A}$. One is free to choose $\nabla \cdot \mathbf{A}$ in any appropriate fashion. From (18), a convenient choice for $\nabla \cdot \mathbf{A}$ is given below such that V can be found from $\mathbf{A}$;

$$\nabla \cdot \mathbf{A} \quad = \quad -j\omega\mu\epsilon V. \tag{19}$$

(19) is consistent with the equation of continuity and it is referred to as the Lorentz condition. From (18) and (19):

$$\nabla^2\mathbf{A} + k^2\mathbf{A} \quad = \quad -\mu\mathbf{J}. \tag{20}$$

(20) is relatively simpler to work with than either (7) or (10). It is also noted that all these three equations are vector wave equations. Note that the divergence of (13) yields

$$\nabla \cdot \mathbf{E} + j\omega\nabla \cdot \mathbf{A} \quad = \quad -\nabla^2 V. \tag{21}$$

Incorporating (3) and (19) into (21) yields

$$\nabla^2 V + k^2 V \quad = \quad -\frac{\rho}{\epsilon}, \tag{22}$$

thus V also satisfies the wave equation.

To obtain a solution for $\mathbf{A}$ in an unbounded homogeneous isotropic media, first consider $\mathbf{A}_\delta$ where

$$\nabla^2\mathbf{A}_\delta + k^2\mathbf{A}_\delta \quad = \quad -\mu\hat{a}\delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{23}$$

Let $\mathbf{A}_\delta \equiv \hat{a}A_\delta$, then

$$(\nabla^2 + k^2)A_\delta \quad = \quad -\mu\delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{24}$$

which is referred to as a scalar wave equation. If $\boldsymbol{r} \neq \boldsymbol{r}'$, then A_δ satisfies

$$(\nabla^2 + k^2)A_\delta \quad = \quad 0; \quad \boldsymbol{r} \neq \boldsymbol{r}'. \tag{25}$$

Let $R \equiv |\boldsymbol{r} - \boldsymbol{r}'|$, then expressing ∇^2 in a coordinate system centered at the source point $\boldsymbol{r}'$, it is not difficult to see that A_δ in unbounded space would be spherically symmetric about the point $\boldsymbol{r}'$. Thus, (25) yields

$$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{\partial A_\delta}{\partial R}) + k^2 A_\delta \quad = \quad 0, \tag{26}$$

i.e.,

$$\frac{1}{R}\frac{\partial^2}{\partial^2 R}(RA_\delta) + k^2 A_\delta \quad = \quad 0, \tag{27}$$

thus,

$$\frac{\partial^2}{\partial^2 R}(RA_\delta) + k^2(RA_\delta) \quad = \quad 0. \tag{28}$$

The general solution to the above equation can be given by

$$A_\delta \quad = \quad C_1\frac{e^{-jkR}}{R} + C_2\frac{e^{jkR}}{R}. \tag{29}$$

From the radiation condition, $C_2 \equiv 0$, thus

$$A_\delta \quad = \quad C_1\frac{e^{-jkR}}{R}. \tag{30}$$

One can find the constant C_1 by integrating (24) within a small spherical volume of radius ζ which is centered at $\boldsymbol{r}'$; this allows the source condition at $\boldsymbol{r}'$ to be fulfilled as $\zeta \to 0$. Therefore in terms of (R, θ', ϕ') with respect to $0'$:

$$\underbrace{\int_0^{2\pi}\int_0^{\pi}\int_0^{\zeta}}_{V\to 0} dRd\theta' d\phi' R^2 \sin\theta' (\nabla^2 + k^2)A_\delta \quad = \quad -\mu \underbrace{\iiint}_{V\to 0} dv\delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{31}$$

As $V \to 0$, the integral on the RHS of (31) yields unity via the property of the delta function (since $\boldsymbol{r}'$ is in V as $V \to 0$). The term $k^2 A_\delta$ on the LHS of (31) vanishes as $\zeta \to 0$ (for $V \to 0$). Thus,

$$\underbrace{\iiint}_{V\to 0} dv\nabla^2 A_\delta \quad = \quad -\mu. \tag{32}$$

However, $\nabla^2 A_\delta = \nabla \cdot \nabla A_\delta$; hence, the above becomes

$$\underbrace{\iiint}_{V\to 0} dv\nabla \cdot \nabla A_\delta \quad = \quad -\mu. \tag{33}$$

Applying the divergence theorem yields

$$\underbrace{\iiint}_{V\to 0} dv\nabla \cdot \nabla A_\delta \quad = \quad \underbrace{\oiint}_{S_V\to 0} d\boldsymbol{s} \cdot \nabla A_\delta; \quad d\boldsymbol{s} = \hat{R}ds. \tag{34}$$

Incorporating the above result into (33) gives

$$\underbrace{\oiint}_{S_V \to 0} ds(\hat{R} \cdot \nabla A_\delta) = \underbrace{\int_0^{2\pi} \int_0^{\pi} d\theta' d\phi' R^2 \sin\theta'}_{ds} \frac{\partial A_\delta}{\partial R}$$
$$= -\mu. \tag{35}$$

From (35) and $A_\delta = C_1 \frac{e^{-jkR}}{R}$,

$$\int_0^{2\pi} \int_0^{\pi} d\theta' d\phi' R^2 \sin\theta' \frac{\partial A_\delta}{\partial R} = 2\pi C_1 \int_0^{\pi} d\theta' R^2 \sin\theta' (\frac{-jk}{R} - \frac{1}{R^2}) e^{-jkR} \tag{36}$$

Taking the limit of the above result as $\zeta \to 0$ yields

$$\lim_{\zeta \to 0} 2\pi C_1 \int_0^{\pi} d\theta' R^2 \sin\theta' (\frac{-jk}{R} - \frac{1}{R^2}) e^{-jkR} \Big|_{R=\zeta} = -\mu, \tag{37}$$

i.e., $-(4\pi)C_1 = -\mu$. Hence,

$$C_1 = \frac{\mu}{4\pi}. \tag{38}$$

The $\frac{e^{-jkR}}{4\pi R}$ is the well-known free-space Green's function, which is used to compute the field due to a point source in free space. In general, if the Green's function pertaining to the electric field is known, the electric field due to any electric current source and magnetic current source can be obtained by

$$\mathbf{E} = \iiint_V \overline{\overline{\Gamma}}^{ee} \cdot \mathbf{J}_V dv, \tag{39}$$

and

$$\mathbf{E} = \iiint_V \overline{\overline{\Gamma}}^{em} \cdot \mathbf{M}_V dv, \tag{40}$$

respectively, where $\overline{\overline{\Gamma}}^{ee}$ and $\overline{\overline{\Gamma}}^{em}$ are the Green's functions pertaining to the electric field due to the electric point source, and the magnetic point source, respectively.